NUMBER OF MISSING LABEL OPERATORS AND UPPER BOUNDS FOR DIMENSIONS OF MAXIMAL LIE SUBALGEBRAS

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We analyze numerically the equation giving the number of missing label operators for reduction chains $\mathfrak{k} \hookrightarrow \mathfrak{g}$ of Lie algebras to obtain information about the maximal possible dimension of certain types of subalgebras, mainly Abelian. Applications to the minimal dimension of faithful representations are given, and the number of invariants of codimension one and two subalgebras is analyzed.

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1. Introduction

In many problems of nuclear physics where Lie algebras and groups are applied, one is confronted with the fact that often the physically important states do not constitute bases of subgroups providing the maximum number of state labels (*e.g.* the $\mathfrak{so}(5)$ seniority model or the Elliot classification of levels in $\mathfrak{su}(3)$ [1]). One possible procedure to solve this problem is to use the common eigenstates of commuting sets of operators as bases. Usually such operators are found integrating systems of partial differential equations related to the corresponding Lie algebra and subalgebra pairs [2]. Among the possible choices for reduction chains, Abelian subalgebras provide a specially useful class. For problems of symmetry breaking of physical systems they play a relevant role, such as the quantum-mechanical frame, where they naturally allow to construct complete sets of commuting operators. The eigenvalues of such operators provide one solution to the labeling of irreducible representations of a Lie group with respect to some (Abelian) subgroup. For the analysis of dynamical systems, Abelian subalgebras are usually employed to determine possible integrals of motion which are in involution [3].

Given a general Lie group of symmetries and its Lie algebra, it is usually not trivial to see which types of subalgebras it admits, as well as the maximal possible dimensions and the number of invariants of subalgebras. It is, therefore, of interest to obtain some information derived from numerical invariants of the Lie algebra, such as its dimension and the number of invariants, as well as their structure. In this work we analyze the equation providing the number of missing labels for a reduction chain $\mathfrak{a} \hookrightarrow \mathfrak{g}$ to derive certain properties of the subalgebras, such as their maximal possible dimension, their number of invariants as a function of its codimension or the existence of common invariants between the algebra and the subalgebra(s).

Given a basis $\{X_1, \ldots, X_n\}$ of the Lie algebra $\mathfrak{g}$ and the structure tensor $\left\{C_{ij}^k\right\}$, $\mathfrak{g}$ can be realized in the space $C^\infty(\mathfrak{g}^*)$ by means of differential operators:

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k$ $(1 \leq i < j \leq n)$ and $\{x_1, \ldots, x_n\}$ is a dual basis of $\{X_1, \ldots, X_n\}$. We say that an analytic function $F \in C^\infty(\mathfrak{g}^*)$ is an invariant of $\mathfrak{g}$ if and only if it is a solution of the system of partial differential equations (PDEs):

$$\widehat{X}_i F = 0, \qquad 1 \leq i \leq n. \tag{2}$$

The cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions (in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis) is easily obtained from the classical criteria for PDEs, and equals:

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \operatorname{rank}\left(C_{ij}^k x_k\right)_{1 \leq i < j \leq \dim \mathfrak{g}}, \tag{3}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the given basis. If we identify the dual space $\mathfrak{g}^*$ with the left-invariant Pfaffian forms on a Lie group whose algebra is isomorphic to $\mathfrak{g}$, we can define an exterior differential d on $\mathfrak{g}^*$ as follows:

$$d\omega(X_i, X_j) = -C_{ij}^k \omega(X_k), \quad \omega \in \mathfrak{g}^*. \tag{4}$$

Therefore, we can rewrite $\mathfrak{g}$ as a system of 2-forms

$$d\omega_k = -C_{ij}^k \omega_i \wedge \omega_j, \quad 1 \leq i < j \leq \dim(\mathfrak{g}), \tag{5}$$

where the Jacobi condition is equivalent to the closure $d^2\omega_i = 0$ for all i. If $\mathcal{L}(\mathfrak{g}) = \mathbb{R}\{d\omega_i\}_{1 \leq i \leq \dim \mathfrak{g}}$ denotes be the linear subspace of $\bigwedge^2 \mathfrak{g}^*$ generated by the 2-forms $d\omega_i$, for any element in general position $\omega = a^i d\omega_i$ $\left(a^i \in \mathbb{R}\right)$ we can find a positive integer $j_0(\omega) \in \mathbb{N}$ such that

$$\bigwedge^{j_0(\omega)} \omega \neq 0\,, \qquad \bigwedge^{j_0(\omega)+1} \omega \equiv 0\,. \tag{6}$$

It follows that $r(\omega) = 2j_0(\omega)$ is the rank of the 2-form ω. The quantity $j_0(\mathfrak{g})$ given by

$$j_0(\mathfrak{g}) = \max\left\{j_0(\omega) \mid \omega \in \mathcal{L}(\mathfrak{g})\right\} \tag{7}$$

is a numerical invariant of $\mathfrak{g}$. In [4] it was shown that equation (3) can be rewritten as:

$$\mathcal{N}(\mathfrak{g}) = \dim \mathfrak{g} - 2j_0(\mathfrak{g})\,. \tag{8}$$

In this context, $j_0(\mathfrak{g})$ has a precise geometrical meaning, namely the number of internal labels needed to label the states of irreducible representations of $\mathfrak{g}$ [5].

