EVALUATING RESIDUES AND INTEGRALS THROUGH NEGATIVE DIMENSIONAL INTEGRATION METHOD (NDIM)

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The standard way of evaluating residues and some real integrals through the residue theorem (Cauchy's theorem) is well-known and widely applied in many branches of Physics. Herein we present an alternative technique based on the negative dimensional integration method (NDIM) originally developed to handle Feynman integrals. The advantage of this new technique is that we need only to apply Gaussian integration and solve systems of linear algebraic equations, with no need to determine the poles themselves or their residues, as well as obtaining a whole class of results for differing orders of poles simultaneously.

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1. Introduction

In a textbook on complex variables we may find real definite integrals of the type

$$I_1 = \int_0^\infty \frac{dx}{x^2+1} \qquad \text{or} \qquad I_2 = \int_0^\infty \frac{dx\, x^2}{(x^2+1)(x^2+4)}\,, \tag{1}$$

to be evaluated using the contour integration in the complex plane, making use of the Cauchy's residue theorem. This is, of course, a simple exercise in complex analysis, and residue theorem is a powerful tool to handle such integrals. However, if we were actually to evaluate them we would do it separately, each integral in its turn, with its own residues of poles summed over to get the final answer. Not so if we use the NDIM technique [1, 2], as we shall shortly see. In NDIM we can integrate both integrals at the same time, and more, without having to do it one by one separately; we simply need to evaluate the general integral

$$I_g^{(m,n,p)} = \int_0^\infty \frac{dx\ (x^2)^m}{(x^2+1)^n(x^2+4)^p} \tag{2}$$

and then work out the result for a particular set of numbers $m, n, p = 0, 1, 2, 3, \ldots$, of which I_1 and I_2 are specific examples.

The concept of negative dimensional integration can be best seen in the following D-dimensional Gaussian integration:

$$G = \int d^D x\, \mathrm{e}^{-\alpha x^2} = \left(\sqrt{\frac{\pi}{\alpha}}\right)^D . \tag{3}$$

On the other hand, expanding in power series the exponential function in the integrand of (3) above, we have

$$\begin{aligned} G &= \int d^D x\, \mathrm{e}^{-\alpha x^2} \\ &= \int d^D x \sum_{n=0}^{\infty} (-)^n \frac{\alpha^n}{n!} (x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^n}{n!} \int d^D x (x^2)^n . \end{aligned} \tag{4}$$

Now, comparing (3) and (4), we conclude that for the equality to hold we must have

$$\mathcal{I} \equiv \int d^D x (x^2)^n = (-1)^{-n} n!\, (\sqrt{\pi})^D\ \delta_{n+\frac{D}{2},0} \tag{5}$$

and since, by construction $n \geq 0$, one is led to assume that D is a negative valued dimension.

This non-trivial result for a D-dimensional integral with pure quadratic integrand elevated to a given power was first pointed out by Ricotta and Halliday in [3], which differed somewhat from all the previous considerations, starting from dimensional regularization scheme developed by 't Hooft and Veltman [4], where all such integrals were set to zero straightforwardly. Since Feynman loop diagrams lead to Feynman integrals in D-dimensions in the dimensional regularization scheme, integrals evaluated with the help of (5) in negative dimensions, must be brought back by analytic continuation to the realm of positive dimensions D. Since the concept of negative dimensional integral arises as a consequence of the positiveness of the polynomial powers in the integrands, the mentioned analytic continuation is achieved by employing a property of the Pocchammer's symbols bearing these parameters labelling powers of the polynomial expressions in the integrands, namely [5]

$$(a)_{-k} = \frac{(-)^k}{(1-a)_k}\,, \tag{6}$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}\,.$$

Of course, here we strict ourselves to one-dimensional integrals such as (2), so that (5) becomes

$$\mathcal{I}_{D=1} \equiv \int dx (x^2)^n = (-1)^{-n} n!\,\sqrt{\pi}\;\delta_{n+\frac{1}{2},0} \tag{7}$$

and all the NDIM formulation developed for D-dimensional Feynman integrals can be applied to evaluate one-dimensional integrals of the type I_1 and I_2.

