ENERGY AND MOMENTUM OF RIGIDLY ROTATING WORMHOLE SPACE-TIME

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This study is purposed to elaborate the problem of energy and momentum distribution of the Rigidly Rotating Wormhole space-time in general theory of relativity. In this connection, we use the energy-momentum definitions of Einstein, Bergmann–Thomson and Tolman. We obtained that the energy and momentum distributions of Einstein, Bergmann–Thomson and Tolman definitions give the same results in Rigidly Rotating Wormhole space-time.

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1. Introduction

It is well known that one of the most interesting and challenging problems of general relativity is the energy and momentum localization. Energy-momentum is an important conserved quantity in any physical theory whose definition has been under investigation for a long time from the General Relativity (GR) viewpoint. The problem is to find an expression which is physically meaningful. The point is that the gravitational field can be made locally vanish and so one is always able to find the frame in which the energy-momentum of gravitational field is zero while in the other frames, it is not true. Unfortunately, there is still no generally accepted definition of energy-momentum for gravitational field. The problem arises with the expression defining the gravitational field energy part.

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In the theory of GR, the energy-momentum conservation laws are given by

$$T^b_{a;b} = 0\,, \quad (a, b = 0, 1, 2, 3)\,, \tag{1}$$

where T^b_a denotes the energy-momentum tensor. In order to change the covariant divergence into an ordinary divergence so that global energy-momentum conservation, including the contribution from gravity, can be expressed in the usual manner as in electromagnetism, Einstein formulated [1] the conservation law in the following form

$$\frac{\partial}{\partial x^b}\left(\sqrt{-g}\left(T^b_a + t^b_a\right)\right) = 0\,. \tag{2}$$

Here t^b_a is not a tensor quantity and is called the gravitational field pseudo-tensor. Schrodinger showed that the pseudo-tensor can be made vanish outside the Schwarzschild radius using a suitable choice of coordinates. There have been many attempts to find a more suitable quantity for describing the distribution of energy and momentum due to matter, non-gravitational and gravitational fields. The proposed quantities which actually fulfill the conservation law of matter plus gravitational parts are called gravitational field pseudo-tensors. The choice of the gravitational field pseudo-tensor is not unique. Because of this, quite a few definitions of these pseudo-tensors have been proposed. The notion of energy-momentum prescriptions was severely criticized for a number reasons. Firstly, the nature of symmetric and locally conserved object is non-tensorial one; thus its physical interpretation appeared obscure [2]. Secondly, different energy-momentum complexes could yield different energy-momentum distributions for the same gravitational background [3]. Finally, energy-momentum complexes were local objects while it was generally believed that the suitable energy-momentum of the gravitational field was only total, *i.e.* it cannot be localized [4]. There have been several attempts to calculate energy-momentum prescriptions associated with different space-times [5, 6].

In order to obtain a meaningful expression for energy, momentum and angular momentum for a general relativistic system, Einstein himself proposed an expression. After Einstein's energy-momentum complex [7], many complexes have been found, for instance, Landau–Lifshitz [8], Tolman [9], Papapetrou [10], Möller [11, 12], Weinberg [13] and Bergmann [14]. Some of these definitions are coordinate dependent while others are not. Also, most of these expression cannot be used to define angular momentum.

The lack of a generally accepted definition of energy-momentum in a curved space-time has led to doubts regarding the idea of energy localization. According to Misner *et al.* [15], energy is localizable only for spherical systems. Cooperstock and Sarracino [16] came up with the view that if energy is localizable for spherical system, then it can be localized for any system. Bondi [17] argued that a non-localizable form of energy is not allowed in GR. After this, an alternative concept of energy, called quasi-local energy, was developed. The use of quasi-local masses to obtain energy-momentum in a curved space-time do not restrict one to use particular coordinate system. A large number of definitions of quasi-local masses have been proposed, those by Penrose and many others [18–20].