Classically, the eigenvalues of Casimir operators of semisimple Lie algebras have been used to label without ambiguity the irreducible representations [5]. In a more general frame, irreducible representations of a Lie algebra $\mathfrak{g}$ are labeled using the eigenvalues of its generalized Casimir invariants [2]. It is often necessary to consider a subalgebra $\mathfrak{h}$ to label the basis states of $\mathfrak{g}$. The reduction chain $\mathfrak{h} \hookrightarrow \mathfrak{g}$ provides $\frac{1}{2}(\dim \mathfrak{h} + \mathcal{N}(\mathfrak{h})) + l'$ labels, where l' is the number of invariants of $\mathfrak{g}$ that depend only on variables of the subalgebra $\mathfrak{h}$ [2]. In order to label irreducible representations of $\mathfrak{g}$ uniquely, it is therefore, necessary to find

$$n = \frac{1}{2}\Big(\dim \mathfrak{g} - \mathcal{N}(\mathfrak{g}) - \dim \mathfrak{h} - \mathcal{N}(\mathfrak{h})\Big) + l' \tag{9}$$

additional operators, which are usually called missing label operators. These are traditionally found integrating the equations of system (2) corresponding to the subalgebra generators. It follows easily from analytical considerations that the total number of available operators is $m = 2n$.

The main purpose of this work is to analyze equation (9) generically, *i.e.*, without taking into account the concrete structure of the Lie algebra $\mathfrak{g}$. This will allow us to deduce some properties about the subalgebras or the relation between dimension and number of invariants.

2. Maximal dimension of Abelian subalgebras

In this section we analyze equation (9) numerically to extract information concerning the relation between the dimension of maximal Abelian subalgebras (short MASA) of $\mathfrak{g}$, and the number of common and total invariants l' and $\mathcal{N}(\mathfrak{g})$, respectively.

In general, if we impose a subalgebra $\mathfrak{a}$ to be Abelian, equation (9) gives the inequality:

$$\frac{1}{2}\Big(\dim\mathfrak{g}-\mathcal{N}(\mathfrak{g})-2\dim\mathfrak{a}\Big)+l'\geq 0\,, \tag{10}$$

thus the dimension of $\mathfrak{a}$ is bounded by

$$\dim\mathfrak{a}\leq\frac{\dim\mathfrak{g}-\mathcal{N}(\mathfrak{g})+2l'}{2}\,. \tag{11}$$

This bound involves however three unknowns, and in order to obtain useful information we have to impose additional conditions on some of these parameters.

Proposition 1 *Let $\mathfrak{a}$ be an Abelian subalgebra of $\mathfrak{g}$ such that $\dim\mathfrak{a}>\frac{1}{2}\dim\mathfrak{g}$. Then $\mathfrak{a}$ and $\mathfrak{g}$ have at least $[\mathcal{N}(\mathfrak{g})/2]$ invariants in common. In particular, $\mathcal{N}(\mathfrak{g})>0$.*

Proof. By assumption, the dimension of $\mathfrak{a}$ exceeds half the dimension of $\mathfrak{g}$. Therefore there is an $\alpha>0$[1] such that $\dim\mathfrak{g}+\alpha=2\dim\mathfrak{a}$. Further, $\mathcal{N}(\mathfrak{a})=\dim\mathfrak{a}$ by abelianity, so that the number of missing labels is given by

$$\begin{aligned} n &= \frac{1}{2}\Big(\dim\mathfrak{g}-\mathcal{N}(\mathfrak{g})-2\dim\mathfrak{a}\Big)+l' \\ &= \frac{1}{2}\Big(-\alpha-\mathcal{N}(\mathfrak{g})\Big)+l'\,. \end{aligned} \tag{12}$$

Since necessarily $n\geq 0$, the latter implies that $2l'\geq\alpha+\mathcal{N}(\mathfrak{g})$, and therefore, at least half the invariants of $\mathfrak{g}$ are also invariants of the subalgebra. More specifically, the lower bound for the number of common invariants is expressed by:

$$l'\geq\frac{\dim\mathfrak{g}+\mathcal{N}(\mathfrak{g})}{2}-\mathrm{codim}_{\mathfrak{g}}\mathfrak{a}\,. \tag{13}$$

Now, if $\mathcal{N}(\mathfrak{g})$ were zero, then $l'=0$, contradicting the non-negativeness of n. ■

In particular, the preceding formula implies that if $l'=0$, then

$$\dim\mathfrak{a}\leq\frac{1}{2}\Big(\dim\mathfrak{g}-\mathcal{N}\left(\mathfrak{g}\right)\Big)\,.$$

for any Abelian subalgebra. If, moreover, $\mathcal{N}(\mathfrak{g})=0$, then the dimension of $\mathfrak{a}$ is at most half the dimension of $\mathfrak{g}$. It is not difficult to construct Lie algebras satisfying the equality in the preceding upper bound. For example, the

[1] More specifically, this number is given by $\alpha=\dim\mathfrak{g}-2\mathrm{codim}_{\mathfrak{g}}\mathfrak{a}$.

solvable Lie algebras of rank one and a Heisenberg nil-radical[2] and having no invariants, classified in [6]. These algebras naturally have a maximal Abelian subalgebra of half its dimension, showing that the bound (13) is sharp. In physical problems solvable algebras of this type are used whenever it is necessary to consider symmetries that go beyond translations and classical Galilean boosts.