2. Review of Cauchy's residue theorem application

Consider, for practical example, the integral

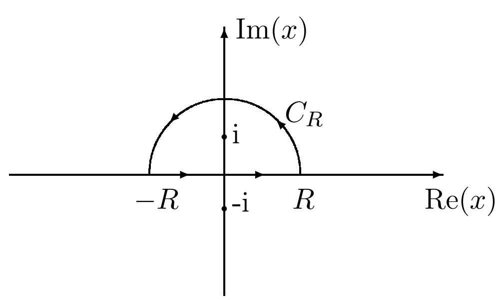
$$I_1 = \int\limits_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}\int\limits_{-\infty}^{+\infty} \frac{dx}{x^2+1}\,. \tag{8}$$

The second integral on the r.h.s. of the above represents an integration along the real axis of the function

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}\,, \tag{9}$$

which is a function of complex variables with simple poles at $z = \pm i$.

Let C_R be the semi-circle with radius $|z| = R$ with $R > 1$ as shown in figure below



By Cauchy's theorem, we have

$$\int_{-R}^{R} dx\, f(x) + \int_{C_R} dz\, f(z) = 2\pi i\, \kappa_1\,, \tag{10}$$

where κ_1 is the residue of $f(z)$ at the (simple) pole $z = i$, which, of course, can be calculated easily by

$$\kappa_1 = (z-i)f(z)|_{z=i} = \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i}\,. \tag{11}$$

Therefore, for $R > 1$, we have from (10)

$$\int_{-R}^{R} \frac{dx}{x^2+1} = \pi - \int_{C_R} \frac{dz}{z^2+1}\,. \tag{12}$$

Now, $|z| = R$ when z is on the semi-circle C_R, so that

$$|z^2+1| \geq |z|^2 - 1 = R^2 - 1\,, \tag{13}$$

and therefore

$$\int_{C_R} \frac{dz}{z^2+1} \leq \int_{C_R} \frac{|dz|}{R^2-1} = \frac{\pi R}{R^2-1} \to 0 \quad \text{for} \quad R \to \infty\,. \tag{14}$$

Then, taking the limit

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \pi\,, \tag{15}$$

or

$$I_1 = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}\,. \tag{16}$$

3. Negative dimension integration method

On the other hand, let us take the Gaussian generating funtional of the negative dimensional integration, namely,

$$\mathcal{I} = \int_{-\infty}^{+\infty} dx\, \mathrm{e}^{-\alpha(x^2+1)}\,, \tag{17}$$

where we note that the argument in the exponential is the denominator of the integrand in (8) multiplied by a real, positive parameter α which is chosen as a converging factor for the integral. Of course, this is the standard Gaussian integral, whose result is

$$\mathcal{I} = \mathrm{e}^{-\alpha}\pi^{\frac{1}{2}}\alpha^{-\frac{1}{2}} = \pi^{\frac{1}{2}}\sum_{j=0}^{\infty}(-)^j\,\frac{(\alpha)^{j-\frac{1}{2}}}{j!}\,. \tag{18}$$

If we project out in power series the integrand of (17) before the integration is done, we have

$$\mathcal{I} = \sum_{n=0}^{\infty}(-)^n\frac{\alpha^n}{n!}\,I_{\mathrm{NDIM}}(n)\,, \tag{19}$$

where the negative dimensional integral $I_{\mathrm{NDIM}}(n)$ is given by

$$I_{\mathrm{NDIM}}(n) \equiv \int_{-\infty}^{+\infty} dx\,(x^2+1)^n\,. \tag{20}$$

Note that for $n=-1$ this is exactly the integral (8) we want to evaluate. Observe, however, that the negative dimensional integral is more general than the one we have in (8) in the sense that the exponent n is not fixed. Note however that (20) is only defined for positive n, so that in order to get the result for (8) we need to make an analytic continuation to negative values of n, a process whereby the integral gets defined into positive dimensionality in (20).