In this paper we calculate the energy and momentum distribution of the rigidly rotating wormhole space-time [24] in the Bergmann–Thomson, Einstein and Tolman prescriptions in general relativity. We will proceed according to the following scheme. In Section 2, we give the Rigidly Rotating Wormhole (RRW) space-time and transformation for given RRW metric. Sections 3, 4, 5 give us the energy-momentum definitions of Bergmann–Thomson, Einstein and Tolman in general relativity, respectively, and we calculate the energy-momentum densities for the Rigidly Rotating Wormhole space-time. Finally, we summarize and discuss our results. Throughout this paper, we use units where $G = c = 1$.

2. Rigidly rotating wormhole space-time

The wormhole solutions of the Einstein equations started with Einstein himself, since he was interested in giving a field representation of particles [21]. The idea was further developed by Ellis [22] and others, where instead of particles, they try to model them as "bridges" between two regions of the space-time. The idea of considering such solutions a actual connections between two separated regions of the Universe has attracted a lot of attention since the seminal work of Morris and Thorne [23]. For the Lorentzian wormhole to be traversable, it requires exotic matter which violates the known energy conditions. To find the reasonable models, the generalized models of the wormhole with other matters and/or in various geometries have been studied. Among the models, the matter or wave in the wormhole geometry as the primary ad auxiliary effects [24]. Recently, the solution for the electrically charged case was also found [25].

Among the models, the rotating wormhole is very interesting to us, since Kerr black hole is the final stationary state of most black holes. The Kerr metric has many insights in black hole physics as the general black hole solution with angular momentum. Likewise, the rotating wormhole is stationary and axially symmetric generalization of the Morris–Thorne wormhole. The reason is that it may be the most genera extension of Morris–Thorne wormhole. Teo [26] derived the rotating wormhole model from the generally axially symmetric space-time and have shown an example with ergoregion and geodesics able to traverse wormhole without encountering any exotic matter.

The Rigidly Rotating Wormhole metric [24] is given by

$$ds^2 = -dt^2 + \frac{1}{1-\frac{b(r)}{r}}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta [d\phi - \omega dt]^2 . \tag{3}$$

where ω is the constant angular velocity and $b(r)$ is the shape function of the wormhole [23,27]. Here, the throat is $r = b_0$ and the shape of the throat is the sphere.

It is well known that the energy-momentum complexes give meaningful result if calculations are performed in quasi-Cartesian coordinates. The line element Eq. (3) may be transformed to quasi-Cartesian coordinates:

$$\begin{aligned} ds^2 \;=\; & \left[\omega^2\left(x^2+y^2\right)-1\right]dt^2 + \frac{Ax^2+y^2+z^2}{r^2}dx^2 \\ & + \frac{Ay^2+x^2+z^2}{r^2}dy^2 + \frac{Az^2+y^2+x^2}{r^2}dz^2 \\ & + \frac{2(A-1)}{r^2}[xydxdy + xzdxdz + yzdydz] \\ & - 2\omega(xdydt - ydxdt) , \end{aligned} \tag{4}$$

where $A = [1 - \frac{b(r)}{r}]^{-1}$ and the coordinates r, θ, ϕ in Eq. (4) and x, y, z are related through

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} , \\ \theta &= \cos^{-1}\left(\frac{z}{r}\right) , \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) . \end{aligned} \tag{5}$$

3. Energy-momentum complexes of RRW model in the Bergmann–Thomson prescription

The energy-momentum prescription of Bergmann–Thomson [14] is given by

$$\Xi^{\mu\nu} = \frac{1}{16\pi}\Pi^{\mu\nu\beta}_{,\beta} , \tag{6}$$

where

$$\Pi^{\mu\nu\beta} = g^{\mu\alpha}V^{\nu\beta}_{\alpha} \tag{7}$$

with

$$V^{\nu\alpha}_{\beta} = -V^{\alpha\nu}_{\beta} = \frac{g_{\beta\xi}}{\sqrt{-g}}\left[-g\left(g^{\nu\xi}g^{\alpha\rho} - g^{\alpha\xi}g^{\nu\rho}\right)\right]_{,\rho} . \tag{8}$$