From Proposition 1 we can also derive an important consequence for Lie algebras $\mathfrak{g}$ satisfying $\mathcal{N}(\mathfrak{g}) = 0$ and admitting a MASA of dimension $\frac{\dim \mathfrak{g}}{2}$. This corollary concerns the minimal dimension of a faithful representation, giving a partial solution to the problem of Ado [7] for this special class of algebras.

Corollary 1 *Let* $\mathfrak{g}$ *be an indecomposable Lie algebra such that* $\mathcal{N}(\mathfrak{g}) = 0$ *and let* $\mathfrak{a}$ *be an Abelian subalgebra such that* $\dim \mathfrak{a} = \frac{1}{2} \dim \mathfrak{g}$. *Then* $\mathfrak{g}$ *does not admit faithful representations of degree* n *such that*

$$n < \begin{cases} \sqrt{2 \dim \mathfrak{g} - 4}\,, & \text{if } \dim \mathfrak{g} \neq 2t^2 + 2,\ t \geq 1\,, \\ \sqrt{2 \dim \mathfrak{g} - 4} - 1\,, & \text{if } \dim \mathfrak{g} = 2t^2 + 2,\ t \geq 1\,. \end{cases}$$

Proof. If $\mathfrak{g}$ admits a faithful representation of degree n, then the Abelian subalgebra $\mathfrak{a}$ is realized by commuting matrices. It is a classical result[3] that any commutative subalgebra of $M_{n\times n}(\mathbb{R})$ has dimension not exceeding $1 + \left[n^2/4\right]$. Therefore, the values for which the following equality

$$1 + \left[\frac{n^2}{4}\right] < \frac{\dim \mathfrak{g}}{2} \tag{14}$$

holds, cannot provide faithful representations of $\mathfrak{g}$. Writing the square n^2 modulo 4, *i.e.*,

$$n^2 = 4q + \alpha\,, \qquad 0 \leq \alpha \leq 1\,,$$

we easily obtain that $\left[n^2/4\right] = q$ and $q < \frac{\dim \mathfrak{g}}{2} - 1$. From this we easily obtain the approximation

$$n < \sqrt{2 \dim \mathfrak{g} - 4}\,.$$

This bound is accurate except for the case where $\dim \mathfrak{g} = 2t^2 + 2$, where we obtain an equality in equation (14)[4]. Therefore, in this particular case we have to take the correction $n < \sqrt{2 \dim \mathfrak{g} - 4} - 1$.

■

[2] *i.e.*, with maximal nilpotent ideal.
[3] See for example reference [8], Sec. 2.
[4] Because then $2 \dim \mathfrak{g} - 4$ is a square.

Proposition 1 has further two interesting consequences for solvable Lie algebras, and more specifically for the construction of solvable algebras having a specific number of invariants [6, 9, 10].

Corollary 2 *If* $\mathfrak{r}$ *is solvable with* $\mathcal{N}(\mathfrak{r}) = 0$*, then its maximal nilpotent ideal* NR *has at most* $\mathrm{codim}_{\mathfrak{r}} NR$ *invariants. In particular* NR *cannot be Abelian.*

Proof. Since $\mathfrak{r}$ is indecomposable, its nil-radical NR satisfies the following strict inequality [11]:

$$\dim NR > \frac{1}{2} \dim \mathfrak{r}\,.$$

Let $\alpha = \mathrm{codim}_{\mathfrak{r}} NR$ be the codimension of NR[5]. Then

$$\dim \mathfrak{r} - \mathcal{N}(NR) - \dim NR = \alpha - \mathcal{N}(NR) \geq 0. \tag{15}$$

In particular, $\alpha < \dim NR$, which proves that $\dim NR > \mathcal{N}(NR)$ strictly, so that NR cannot be an Abelian algebra.
■

This result can be sharpened without imposing that the algebra $\mathfrak{r}$ has no invariants.

Corollary 3 *Let* $\mathfrak{r}$ *be a solvable indecomposable Lie algebra. If it has no common invariants with its nil-radical* NR*, then* NR *cannot be Abelian.*

Proof. By indecomposability, we have $\dim \mathfrak{r} = \dim NR + \alpha$, where $\alpha < \dim NR$. If $\mathfrak{r}$ and NR have no common invariants, then $l' = 0$, and applying the formula (9) we obtain

$$\frac{1}{2} \left(\alpha - \mathcal{N}(\mathfrak{r}) - \mathcal{N}(NR)\right) \geq 0. \tag{16}$$

If NR were Abelian, then $\mathcal{N}(NR) = \dim NR$, but then $\alpha - \mathcal{N}(NR) < 0$, contradicting equation (16).
■

The opposite case to $l' = 0$ consists of the case where all the invariants of the Lie algebra $\mathfrak{g}$ belong to some Abelian subalgebra. Semidirect products $\mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ of a semisimple algebra $\mathfrak{s}$ and solvable algebra $\mathfrak{r}$ have this property for representations R of $\mathfrak{s}$ of high dimension (in general, that exceed the dimension of the Levi subalgebra [12]).