Comparing the two series expansion (18) with (19), we observe that in order to both series be equivalent, we must have $n = j - \frac{1}{2}$, so that

$$\begin{aligned} I_{\mathrm{NDIM}}(n) \;&\equiv\; \int_{-\infty}^{+\infty} dx\,(x^2+1)^n = (-)^{-n}\,n!\,\pi^{\frac{1}{2}}\frac{(-)^{n+\frac{1}{2}}}{(n+\frac{1}{2})!} \\ &=\; (-\pi)^{\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2}+1)} \\ &=\; \frac{(-\pi)^{\frac{1}{2}}}{(n+1)_{\frac{1}{2}}}\,. \end{aligned} \tag{21}$$

Now, analytic continuing (AC) the exponent n into negative values using the property (6), we have

$$I^{(\mathrm{AC})}_{\mathrm{NDIM}}(n) = \int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^{-n}} = \pi^{\frac{1}{2}}\,(-n)_{-\frac{1}{2}} = \pi^{\frac{1}{2}}\frac{\Gamma(-n-\frac{1}{2})}{\Gamma(-n)}\,. \tag{22}$$

Now, substituting $n = -1, -2, -3, \ldots,$ we get

$$
\begin{aligned}
I^{(\mathrm{AC})}_{\mathrm{NDIM}}(-1) &= \int_{-\infty}^{+\infty} \frac{dx}{x^2+1} = \pi\,, && (23)\\
I^{(\mathrm{AC})}_{\mathrm{NDIM}}(-2) &= \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{2}\,, && (24)\\
I^{(\mathrm{AC})}_{\mathrm{NDIM}}(-3) &= \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}\,, && (25)\\
\vdots \quad &= \quad \vdots \quad = \vdots \quad ,
\end{aligned}
$$

and so on and so forth. Of course, for the original integrals, which range from $[0, \infty)$, the corresponding values are half of the values quoted above.

A more interesting case is the evaluation of the other integral, namely,

$$
\begin{aligned}
I_2 &= \int_0^{\infty} \frac{dx\; x^2}{(x^2+1)(x^2+4)}\\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx\; x^2}{(x^2+1)(x^2+4)}\,. && (26)
\end{aligned}
$$

Evaluation of this integral by the residue technique is similar to the previous one where now we have two poles in the upper hemisphere, $z = i$ and $z = 2i$, so that the contour C_R is such that its radius must be $|z| = R$, $R > 2$, before taking the limit $R \to \infty$. The result, after summing the residues of both poles is

$$
I_2 = \frac{1}{2}\, 2\pi i \left\{ -\frac{1}{6i} + \frac{1}{3i} \right\} = \frac{1}{2} \left\{ -\frac{\pi}{3} + \frac{2\pi}{3} \right\} = \frac{\pi}{6}\,. \tag{27}
$$

We have written down the explicit contributions of each residue, the first one corresponding to the residue at the simple pole $z = i$ and the second one to the residue at $z = 2i$, to show that these residues correlate with each of the *"basis"* (linearly independent) solutions (with a word borrowed from the language of basis vectors in a vector space) in NDIM.

The Gaussian generating functional of the negative dimensional integral to this case is

$$
\begin{aligned}
\mathcal{J} &\equiv \int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2-\beta(x^2+1)-\gamma(x^2+4)} = \mathrm{e}^{-\beta-4\gamma}\int_{-\infty}^{+\infty} dx\, e^{-(\alpha+\beta+\gamma)x^2} \\
&= e^{-\beta-4\gamma}\frac{\pi^{\frac{1}{2}}}{(\alpha+\beta+\gamma)^{\frac{1}{2}}} \\
&= \frac{\pi^{\frac{1}{2}}}{(\alpha+\beta+\gamma)^{\frac{1}{2}}}\sum_{r,s=0}^{\infty}(-)^{r+s}\frac{\beta^r}{r!}4^s\frac{\gamma^s}{s!} \\
&= \pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\sum_{r,s,a,b,c}^{a+b+c=-\frac{1}{2}}(-)^{r+s}4^s\frac{\alpha^a\beta^{b+r}\gamma^{c+s}}{a!\,b!\,c!\,r!\,s!}\,,
\end{aligned}
\tag{28}
$$

where in the last line of the above we have employed the standard multinomial expansion for $(\alpha+\beta+\gamma)^{-\frac{1}{2}}$.