Ξ_0^0 is the energy density, Ξ_μ^0 are the momentum density components, and Ξ_0^μ are the components of the energy current density. The Bergmann–Thomson energy-momentum definition satisfies the following local conservation laws

$$\frac{\partial \Xi^{\mu\nu}}{\partial x^\nu} = 0 \tag{9}$$

in any coordinate system. The energy and momentum components are given by

$$P^\mu = \int\int\int \Xi^{\mu 0} dxdydz\,. \tag{10}$$

Further Gauss's theorem furnishes

$$P^\mu = \frac{1}{16\pi}\int\int \Pi^{\mu 0\alpha}\kappa_\alpha dS\,. \tag{11}$$

κ_α stands for the 3-components of unit vector over an infinitesimal surface element dS. The quantities P^i for $i = 1, 2, 3$ are the momentum components, while P^0 is the energy.

Considering the line element Eq. (4) for Eqs. (7) and (8), required components of $\Pi^{\mu\nu\alpha}$ are

$$\Pi^{001} = -4\sqrt{\frac{(x^2+y^2)(r-b)}{r}}\,, \tag{12}$$

$$\Pi^{301} = -4\omega\sqrt{\frac{(x^2+y^2)(r-b)}{r}}\,, \tag{13}$$

$$\Pi^{302} = -\frac{2\omega z}{\sqrt{r(r-b)}}\,. \tag{14}$$

Substituting these results into Eq. (6), we find that energy and momentum densities, respectively.

$$\Xi^{00} = \Xi^{03} = \frac{1}{8\pi}\frac{\sqrt{x^2+y^2}(xb_x + yb_y + zb_z + b - r)}{r\sqrt{r-b}}\,, \tag{15}$$

$$\Xi_1^0 = \Xi_2^0 = 0\,, \tag{16}$$

where subscripts (x, y, z) denote partial differentiation.

4. Energy-momentum complexes of RRW model in the Einstein prescription

The energy-momentum complex as defined by Einstein [12] is given by

$$\Upsilon_\mu^\nu = \frac{1}{16\pi} H_{\mu,\alpha}^{\nu\alpha}\,, \tag{17}$$

where

$$H_\mu^{\nu\alpha} = \frac{g_{\mu\beta}}{\sqrt{-g}} \left[-g\left(g^{\nu\beta}g^{\alpha\xi} - g^{\alpha\beta}g^{\nu\xi}\right)\right]_{,\xi}\,. \tag{18}$$

Υ_0^0 is the energy density, Υ_α^0 are the momentum density components, and Υ_0^α are the components of energy current density. The Einstein energy and momentum density satisfies the local conservation laws

$$\frac{\partial \Upsilon_\mu^\nu}{\partial x^\nu} = 0 \tag{19}$$

and energy and momentum components are given by

$$P^\mu = \int\int\int \Upsilon_\mu^0 dxdydz\,. \tag{20}$$

Further Gauss's theorem furnishes

$$P^\mu = \frac{1}{16\pi}\int\int H_\mu^{0\alpha}\eta_\alpha dS\,, \tag{21}$$

η_α stands for the 3-components of unit vector over an infinitesimal surface element dS. The quantities P^i for i=1,2,3 are the momentum components, while P^0 is the energy.

The required components of $H_\mu^{\nu\alpha}$ from Eqs. (4) and (18) are

$$H_0^{01} = 4\sqrt{\frac{(x^2+y^2)(r-b)}{r}}\,, \tag{22}$$

$$H_0^{02} = \frac{2z}{\sqrt{r(r-b)}}\,. \tag{23}$$

Substituting these results into Eq. (17), we get following energy and momentum densities in the form

$$\Upsilon^{00} = \frac{1}{8\pi}\frac{\sqrt{x^2+y^2}(xb_x+yb_y+zb_z+b-r)}{r\sqrt{r-b}}\,, \tag{24}$$