[5] We recall that $\mathrm{codim}_{\mathfrak{r}}(NR) = \dim \mathfrak{r} - \dim NR$.

Proposition 2 *Let $\mathfrak{g}$ be an indecomposable Lie algebra and $\mathfrak{a}$ an Abelian subalgebra such that any invariant of $\mathfrak{g}$ is an invariant of $\mathfrak{a}$. Then the dimension of $\mathfrak{a}$ is bounded by*

$$\dim \mathfrak{a} \leq \frac{1}{2} \left(\dim \mathfrak{g} + \mathcal{N}(\mathfrak{g}) \right) . \tag{17}$$

The proof follows at once applying equation (9). We observe that this bound is indeed attainable. In fact, the indecomposable Lie algebras underlying eleven dimensional spatially homogeneous space-times containing a compact subalgebra of at least dimension seven [13] satisfy the preceding proposition[6]. For the physical models related to these indecomposable algebras, the result above tells that the eigenvalues of the invariants of the algebra are determined by the Abelian subalgebra, and therefore, the labeling of the states are described by the maximal compact subalgebra.

3. Algebras with Heisenberg radical

Semidirect products of simple and Heisenberg algebras constitute another important type of algebras in physics, for example in the microscopic theory of collective motions in nuclei [14].Their general structure and invariants were analyzed in [15]. In this section we obtain a bound for MASAs that is attainable only for Levi parts isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$, and indicates that for semidirect products with higher rank Levi parts the structure of Abelian subalgebras is related to the branching rules of representations of simple algebras with respect to rank one simple subalgebras.

Proposition 3 *Let $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_N$ be the semidirect product of a semisimple Lie algebra and a Heisenberg algebra $\mathfrak{h}_N$. Then the dimension of a MASA $\mathfrak{a}$ is bounded by*

$$\dim \mathfrak{a} \leq 1 + N + j_0 \left(\mathfrak{s} \right) .$$

Proof. Since $\mathfrak{a}$ is maximal Abelian, it necessarily contains the centre of $\mathfrak{g}$, which is generated by the centre of its radical $\mathfrak{h}_N$. This fact proves that the minimal dimension of $\mathfrak{a}$ equals $N+1$. On the other hand, since $\mathcal{N}(\mathfrak{g}) = \mathcal{N}(\mathfrak{s}) + 1$ [15], one of the invariants being the centre generator, we obtain that $l' \geq 1$. As shown in [16], the remaining invariants of $\mathfrak{g}$ depend on all the generators of $\mathfrak{g}$, thus we have $l' = 1$. By formula (9)

$$\begin{aligned} & \frac{1}{2}\Big(\dim \mathfrak{g} - \mathcal{N}(\mathfrak{s}) - 1 - \dim \mathfrak{a} - \mathcal{N}(k) \Big) + 1 \\ = \ & \frac{1}{2}\Big(\dim \mathfrak{s} + 2N + 1 - \mathcal{N}(s) - 1 - 2\dim \mathfrak{a} \Big) + 1 \geq 0 . \end{aligned} \tag{18}$$

[6] Among the forty classes of Lie algebras obtained, only six of them are indecomposable Lie algebras, *i.e.*, do not split into a direct sum of lower dimensional algebras (see [13] for details on this classification).

Now $\dim \mathfrak{s} = 2j_0(\mathfrak{s}) + \mathcal{N}(\mathfrak{s})$ by formula (8), and therefore, we obtain

$$\frac{1}{2}\left(2j_0(\mathfrak{s}) + 2N - 2\dim \mathfrak{a}\right) + 1 \geq 0\,,$$

from which the inequality

$$\dim \mathfrak{a} \leq 1 + N + j_0(\mathfrak{s})$$

follows. ∎

In general, the upper bound above is sharp, as shows the following example. Consider the 8-dimensional Lie algebra $L_{8,6} = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_{\frac{1}{2}} \oplus 3D_0} \mathfrak{h}_2$ given by the structure tensor

$$\begin{array}{lll} C_{12}^{2} &=& 2\,,\ C_{13}^{3} = -2\,,\ C_{23}^{1} = 1\,,\ C_{14}^{4} = 1\,,\ C_{15}^{5} = -1\,, \\ C_{25}^{4} &=& 1\,,\ C_{34}^{5} = 1\,, \quad C_{45}^{8} = 1\,,\ C_{67}^{8} = 1\,, \end{array} \tag{19}$$

over the basis $\{X_1, \ldots, X_8\}$[7]. We thus have $N = 2$. A maximal Abelian subalgebra of dimension 4 is given for example by $\{X_2, X_4, X_6, X_8\}$. Since the Levi part is $\mathfrak{sl}(2,\mathbb{R})$, we have $j_0(\mathfrak{s}) = 1$.