On the other hand, direct expansion in power series of the integrand yields

$$
\begin{aligned}
\mathcal{J} &= \sum_{j,l,m=0}^{\infty}(-)^{j+l+m}\frac{\alpha^j\beta^l\gamma^m}{j!\,l!\,m!}\int_{-\infty}^{+\infty}dx\,(x^2)^j(x^2+1)^l(x^2+4)^m \\
&= \sum_{j,l,m=0}^{\infty}(-)^{j+l+m}\frac{\alpha^j\beta^l\gamma^m}{j!\,l!\,m!}\,I_{\mathrm{NDIM}}(j,l,m)\,.
\end{aligned}
\tag{29}
$$

Comparing (28) with (29) we see that we must have

$$
\begin{aligned}
j &= a\,, \\
l &= b+r\,, \\
m &= c+s\,, \\
a+b+c &= -\frac{1}{2}\,.
\end{aligned}
\tag{30}
$$

Also, when we compare the two series, the one in (28) has five summation indices with one constraint, whereas the other one (29) has three summation indices. Then our result for $I_{\mathrm{NDIM}}(,j,l,m)$ will be given in terms of $5-3-1=1$, that is, a single summation index. However, since we have five indices in one and three in the other, with one constraint equation among them, one can in principle have the following combinatorial possibilities:

$$
{}_5C_4 = \frac{5!}{4!\,1!} = 5 \tag{31}
$$

for the remaining series index.

Letting r be the summation index in the result, we have then the following conditions:

$$
\begin{aligned}
a &= j\,, \\
b &= l - r\,, \\
c &= -j - l - \frac{1}{2} + r\,, \\
s &= j + l + m + \frac{1}{2} - r\,,
\end{aligned} \tag{32}
$$

so that the negative dimensional integral will be given by

$$
\begin{aligned}
I^{(r)}_{\rm NDIM}(j,l,m) &= (-)^{-j-l-m} j!\,l!\,m!\,\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}) \\
&\quad \times \sum_{r=0}^{\infty} \frac{(-)^{j+l+m+\frac{1}{2}}\,4^{\,j+l+m+\frac{1}{2}-r}}{j!\,(l-r)!\,(-j-l-\frac{1}{2}+r)!\,r!\,(j+l+m+\frac{1}{2}-r)!} \\
&= (-\pi)^{\frac{1}{2}} 4^{\,j+l+m+\frac{1}{2}} \frac{\Gamma(\frac{1}{2})\Gamma(1+m)}{\Gamma(1-j-l-\frac{1}{2})\Gamma(1+j+l+m+\frac{1}{2})} \\
&\quad \times \sum_{r=0}^{\infty} \frac{1}{(1+l)_{-r}(1-j-l-\frac{1}{2})_r(1+j+l+m+\frac{1}{2})_{-r}4^r r!} \\
&= \frac{(-\pi)^{\frac{1}{2}}\,4^{j+l+m+\frac{1}{2}}}{(\frac{1}{2})_{-j-l}\,(1+m)_{j+l+\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{(-l)_r\,(-j-l-m-\frac{1}{2})_r}{(1-j-l-\frac{1}{2})_r} \frac{(\frac{1}{4})^r}{r!} \\
&= \frac{(-\pi)^{\frac{1}{2}}\,4^{j+l+m+\frac{1}{2}}}{(\frac{1}{2})_{-j-l}\,(1+m)_{j+l+\frac{1}{2}}}\ {}_2 \\
&\quad \times F_1\left(\left. \begin{array}{c} -l\,,\ -j-l-m-\frac{1}{2} \\ 1-j-l-\frac{1}{2} \end{array} \right| \frac{1}{4} \right).
\end{aligned} \tag{33}
$$

