CONSTRUCTING THE OFF-DIAGONAL PART OF ACTIVE-NEUTRINO MASS MATRIX FROM ANNIHILATION AND CREATION MATRICES IN NEUTRINO-GENERATION SPACE*

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The off-diagonal part of the active-neutrino mass matrix is constructed from two 3×3 matrices playing the role of annihilation and creation matrices acting in the neutrino-generation space of ν_1, ν_2, ν_3. The construction leads to a new relation, $M_{\mu\tau} = 4\sqrt{3}M_{e\mu}$, which *predicts* in the case of tribimaximal neutrino mixing that $m_3 - m_1 = \eta\,(m_2 - m_1)$ with $\eta = 5.28547$. Then, the maximal possible value of $\Delta m^2_{32}/\Delta m^2_{21}$ is equal to $\eta^2 - 1 = 26.9362$ and gives $m_1 = 0$. With the experimental estimate $\Delta m^2_{21} \sim 8.0 \times 10^{-5}$ eV^2, this maximal value, if realized, *predicts* $\Delta m^2_{32} \sim 2.2 \times 10^{-3}$ eV^2, near to the popular experimental estimation $\Delta m^2_{32} \sim 2.4 \times 10^{-3}$ eV^2.

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As is well known, the so-called tribimaximal mixing matrix [1]

$$U = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{1}$$

describes reasonably well the active-neutrino mixing

$$\nu_\alpha = \sum_i U_{\alpha i}\,\nu_i \;\; (\alpha = e, \mu, \tau\,,\; i = 1, 2, 3) \tag{2}$$

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in all confirmed neutrino oscillation experiments [2]. Here, $c_{12} = \sqrt{2/3}$, $s_{12} = 1/\sqrt{3}$, $c_{23} = 1/\sqrt{2} = s_{23}$ and $s_{13} = 0$. If the charged-lepton mass matrix is diagonal, the neutrino mixing matrix $U = (U_{\alpha i})$ is at the same time the diagonalizing matrix for the active-neutrino mass matrix $M = (M_{\alpha\beta})$ whose elements

$$M_{\alpha\,\beta} = \sum_i U_{\alpha i} m_i U^*_{\beta i} \;\; (\alpha, \beta = e, \mu, \tau) \tag{3}$$

get explicitly the form

$$\begin{aligned} M_{e\,e} &= \frac{1}{3}(2m_1 + m_2)\,, \\ M_{\mu\,\mu} &= M_{\tau\,\tau} = \frac{1}{6}(m_1 + 2m_2 + 3m_3)\,, \\ M_{e\,\mu} &= -M_{e\,\tau} = -\frac{1}{3}(m_1 - m_2)\,, \\ M_{\mu\,\tau} &= -\frac{1}{6}(m_1 + 2m_2 - 3m_3) \end{aligned} \tag{4}$$

with m_i denoting the active-neutrino masses. Thus,

$$m_1 = M_{e\,e} - M_{e\,\mu}\;,\; m_2 = M_{e\,e} + 2M_{e\,\mu}\;,\; m_3 = M_{\mu\,\mu} + M_{\mu\,\tau}\;. \tag{5}$$

In the present paper, we will use the mass matrix (4), valid in the case of tribimaximal neutrino mixing, as a reasonable approximation.

Recently, we have proposed for active neutrinos of three generations $i = 1, 2, 3$ the following empirical mass formula [3]:

$$m_i = \mu\,\rho_i \left[1 - \frac{1}{\xi}\left(N_i^2 + \frac{\varepsilon - 1}{N_i^2}\right)\right] \;\; (i = 1, 2, 3) \tag{6}$$

or, rewritten explicitly,

$$\begin{aligned} m_1 &= \frac{\mu}{29}\left(1 - \frac{\varepsilon}{\xi}\right)\,, \\ m_2 &= \frac{\mu}{29}\,4\left[1 - \frac{1}{9\xi}(80 + \varepsilon)\right]\,, \\ m_3 &= \frac{\mu}{29}\,24\left[1 - \frac{1}{25\xi}\,(624 + \varepsilon)\right]\,. \end{aligned} \tag{7}$$

Here, $\mu > 0\,,\, \varepsilon > 0$ and $\xi > 0$ are three free parameters, while

$$N_1 \;=\; 1\,, \quad N_2 = 3\,, \quad N_3 = 5 \tag{8}$$

and

$$\rho_1 \;=\; \frac{1}{29}\,, \quad \rho_2 = \frac{4}{29}\,, \quad \rho_3 = \frac{24}{29}\,. \tag{9}$$

($\sum_i \rho_i \;=\; 1$). The latter numbers have been called generation-weighting factors. The empirical mass formula (6) can be supported by an intuitive model of formal intrinsic interactions which might work within leptons and quarks [4]. The factors (9) appear also in the charged-lepton mass formula, predicting the correct m_τ, consistently with m_e and m_μ [3].