This example shows moreover, that the upper bound for MASA depends essentially on the structure of the representation R. In particular, for the simple Levi part of rank one we obtain the following result:

Proposition 4 *If* $\mathrm{rank}(\mathfrak{s}) = 1$ *and* $R = R' \oplus D_0$*, where* R' *is an irreducible representation of the Levi part* $\mathfrak{s}$ *and* D_0 *the trivial representation, then an Abelian subalgebra has to most dimension* $N+1$*. Moreover, a MASA of* $\mathfrak{h}_N$ *is also a MASA of* $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{h}_N$*.*

Proof. We separate the proof for the two simple algebras of rank one. For the normal real form $\mathfrak{sl}(2,\mathbb{R})$ with brackets

$$[H, X_{\pm}] = 2(-1)^{\pm 1} X_{\pm},\ [X_+, X_-] = H\,,$$

over the basis $\{H, X_+, X_-\}$, the irreducible representations D_j $\left(j \in \frac{\mathbb{N}}{2}\right)$ of dimension $2j+1$ are given by the matrices:

[7] See reference [17] for the notation on the isomorphism classes.

$$D_j\left(H\right)=\begin{pmatrix} 2j & & & & \\ & 2j-2 & & & \\ & & \ddots & & \\ & & & -2j-2 & \\ & & & & -2j \end{pmatrix},$$

$$D_j\left(X_+\right)=\begin{pmatrix} 0 & 2j & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix},$$

$$D_j\left(X_-\right)=\begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & 2j & 0 \end{pmatrix}.$$

By assumption, R' is an irreducible representation. Since it is compatible with a Heisenberg algebra, this implies that j must be of the form $j = 2p+1/2$ for some $p \geq 0$ [15]. Therefore, the diagonal matrix $D_j(H)$ has not zero as eigenvalue. It follows that

$$\dim\ker D_j\left(H\right)=0\,,\quad \dim\ker D_j\left(X_\pm\right)=1\,,$$

for all $p \geq 0$. Now let $\mathfrak{a}$ be a maximal Abelian subalgebra of the radical $\mathfrak{h}_N$. It satisfies $\dim\mathfrak{a} = N+1$. The preceding equation shows that

$$\dim\left(C_{\mathfrak{g}}\left(X\right)\cap\mathfrak{a}\right)\leq 1$$

for the centralizer $C_{\mathfrak{g}}\left(X\right)$ in $\mathfrak{g}$ of an arbitrary element X of the Levi subalgebra $\mathfrak{sl}\left(2,\mathbb{R}\right)$. Therefore, a maximal Abelian algebra has dimension $N+1$, and is contained in the radical of $\mathfrak{g}$.

For the compact case $\mathfrak{s} = \mathfrak{so}\left(3\right)$, the above matrices (20) provide complex representations, given by the following expressions[8]

[8] Here we have followed the notation used by Turkowski in [13].

$$\begin{aligned}
D^j\left(\overline{H}\right) &= \frac{i}{2}\left(D_j\left(X_+\right)+D_j\left(X_-\right)\right),\\
D^j\left(\overline{X}_+\right) &= -\frac{1}{2}D_j\left(X_+\right)+\frac{1}{2}D_j\left(X_-\right),\\
D^j\left(\overline{X}_-\right) &= \frac{i}{2}D_j\left(H\right),
\end{aligned}$$

where $\left\{\overline{H},\overline{X}_+,\overline{X}_-\right\}$ is a basis of $\mathfrak{so}(3)$. In particular, $D^j\left(\overline{H}\right)$ and $D^j\left(\overline{X}_+\right)$ are matrices of full rank, while the rank of $D^j\left(\overline{X}_-\right)$ coincides with that of $D_j\left(H\right)$. Since $j=\frac{2p+1}{2}$, the corresponding real representations are given by the matrices of double size

$$\begin{aligned}
D^{II,j}\left(\overline{H}\right) &= \frac{1}{2}\begin{pmatrix} 0 & -D_j\left(X_+\right)-D_j\left(X_-\right)\\ D_j\left(X_+\right)+D_j\left(X_-\right) & 0\end{pmatrix},\\
D^{II,j}\left(\overline{X}_+\right) &= \frac{1}{2}\begin{pmatrix} -D_j\left(X_+\right)+D_j\left(X_-\right) & 0\\ 0 & -D_j\left(X_+\right)+D_j\left(X_-\right)\end{pmatrix},\\
D^{II,j}\left(\overline{X}_-\right) &= \frac{1}{2}\begin{pmatrix} 0 & -D_j\left(H\right)\\ D_j\left(H\right) & 0\end{pmatrix}.
\end{aligned}$$

By the observation on the rank of the block matrices we deduce that

$$\dim\ker D^{II,j}\left(\overline{H}\right)=\dim\ker D^{II,j}\left(\overline{X}_\pm\right)=0,$$

showing that no element of $\mathfrak{so}(3)$ can belong to a maximal Abelian subalgebra of $\mathfrak{so}_(3)\overrightarrow{\oplus}_R\mathfrak{h}_N$. ∎

We remark that, in general, this result is false if the rank of the Levi part $\mathfrak{s}$ is greater or equal to 2. Consider the simple algebra $\mathfrak{sp}\left(4,\mathbb{R}\right)$ given by its standard boson representation

$$X_{i,j}=a_i^\dagger a_j,\quad X_{-i,j}=a_i^\dagger a_j^\dagger,\quad X_{i,-j}=a_ia_j \tag{20}$$

for $1\leq i,j\leq 2$. Consider the matrix

$$A:=\begin{pmatrix} x_{1,1} & x_{1,2} & x_{-1,1} & x_{-1,2}\\ x_{2,1} & x_{2,2} & x_{-1,2} & x_{-2,2}\\ x_{1,-1} & x_{1,-2} & -x_{1,1} & -x_{,21}\\ x_{1,-2} & x_{2,-2} & -x_{1,2} & -x_{2,2}\end{pmatrix}. \tag{21}$$