Analogous calculations can be performed for s, b, c indices, whereas for the index a there is no solution except the trivial one (this is the case when the determinant of the system of linear equations vanishes). For the remaining non-vanishing determinants, we have then the solutions: For s remaining summation index, we have the conditions

$$
\begin{aligned}
a &= j\,,\\
b &= -j-m-\frac{1}{2}+s\,,\\
c &= -m-s\,,\\
r &= j+l+m+\frac{1}{2}-s\,,
\end{aligned}
\tag{34}
$$

and the solution yields:

$$
\begin{aligned}
I^{(s)}_{\mathrm{NDIM}}(j,l,m) &= \frac{(-\pi)^{\frac{1}{2}}}{(\frac{1}{2})_{-j-m}(1+l)_{j+m+\frac{1}{2}}}\sum_{s=0}^{\infty}\frac{(-m)_s(-j-l-m-\frac{1}{2})_s}{(1-j-m-\frac{1}{2})_r}\frac{4^s}{s!}\\
&= \frac{(-\pi)^{\frac{1}{2}}}{(\frac{1}{2})_{-j-m}\,(1+l)_{j+m+\frac{1}{2}}}\,{}_2F_1\left(\begin{array}{c} -m\,,-j-l-m-\frac{1}{2}\\ 1-j-m-\frac{1}{2}\end{array}\middle|\,4\right).
\end{aligned}
\tag{35}
$$

For b summation, we have

$$
\begin{aligned}
a &= j\,,\\
r &= l-b\,,\\
s &= j+m+\frac{1}{2}+b\,,\\
c &= -j-\frac{1}{2}-b\,,
\end{aligned}
\tag{36}
$$

and the solution yields:

$$
I^{(b)}_{\mathrm{NDIM}}(j,l,m) = \frac{(-\pi)^{\frac{1}{2}}\,4^{j+m+\frac{1}{2}}}{(\frac{1}{2})_{-j}\,(1+m)_{j+\frac{1}{2}}}\,{}_2F_1\left(\begin{array}{c} j+\frac{1}{2}\,,\,-l\\ 1+j+m+\frac{1}{2}\end{array}\middle|\,4\right). \tag{37}
$$

Finally, for c summation we have

$$
\begin{aligned}
a &= j\,,\\
r &= j+l+\frac{1}{2}+c\,,\\
s &= m-c\,,\\
b &= -j-\frac{1}{2}-c\,,
\end{aligned}
\tag{38}
$$

and the solution yields:

$$
I^{(c)}_{\mathrm{NDIM}}(j,l,m) = \frac{(-\pi)^{\frac{1}{2}}\,4^{m}}{(\frac{1}{2})_{-j}\,(1+l)_{j+\frac{1}{2}}}\,{}_2F_1\left(\begin{array}{c} j+\frac{1}{2}\,,\,-m\\ 1+j+l+\frac{1}{2}\end{array}\middle|\,\frac{1}{4}\right). \tag{39}
$$

Now, observe that we have four non-vanishing solutions arising from the solving of systems of linear equations, which here clearly come in pairs when we look at the argument of the hypergeometric functions in the results. The solution for the integral is then the sum of the pairs with the same argument (linear combination), namely,

$$\begin{aligned} I_{\rm NDIM}(j,l,m) &= A\,{}_2F_1(a,b;c|z^{-1}) + B\,{}_2F_1(d,e;f|z^{-1}) \\ &= A'\,{}_2F_1(a',b';c'|z) + B'\,{}_2F_1(d',e';f'|z)\,, \end{aligned} \tag{40}$$

where the coefficients A, B, A' and B' are given by

$$\begin{aligned} A &= \frac{(-\pi)^{\frac{1}{2}}\,4^{j+l+m+\frac{1}{2}}}{(\frac{1}{2})_{-j-l}\,(1+m)_{j+l+\frac{1}{2}}}\,, \\ B &= \frac{(-\pi)^{\frac{1}{2}}\;4^{m}}{(\frac{1}{2})_{-j}\;(1+l)_{j+\frac{1}{2}}}\,, \\ A' &= \frac{(-\pi)^{\frac{1}{2}}}{(\frac{1}{2})_{-j-m}\,(1+l)_{j+m+\frac{1}{2}}}\,, \\ B' &= \frac{(-\pi)^{\frac{1}{2}}\;4^{j+m+\frac{1}{2}}}{(\frac{1}{2})_{-j}\;(1+m)_{j+\frac{1}{2}}}\,, \end{aligned} \tag{41}$$