For normal hierarchy of neutrino masses $m_1^2 \ll m_2^2 \ll m_3^2$, when taking the lowest mass lying in the range

$$m_1 \sim (0 \text{ to } 10^{-3})\text{ eV}\,, \tag{10}$$

we obtain from the popular experimental estimates [2]

$$|m_2^2 - m_1^2| \sim 8.0 \times 10^{-5}\text{ eV}^2\,,\ |m_3^2 - m_2^2| \sim 2.4 \times 10^{-3}\text{ eV}^2\,, \tag{11}$$

two higher masses

$$m_2 \sim (8.9 \;\text{ to }\; 9.0) \times 10^{-3}\text{ eV}\,,\ m_3 \sim 5.0 \times 10^{-2}\text{ eV}\,. \tag{12}$$

Then, we can determine the following parameter values in Eq. (6):

$$\mu \sim (7.9 \text{ to } 7.5) \times 10^{-2}\text{ eV}\,,\ \frac{\varepsilon}{\xi} \sim (1 \text{ to } 0.61)\,,\ \frac{1}{\xi} \sim (8.1 \text{ to } 6.9) \times 10^{-3}\,. \tag{13}$$

So, the parameter $1/\xi$ in Eq. (6) is small *versus* 1 and ε/ξ.

One may try to conjecture that in Eq. (6) $1/\xi = 0$ exactly [3]. Then, one *predicts* $m_3 = (6/25)(27m_2 - 8m_1)$, implying from the estimates (11) the inverse order of m_1 and m_2: $m_1 \sim 1.5 \times 10^{-2}$ eV, $m_2 \sim 1.2 \times 10^{-2}$ eV, $m_3 \sim 5.1 \times 10^{-2}$ eV. In this case, $\mu \sim 4.5 \times 10^{-2}$ eV and $\varepsilon/\xi \sim -8.8 < 0$.

Instead, we shall try in the present paper to relate the empirical mass formula (6) to the structure of active-neutrino mass matrix defining also the neutrino mixing, what will lead to another, more systematic, *prediction* of m_3 in terms of m_1 and m_2 (Eq. (24)). To this end, let us introduce the matrices [5]

$$N = 2n + \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

and

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{15}$$

where

$$[a\,,\, n] = a\,, \quad [a^\dagger\,,\, n] = -a^\dagger\,, \quad n = a^\dagger a\,, \quad a^3 = 0\,, \quad a^{\dagger\,3} = 0\,, \tag{16}$$

the latter, a and $a^\dagger$, playing the role of annihilation and creation 3×3 matrices, although

$$[a\,,\, a^\dagger] = \mathbf{1} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \neq \mathbf{1}\,. \tag{17}$$

Note that

$$a^2 = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a^{\dagger\,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}. \tag{18}$$

Due to Eqs. (16), a and $a^\dagger$ change the eigenvalues $n_i = 0, 1, 2$ (or $N_i = 1, 3, 5$) of the matrix n (or N) by -1 and $+1$ (or -2 and $+2$), respectively, *within* the range 0 to 2 (or 1 to 5) [1].

In the formalism of a and $a^\dagger$, it is natural to conjecture tentatively that the active-neutrino mass matrix has the form $M = (M_{\alpha\beta})$ with $M_{\alpha\beta}$ as given in Eqs. (4) and (7), *but* its off-diagonal part can be presented — more restrictively — in terms of our annihilation and creation matrices in the following way:

[1] Three generations $i = 1, 2, 3$ may be also labeled by $n_i = 0, 1, 2$ or by $N_i = 1 + 2n_i = 1, 3, 5$. In the model of three fundamental-fermion generations based on the generalized Dirac equation proposed some years ago [4,5], the label N_i is the number of bispinor indices which appear in three generalized Dirac wave functions describing fundamental fermions of three generations. It is conjectured that among these N_i bispinor indices ("algebraic partons") there are $N_i - 1$ undistinguishable and antisymmetrized (so, obeying "Pauli principle realized intrinsically"), while there is only one distinguished by its coupling to the Standard Model gauge interactions. The antisymmetrized bispinor indices appear in pairs: the label $n_i = (N_i - 1)/2$ is the number of such pairs present in three generations. The number $N_i - 1$ of antisymmetrized bispinor indices, as taking four values each, cannot exceed four. Thus, we conclude that N_i is necessarily equal to 1 or 3 or 5, what explains the existence of exactly three generations of fundamental fermions.