It is easy to see that A can be rewritten as

$$A=\sum_{i,j}x_{i,j}\Gamma_{\omega_1}(X_{i,j}), \tag{22}$$

where $\Gamma_{\omega_1}(X_{i,j})$ is the matrix corresponding to the generator $X_{i,j}$ by the standard representation Γ_{ω_1} of $\mathfrak{sp}(4,\mathbb{R})$. Since the representation Γ_{ω_1} satisfies $\bigwedge^2 \Gamma_{\omega_1} \supset \Lambda_0$, where Λ_0 is the zero representation of $\mathfrak{sp}(4,\mathbb{R})$, Γ_{ω_1} is compatible with the Heisenberg algebra $\mathfrak{h}_2$ of dimension five [18]. We can, therefore, construct the 15 dimensional Lie algebra $w\mathfrak{sp}(2,\mathbb{R}) = \mathfrak{sp}(4,\mathbb{R}) \overrightarrow{\oplus}_{\Gamma_{\omega_1}\oplus\Lambda_0} \mathfrak{h}_2$. Let $\{P_1, P_2, P_3, P_4, P_5\}$ be a basis of the radical[9]. It follows from (21) that the generators $X_{-1,1}, X_{-1,2}$ and $X_{-2,2}$ act trivially on $\{P_1, P_2\}$. Therefore, the subalgebra of $w\mathfrak{sp}(2,\mathbb{R})$ generated by the elements

$$\{X_{-1,1}, X_{-1,2}, X_{-2,2}, P_1, P_2, P_5\}$$

is Abelian of dimension $6 > N + 1 = 3$. This fact is explained by means of the branching rules of $\mathfrak{sp}(4,\mathbb{R})$ representations. Take for example the regular subalgebra $\mathfrak{s}_1$ generated by $\{X_{1,1}, X_{-1,1}, X_{1,-1}\}$, which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The branching rule gives the decomposition

$$\Gamma_{\omega_1}|_{\mathfrak{s}_1} = D_{\frac{1}{2}} \oplus 2D_0\,,$$

which shows the existence of elements in the simple subalgebra $\mathfrak{s}_1$ of $\mathfrak{sp}(4,\mathbb{R})$ that commute with a maximal Abelian subalgebra of the radical $\mathfrak{h}_2$.

In general, the branching rule with respect to simple subalgebras $\mathfrak{s}_1$ of rank one gives information about the elements of $\mathfrak{s}_1$ that commute with (some) elements of the radical, and can be used to construct Abelian subalgebras (not necessarily maximal) that are not contained in the radical.

4. Number of invariants for codimension one and two subalgebras

In this section we analyze a special case of importance for applications, namely the structure of Lie subalgebras of codimension one or two, and their possible number of invariants with respect to the total number of invariants. This case is related to expansion problem of algebras, and more concretely with the existence of central extensions.

If $\mathrm{codim}_{\mathfrak{g}}\mathfrak{k} = 1$, the equality (9) gives the relation:

$$1 - \mathcal{N}(\mathfrak{g}) - \mathcal{N}(\mathfrak{k}) + 2l' \geq 0\,,$$

and therefore,

$$1 - \mathcal{N}(\mathfrak{g}) + 2l' \geq \mathcal{N}(\mathfrak{k}) \geq 0\,. \tag{23}$$

Proposition 5 *Let $\mathfrak{k}$ be a codimension one subalgebra of $\mathfrak{g}$. Then $\mathfrak{k}$ and $\mathfrak{g}$ have at least $\left[\frac{\mathcal{N}(\mathfrak{g})}{2}\right]$ invariants in common. More specifically,*

[9] The brackets of the Heisenberg radical would be given by $[P_1, P_3] = [P_2, P_4] = P_5$ over the chosen basis.

1. *If $\mathcal{N}(\mathfrak{g}) = 0$, then $\mathfrak{k}$ has exactly one invariant.*

2. *If $\mathcal{N}(\mathfrak{g}) = 1$, then either $\mathcal{N}(\mathfrak{k}) = 0$ or $\mathcal{N}(\mathfrak{k}) = 2$, and in this case the invariant of $\mathfrak{g}$ is necessarily an invariant of $\mathfrak{k}$.*

3. *If $l' = \mathcal{N}(\mathfrak{g})$, then $\mathfrak{k}$ has exactly $1 + \mathcal{N}(\mathfrak{g})$ invariants.*

Proof. We distinguish two cases, according to the parity of $\mathcal{N}(\mathfrak{g})$.