and the hypergeometric function parameters and variables are:

a	b	c	d	e	f	z^{-1}
$-l$	$-j-l-m-\frac{1}{2}$	$1-j-l-\frac{1}{2}$	$-m$	$j+\frac{1}{2}$	$1+j+l+\frac{1}{2}$	$\frac{1}{4}$
a'	b'	c'	d'	e'	f'	z
$-m$	$-j-l-m-\frac{1}{2}$	$1-j-m-\frac{1}{2}$	$-l$	$j+\frac{1}{2}$	$1+j+m+\frac{1}{2}$	4

Before proceeding, let us demonstrate that the two sets of solutions, namely the primed and unprimed ones are totally equivalent. To do this, we employ the following analytic continuation property of hypergeometric functions [7]

$$\begin{aligned} {}_2F_1(a,b;c|z) &= \frac{z^{-a}}{(1-b)_a\;(c)_{-a}}\,{}_2F_1(a,1+a-c;1+a-b|z^{-1}) \\ &\quad+\frac{z^{-b}}{(1-a)_b\;(c)_{-b}}\,{}_2F_1(b,1+b-c;1+b-a|z^{-1})\,, \\ &\quad\text{where}\quad |\arg(-z)|<\pi\,, \end{aligned} \tag{42}$$

to one of the basis solutions, say of the unprimed set

$$ {}_2F_1\left(-l,\, -j-l-m-\tfrac{1}{2};\, 1-j-l-\tfrac{1}{2}\,|\,\tfrac{1}{4}\right) $$

to get the primed result. Of course, we could have used the transformation property above to the other basis solution ${}_2F_1(-m,\, j+\frac{1}{2};\, 1+j+l+\frac{1}{2}\,|\frac{1}{4})$, and we would get the same primed result. We need to apply only to one of the basis solutions, since the transformation property above referred to cannot produce neither new nor any more than two linearly independent hypergeometric functions.

Now, we need to analytic continue the results to negative values of exponents and positive dimension. Observe that the result for $I_{\rm NDIM}$ contains two factors: One is the coefficients, given by ratios of gamma functions and the other is the functional part, given by the hypergeometric functions. For the coefficients, which contain ratios of gamma functions given in terms of Pocchhammers symbols, we employ (6), and for the functional part, just let the exponents go to negative valued parameters. Note that we need to be aware of which exponent should be continued to negative values [6]. Then, our final result for the integral reads:

$$ \begin{aligned} I^{\rm AC}_{\rm NDIM}(j,l,m) &= A^{\rm AC}\ {}_2F_1(a,b;c|z^{-1}) + B^{\rm AC}\ {}_2F_1(d,e;f|z^{-1}) \\ &= A'^{\rm AC}\ {}_2F_1(a',b';c'|z) + B'^{\rm AC}\ {}_2F_1(d',e';f'|z)\,, \end{aligned} \tag{43} $$

where

$$ \begin{aligned} A^{\rm AC} &= \pi^{\frac{1}{2}}\, 4^{j+l+m+\frac{1}{2}} \left(\frac{1}{2}\right)_{j+l} (-m)_{-j-l-\frac{1}{2}}\,, \\ B^{\rm AC} &= \pi^{\frac{1}{2}}\, 4^{m} \left(\frac{1}{2}\right)_{j} (-l)_{-j-\frac{1}{2}}\,, \\ A'^{\rm AC} &= \pi^{\frac{1}{2}} \left(\frac{1}{2}\right)_{j+m} (-l)_{-j-m-\frac{1}{2}}\,, \\ B'^{\rm AC} &= \pi^{\frac{1}{2}}\, 4^{j+m+\frac{1}{2}} \left(\frac{1}{2}\right)_{j} (-m)_{-j-\frac{1}{2}}\,. \end{aligned} \tag{44} $$

One interesting thing about the NDIM technology is that it allows us to write the correct answer in as many equivalent ways as it is possible to do. For the case in question, we have two equivalent answers, namely, the unprimed and primed answers.