$$\begin{pmatrix} 0 & M_{e\,\mu} & -M_{e\,\mu} \\ M_{e\,\mu} & 0 & M_{\mu\,\tau} \\ -M_{e\,\mu} & M_{\mu\,\tau} & 0 \end{pmatrix} = \mu\,\rho^{1/2}\left[g(a+a^\dagger) - g'(a^2+a^{\dagger\,2})\right]\rho^{1/2}$$

$$= \frac{\mu}{29}\begin{pmatrix} 0 & 2g & -4\sqrt{3}\,g' \\ 2g & 0 & 8\sqrt{3}\,g \\ -4\sqrt{3}\,g' & 8\sqrt{3}\,g & 0 \end{pmatrix}, \quad (19)$$

where $g > 0$ and $g' > 0$ are free parameters (multiplied by the mass scale μ introduced in Eqs. (7)), while

$$\rho^{1/2} = \begin{pmatrix} \rho_1^{1/2} & 0 & 0 \\ 0 & \rho_2^{1/2} & 0 \\ 0 & 0 & \rho_3^{1/2} \end{pmatrix} = \frac{1}{\sqrt{29}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix}. \quad (20)$$

Here, ρ_i ($i = 1, 2, 3$) are the generation-weighting factors defined in Eqs. (9) in the context of mass formula (6). Formally, we have in Eq. (19) $a = (a_{ij})$ and $\rho^{1/2} = (\delta_{\alpha i}\rho_i^{1/2})$, where $\delta_{\alpha i} = 1$ for $\alpha i = e1, \mu 2, \tau 3$ and 0 otherwise. We can see from Eqs. (4) and the equality (19) that

$$-\frac{1}{3}(m_1 - m_2) = M_{e\,\mu} = \frac{\mu}{29}2g = \frac{\mu}{29}4\sqrt{3}g' \quad (21)$$

and

$$-\frac{1}{6}(m_1 + 2m_2 - 3m_3) = M_{\mu\,\tau} = \frac{\mu}{29}8\sqrt{3}\,g = 4\sqrt{3}\,M_{e\,\mu} = -\frac{4}{\sqrt{3}}(m_1 - m_2)\,. \quad (22)$$

Thus[2], Eq. (21) determines g and g' through $m_2 - m_1$:

$$0 < g = 2\sqrt{3}\,g' = \frac{29}{6\mu}\,(m_2 - m_1)\,, \quad (23)$$

and Eq. (22) *predicts* m_3 in terms of m_1 and m_2:

$$m_3 = \eta\,m_2 - (\eta - 1)\,m_1 = \eta(m_2 - m_1) + m_1 \quad (24)$$

[2] Note that the relations $g = 2\sqrt{3}\,g'$ and $M_{\mu\tau} = 4\sqrt{3}\,M_{e\mu}$ following from the conjecture (19) are valid more generally for the bilarge mixing matrix U, where $c_{23} = 1/\sqrt{2} = s_{23}$ and $s_{13} = 0$, while the angle θ_{12} in c_{12} and s_{12} is a free parameter determined from the experimental data. Then, $0 < g = 2\sqrt{3}\,g' = (29/\sqrt{8}\,\mu)c_{12}s_{12}(m_2 - m_1)$ and the coefficient η in Eq. (24) is $\eta \equiv c_{12}^2(1 + 8\sqrt{3/2}\,t_{12})$ with $t_{12} \equiv \tan\theta_{12}$.

with

$$\eta \equiv \frac{2}{3}\,(4\sqrt{3}+1) = 5.28547\,. \tag{25}$$

Notice that Eq. (24) allows in particular for the limiting option of exact degeneracy $m_1 = m_2 = m_3$ that is excluded experimentally. Generically, from the restrictive relation (24) we get the equation

$$[\eta\, m_2 - (\eta-1)m_1]^2 - m_2^2 = m_3^2 - m_2^2 = \lambda(m_2^2 - m_1^2) \tag{26}$$

or

$$[\eta - (\eta-1)r]^2 - 1 - \lambda(1-r^2) = 0\,, \tag{27}$$

where

$$r \equiv \frac{m_1}{m_2} \tag{28}$$

and

$$\lambda \equiv \frac{m_3^2 - m_2^2}{m_2^2 - m_1^2} \sim \frac{2.4\times 10^{-3}}{8.0\times 10^{-5}} = 30\,, \tag{29}$$

the latter value being valid in the case of popular experimental best fit [2] (here $m_1 < m_2 < m_3$).