1. $\mathcal{N}(\mathfrak{g}) = 2k + 1$ with $k \geq 0$.
   In this case $\mathcal{N}(\mathfrak{g}) = 2p$ with $p \geq 0$ and equation (23) takes the form

$$2(l' - k) \geq \mathcal{N}(\mathfrak{k}) = 2p \geq 0\,. \tag{24}$$

   It follows at once that $l' \geq k$, which shows that at least $[\mathcal{N}(\mathfrak{g})/2]$ invariants are in common. If $l' = k$, then equation (24) implies

$$0 \geq \mathcal{N}(\mathfrak{k}) = 2p \geq 0\,,$$

   and therefore, $\mathcal{N}(\mathfrak{g}) = 1$, $\mathcal{N}(\mathfrak{k}) = 0$.
   If $l' = \mathcal{N}(\mathfrak{g})$, then $2p \geq 2k + 2$, and from equation (24) it follows that

$$2k + 2 \geq \mathcal{N}(\mathfrak{k}) = 2p \geq 0\,,$$

   which necessarily implies that $\mathcal{N}(\mathfrak{k}) = 1 + \mathcal{N}(\mathfrak{g})$. In particular, if $k = 0$ then $\mathfrak{k}$ has exactly two invariants.

2. $\mathcal{N}(\mathfrak{g}) = 2k$ with $k \geq 0$. By parity, $\mathcal{N}(\mathfrak{k}) = 2p + 1$ with $p \geq 0$. In this case (24) is rewritten as

$$1 + 2(l' - k) \geq \mathcal{N}(\mathfrak{k}) = 2p + 1 \geq 1\,. \tag{25}$$

   Again the condition $l' \geq k$ holds, showing that $\mathfrak{k}$ and $\mathfrak{g}$ have $[\mathcal{N}(\mathfrak{g})/2]$ invariant in common. If $l' = k$, then $\mathcal{N}(\mathfrak{k}) = 2p + 1 \leq 1$, thus $p = 0$. Two cases are possible:

   (a) $l' = 0$: $\mathcal{N}(\mathfrak{g}) = 0$ and $\mathcal{N}(\mathfrak{k}) = 1$.
   (b) $l' = 1$: $\mathcal{N}(\mathfrak{g}) = 2$ and $\mathcal{N}(\mathfrak{k}) = 1$.

   If $l' = \mathcal{N}(\mathfrak{g})$, then $2p + 1 \geq 2k + 1$ and from (25) we obtain

$$2k + 1 \geq \mathcal{N}(\mathfrak{k}) = 2p + 1 \geq 1\,, \tag{26}$$

   showing that $\mathcal{N}(\mathfrak{k}) = \mathcal{N}(\mathfrak{g}) + 1$.

■

As an application of this result we can easily deduce that a simple Lie algebra $\mathfrak{s}$ of rank ≥ 2 has no codimension one subalgebra. In fact, since $\mathfrak{s}$ is simple this implies that $l' = 0$ for its subalgebras [19], thus a codimension one $\mathfrak{k}$ would satisfy

$$(1 - \mathcal{N}(\mathfrak{s}) - \mathcal{N}(\mathfrak{k})) \geq 0\,,$$

and in particular that $1 - \mathcal{N}(\mathfrak{s}) \geq 0$. Since the number of invariants of a semisimple Lie algebra is its rank, we get that $\mathfrak{s}$ must have rank one[10].

For codimension 2 subalgebras a similar result can be obtained:

Proposition 6 *Let $\mathfrak{k}$ be a codimension 2 Lie subalgebra of $\mathfrak{g}$.*

1. *If $\mathcal{N}(\mathfrak{g})$ is odd, then $\mathfrak{g}$ has at least $[\mathcal{N}(\mathfrak{g})/2]$ invariants in common with $\mathfrak{k}$.*
2. *If $\mathcal{N}(\mathfrak{g})$ is even, then $l' \geq \frac{\mathcal{N}(\mathfrak{g})}{2} - 1$, and if the equality holds, then $\mathcal{N}(\mathfrak{g}) = 2$ and $\mathcal{N}(\mathfrak{k}) = 0$.*

In particular, if $l' = \mathcal{N}(\mathfrak{g})$, then $\mathcal{N}(\mathfrak{k}) = \mathcal{N}(\mathfrak{g})$ or $\mathcal{N}(\mathfrak{g}) + 2$.

Proof. In this case, the number of invariants of $\mathfrak{k}$ is bounded by

$$2 + 2l' - \mathcal{N}(\mathfrak{g}) \geq \mathcal{N}(\mathfrak{k})\,. \tag{27}$$

We also distinguish two cases:

1. If $\mathcal{N}(\mathfrak{g}) = 2k + 1$ with $k \geq 0$, then $\mathcal{N}(\mathfrak{g})$ has the same parity. From (27) it follows that

$$1 + 2(l' - k) \geq \mathcal{N}(\mathfrak{k}) = 2p + 1 \geq 1\,. \tag{28}$$

The number of common invariants l' is at least k, in order that the left side of the preceding equation is nonnegative. If $l' = k$, then $p = 0$ and $\mathcal{N}(\mathfrak{k}) = 1$. There are two possibilities for k:

(a) If $k = 0$ then $\mathcal{N}(\mathfrak{g}) = 1$.

(b) If $k = 1$ then $\mathcal{N}(\mathfrak{g}) = 3$.