So, for particular values of the exponents, say, $j=1$ and $l=m=-1$, which is the case for the I_2 integral mentioned in the introduction, we have:

$$
\begin{aligned}
& I_{\rm NDIM}^{\rm AC}(1,-1,-1) \\
& = \pi^{\frac{1}{2}}\left\{4^{-\frac{1}{2}}\,(1)_{-\frac{1}{2}}\,{}_2F_1\left(1,\,\tfrac{1}{2};\,\tfrac{1}{2}|\tfrac{1}{4}\right)+4^{-1}\,(\tfrac{1}{2})_1(1)_{-\frac{3}{2}}\,{}_2F_1\left(1,\,\tfrac{3}{2};\,\tfrac{3}{2}|\tfrac{1}{4}\right)\right\} \quad (45) \\
& = \pi^{\frac{1}{2}}\left\{(1)_{-\frac{1}{2}}\,{}_2F_1\left(1,\,\tfrac{1}{2};\,\tfrac{1}{2}|4\right)+4^{\frac{1}{2}}\,(\tfrac{1}{2})_1\,(1)_{-\frac{3}{2}}\,{}_2F_1\left(1,\,\tfrac{3}{2};\,\tfrac{3}{2}|4\right)\right\}. \quad (46)
\end{aligned}
$$

Finally, using the fact that [7]

$$
{}_2F_1(a,b;b|z) = (1-z)^{-a} \tag{47}
$$

the two results (45) and (46) coalesce into one:

$$
I_{\rm NDIM}^{\rm AC}(1,-1,-1) = -\frac{\pi}{3} + \frac{2\,\pi}{3} = \frac{\pi}{3}\,, \tag{48}
$$

so that each of the basis solution corresponds exactly to the residue of the poles at $z = i$ and $z = 2i$. Finally,

$$
I_2 = \frac{1}{2} I_{\rm NDIM}^{\rm AC}(1,-1,-1) = \frac{\pi}{6}\,. \tag{49}
$$

Other particular cases, such as $j = m = 0$, $l = -1$, $j = 0$, $l = m = -1$ and $j = 0$, $l = -1$, $m = -2$ can be calculated from the general solution, yielding

$$
\begin{aligned}
I_1 &= \tfrac{1}{2}\, I_{\rm NDIM}^{\rm AC}(0,-1,0) = \tfrac{\pi}{2}\,, \\
I_2^{(0,1,1)} &= \frac{1}{2}\, I_{\rm NDIM}^{\rm AC}(0,-1,-1) = \frac{\pi}{12}\,, \\
I_2^{(0,1,2)} &= \frac{1}{2}\, I_{\rm NDIM}^{\rm AC}(0,-1,-2) = \frac{5\,\pi}{288}\,.
\end{aligned}
$$

Note that for all the cases where either $l = 0$ or $m = 0$ the general solution is such that one of the terms in the overall result vanishes because we have a term proportional to $1/\Gamma(0) = 0$, and the answer is then given by a single hypergeometric function.

4. Conclusions

Using the NDIM technique, we evaluated some sample real definite integrals which may be calculated by the Cauchy residue theorem in the complex plane. The alternative methodology here presented gives us the bonus in that all the generic exponents of integrands can be calculated at once, from where particular solutions can be drawn. There is no difficulty in the performing of the integration since the integration involved is of the Gaussian type and the technique requires only series comparison term by term and the solving of systems of algebraic linear equations resulting from such a comparison. We showed that the results for different variables are obtained simultaneously and they are equivalent to each other. Moreover, for each set of basis solutions correspond the residue of a given pole. The strength of this new technique also can be envisaged in that simple, double or higher order poles can be evaluated all at once.

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