With $\lambda > \eta - 1 = 4.28547$, Eq. (27) for r gets two solutions

$$r^{(1,2)} = \frac{\eta(\eta-1)}{(\eta-1)^2+\lambda} \pm \frac{|\eta-1-\lambda|}{(\eta-1)^2+\lambda} = \begin{cases} 1 \\ \frac{\eta^2-1-\lambda}{(\eta-1)^2+\lambda} \end{cases}. \tag{30}$$

The first solution $r^{(1)} = 1$ corresponds to the limiting option of exact neutrino mass degeneracy $m_1 = m_2 = m_3$, what is not the case realized experimentally. The second solution is nonnegative, $r^{(2)} \geq 0$ (giving $m_1 \geq 0$), *only* if $\lambda \leq \eta^2 - 1 = 26.9362$ (while the popular experimental best fit is $\lambda \sim 30$). With the use of experimental estimate $m_2^2 - m_1^2 \sim 8.0\times 10^{-5}$ eV^2, this solution gives $m_3^2 - m_2^2 \lesssim 2.2\times 10^{-3}$ eV^2.

For the maximal allowed value $\lambda = \eta^2 - 1 = 26.9362$ we get $r^{(2)} = 0$ and so, we *predict* $m_1 = 0$ and $m_3^2 - m_2^2 \sim 2.2\times 10^{-3}$ eV^2 (while the popular experimental estimate is $m_3^2 - m_2^2 \sim 2.4\times 10^{-3}$ eV^2). Such a smaller value is still consistent with the present data within experimental limits. With $m_1 = 0$, we now obtain

$$m_2 \sim 8.9\times 10^{-3}\ \text{eV}\,, \quad m_3 \sim 4.7\times 10^{-2}\ \text{eV} \tag{31}$$

and infer from Eq. (23) that

$$g = 2\sqrt{3}\,g' \sim 0.53\,. \tag{32}$$

In the last estimation, the value $\mu \sim 8.1 \times 10^{-2}$ eV (Eq. (33)) is applied. From Eq. (31) we can determine with $m_1 = 0$ the following parameter values in Eq. (6) in place of the previous values (13):

$$\mu \sim 8.1 \times 10^{-2}\ \text{eV}\,, \quad \frac{\varepsilon}{\xi} = 1\,, \quad \frac{1}{\xi} = 1.03311 \times 10^{-2}. \tag{33}$$

Here, the parameter $1/\xi$ in Eq. (6) is still small *versus* 1 and ε/ξ, though it is larger than previously. In contrast to the value of μ, the value of $1/\xi$ is independent of the input of experimental estimate $m_2^2 - m_1^2 \sim 8.0 \times 10^{-5}$ eV^2, since $1/\xi$ is calculated from the relation $m_3 = \eta\, m_2$, where Eqs. (7) are used with $\varepsilon/\xi = 1$.

Concluding, we have constructed in this note the off-diagonal part of the active-neutrino mass matrix with the use of two 3×3 matrices playing the role of annihilation and creation matrices acting in the neutrino-generation space of $\nu_1\,, \nu_2\,, \nu_3$. The construction leads to the new relation $M_{\mu\tau} = 4\sqrt{3}\, M_{e\mu}$ (Eq. (22)) which *predicts* in the case of tribimaximal neutrino mixing that $m_3 - m_1 = \eta(m_2 - m_1)$ with $\eta = 5.28547$. Then, the maximal possible value of $\lambda \equiv (m_3^2 - m_2^2)/(m_2^2 - m_1^2)$ is equal to $\eta^2 - 1 = 26.9362$ and gives $m_1 = 0$. With the experimental estimate $m_2^2 - m_1^2 \sim 8.0 \times 10^{-5}$ eV^2, this maximal value, if realized, *predicts* $m_3^2 - m_2^2 \sim 2.2 \times 10^{-3}$ eV^2, near to the popular experimental estimation $m_3^2 - m_2^2 \sim 2.4 \times 10^{-3}$ eV^2.

Formally, one may say that our neutrino mass formula (6), which primarily is a special transformation of three masses m_1, m_2, m_3 into three parameters μ, ε, ξ, becomes a special function predicting $m_1 = 0$, m_2 and $m_3 = \eta\, m_2$ in terms of μ and $\varepsilon = \xi$, when the conjecture (19) about the off-diagonal part of neutrino mass matrix is made and the value $\lambda \equiv (m_3^2 - m_2^2)/(m_2^2 - m_1^2) \leq \eta^2 - 1$ assumed to be maximal: $\eta^2 - 1$. More generally, without the latter requirement, still $m_3 = \eta\, m_2 - (\eta - 1)m_1$ is predicted in terms of m_1 and m_2, what, due to Eqs. (7), imposes a relation between ε and ξ, namely $1/\xi = 0.0173763 - 0.0070452\,\varepsilon/\xi$. This gives $1/\xi = 0.0103311$ in the case of our maximality requirement (when $m_1 = 0$ *i.e.*, $\varepsilon = \xi$). Of course, when $m_1 \gtrsim 0$ *i.e.*, $\varepsilon \lesssim \xi$, then $\lambda \equiv (m_3^2 - m_2^2)/(m_2^2 - m_1^2) \lesssim \eta^2 - 1$ and $1/\xi \gtrsim 0.0103311$.

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