For the maximal number of common invariants, $l' = \mathcal{N}(\mathfrak{g})$, we obtain on one side that $2p + 1 \geq 2k + 1$, and considering (27), that

$$2k + 3 \geq \mathcal{N}(\mathfrak{k}) = 2p + 1 \geq 1\,.$$

There are thus two possibilities, either $\mathcal{N}(\mathfrak{k}) = \mathcal{N}(\mathfrak{g})$ or $\mathcal{N}(\mathfrak{g}) + 2 = \mathcal{N}(\mathfrak{k})$.

[10] Although the compact algebra $\mathfrak{so}(3)$ has no subalgebra of dimension two, this fact cannot be deduced from the previous argument, based only on the rank of the algebra.

2. If $\mathcal{N}(\mathfrak{g}) = 2k$ is even, then the subalgebra also has an even number of invariants. Here we obtain

$$2 + 2(l' - k) \geq \mathcal{N}(\mathfrak{k}) = 2p \geq 0\,. \tag{29}$$

In any case we must have $l' \geq k-1$. We separate the different possibilities:

- If $l' = k-1$, then (29) implies that $\mathcal{N}(\mathfrak{k}) = 0$, thus $l' = k-1 = 0$, and therefore, $\mathcal{N}(\mathfrak{g}) = 2$.
- If $l' = k$, then $2 \geq \mathcal{N}(\mathfrak{k}) = 2p \geq 0$, thus either $p = 0, 1$. Following cases can appear:
  (a) If $p = 0$ then $\mathcal{N}(\mathfrak{k}) = 0$ and thus $\mathcal{N}(\mathfrak{g}) = 0$.
  (b) If $p = 1$ then $\mathcal{N}(\mathfrak{k}) = 2$ and therefore, $l' \leq 2$. Three cases arise:
    (i) $l' = k = 0$ implies $\mathcal{N}(\mathfrak{g}) = 0\,.$
    (ii) $l' = k = 1$ implies $\mathcal{N}(\mathfrak{g}) = 2\,.$
    (iii) $l' = k = 2$ implies $\mathcal{N}(\mathfrak{g}) = 4\,.$

3. If $l' = 2k = \mathcal{N}(\mathfrak{g})$ then $2p \geq 2k$ and from (29) we get $2+2k \geq \mathcal{N}(\mathfrak{k}) = 2p \geq 0$. Thus either $\mathcal{N}(\mathfrak{g}) = \mathcal{N}(\mathfrak{k})$ or $\mathcal{N}(\mathfrak{g}) + 2 = \mathcal{N}(\mathfrak{k})$ holds. In particular, if $k = 0$ then $\mathcal{N}(\mathfrak{k})$ equals 0 or 2.

■

Corollary 4 *Let $\mathfrak{k}$ be a codimension 2 Lie subalgebra of $\mathfrak{g}$ such that $\mathfrak{k}$ and $\mathfrak{g}$ have no common invariants. Then $\mathcal{N}(\mathfrak{g}) \leq 2$, and one of the following cases holds:*

1. *If $\mathcal{N}(\mathfrak{g}) = 0$, then $\mathfrak{k}$ has at most 2 invariants.*
2. *If $\mathcal{N}(\mathfrak{g}) = 1$, then $\mathcal{N}(\mathfrak{k}) = 1$.*
3. *If $\mathcal{N}(\mathfrak{g}) = 2$, then $\mathfrak{k}$ has no invariants.*

The proof is immediate. We remark that a slight generalization of these results allows us, for the case of simple Lie algebra, to obtain a lower bound for the codimension of a subalgebra. As commented, we have $l' = 0$, since the Casimir operators of a simple algebra depends on all its generators. Given a subalgebra $\mathfrak{k}$ of codimension α, by formulae (9) and (8) we have

$$\alpha - \mathcal{N}(\mathfrak{g}) - \mathcal{N}(\mathfrak{k}) \geq 0\,,$$

thus $\alpha - \mathcal{N}(\mathfrak{g}) \geq 0$. As a consequence, the codimension of $\mathfrak{k}$ must be at least the rank of $\mathfrak{s}$[11].

[11] For semisimple Lie algebras this must no longer be true, as shows the complexification of the Lorentz algebra. The reason is that for this particular case we can have $l' > 0$ for some subalgebras.

5. Conclusions

We have shown that the numerical formula providing the number of missing operators for a reduction chain can be applied to the study of the maximal dimension of subalgebras, depending on their structure, as well as to obtain upper bounds for the number of invariants of subalgebras in dependence of their codimension. These bounds are useful for the classification problem of subalgebras, since they allow to decide which inclusions are forbidden, and also find applications in the analysis of the minimal dimension of faithful representations of Lie algebras, as shown for algebras admitting a MASA of half its dimension. For the semidirect products of simple and Heisenberg algebras, we have seen that the existence of Abelian subalgebras that contain the maximal Abelian ideal of the radical is related to the branching rule problem of the representation that describes the semidirect sum with respect to simple subalgebras of rank one. Finally, the analysis of codimension one and two subalgebras is useful for the classification of Lie algebras in low dimension, such as the indecomposable Lie algebras with nontrivial Levi decomposition in dimension ten, which are classified only partially [13]. Other potential Lie algebras to which the results obtained can be applied are the inhomogeneous Lie algebras and the symmetry algebras of differential equations, as well as isolated algebras appearing in specific physical problems [20–22].

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