$$\Upsilon^{01} = \Upsilon^{02} = \Upsilon^{03} = 0\,. \tag{25}$$

5. Energy-momentum complexes of RRW model in the Tolman prescription

The energy-momentum complex of Tolman [9] is

$$\mathrm{Im}_k^i = \frac{1}{8\pi} U_{k,j}^{ij}\,, \tag{26}$$

where

$$U_k^{ij} = \sqrt{-g}[-g^{pi}(-\Gamma_{kp}^j + \tfrac{1}{2}\delta_k^j \Gamma_{ap}^a + \tfrac{1}{2}\delta_p^i \Gamma_{ak}^a) + \tfrac{1}{2}\delta_k^i g^{pm}(-\Gamma_{pm}^j + \tfrac{1}{2}\delta_p^i \Gamma_{am}^a + \tfrac{1}{2}\delta_m^j \Gamma_{ap}^a)], \tag{27}$$

where Im_0^0 is the energy density, Im_0^α are the components of the energy current density and Im_α^0 are the momentum density components. The energy-momentum complex Im_k^i satisfies the local conservation law

$$\frac{\partial \mathrm{Im}_k^i}{\partial x^i} = 0\,. \tag{28}$$

The energy distribution in the Tolman definition E_{Tol} is given by

$$E_{\mathrm{Tol}} = \int\!\!\int\!\!\int \mathrm{Im}_0^0 dxdydz\,. \tag{29}$$

Noticing that the space-time under consideration is static, one has

$$E_{\mathrm{Tol}} = \frac{1}{8\pi} \int\!\!\int U_0^{0\beta} n_\beta^{(\alpha)} ds_{(\alpha)}\,, \tag{30}$$

where n_β is the unit vector over an infinitesimal surface element, ds. The required component of $U_\mu^{\nu\alpha}$ from Eqs. (4) and (27) are

$$U_0^{01} = 2\sqrt{\frac{(x^2+y^2)(r-b)}{r}}\,, \tag{31}$$

$$U_0^{02} = \frac{z}{\sqrt{r(r-b)}} \tag{32}$$

and if we take this result into from Eqs. (26), we obtain following energy and momentum densities

$$\mathrm{Im}^{00} = \frac{1}{8\pi} \frac{\sqrt{x^2+y^2}(xb_x + yb_y + zb_z + b - r)}{r\sqrt{r-b}}\,, \tag{33}$$

$$\mathrm{Im}^{01} = \mathrm{Im}^{02} = \mathrm{Im}^{03} = 0\,. \tag{34}$$

6. Summary and discussion

The subject of energy-momentum localization in the general theory of relativity has been very exciting and interesting; however it has been associated with some debate. Recently, some researchers have been interested in studying the energy content of the universe in various models.

The object of present paper is to show that it is possible to solve the problem of localization of energy in general relativity by using the energy and momentum complexes. In this paper, we get some energy distributions of the Rigidly Rotating Wormhole cosmological model, we have considered three different energy and momentum complexes in general relativity: *e.g.* Bergmann–Thomson, Einstein, and Tolman.

TABLE

The energy and momentum densities in Bergmann–Thomson, Einstein, Tolman definitions for RRW space-time.

Prescription	Energy density	Momentum density
Bergmann–Thomson	$\varXi^{00} = \frac{1}{8\pi}\frac{\sqrt{x^2+y^2}(xb_x+yb_y+zb_z+b-r)}{r\sqrt{r-b}}$	$P^1 = P^2 = 0, P^3 \neq 0$
Einstein	$\varUpsilon^{00} = \frac{1}{8\pi}\frac{\sqrt{x^2+y^2}(xb_x+yb_y+zb_z+b-r)}{r\sqrt{r-b}}$	$P^1 = P^2 = P^3 = 0$
Tolman	$\mathrm{Im}^{00} = \frac{1}{8\pi}\frac{\sqrt{x^2+y^2}(xb_x+yb_y+zb_z+b-r)}{r\sqrt{r-b}}$	$P^1 = P^2 = P^3 = 0$

We found that; The energy (due to matter plus field) distribution in Einstein, Bergmann–Thomson and Tolman prescriptions of RRW are the same and non zero. The momentum distribution of RRW in Einstein and Tolman prescriptions are the same and zero, but some components of Bergmann–Thomson momentum distribution of RRW is zero while momentum density ($P^3 \neq 0$) is different from zero. The results advocate the importance of energy-momentum complexes.

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