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A NONLOCAL INTEGRABLE GENERALIZATION OF THE FRENKEL–KONTOROVA MODEL OF DISLOCATION

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A new simple nonlocal generalization of the Frenkel–Kontorova model of dislocation in solid body as a type of the nonlocal *sine*-Gordon equation with the generalized interaction term is suggested. Its limit cases, symmetries and exact analytical solutions are obtained. This type of the nonlocal *sine*-Gordon equation is shown to possess one-solitonic solutions which are a nonlocal deformation of the corresponding classical solutions of the *sine*-Gordon equation. Exact analytical solutions of this equation and its Lagrangian integrability and geometrical approach are considered.

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1. Introduction

It seems reasonable, of all the possible nonlinear evolution equations, to pick up rather "nice" ones to be analyzed in detail. This requirement of "niceness" is met by the so-called integrable nonlinear equations in which additional symmetry allows application of the newly coined methods, such as inverse scattering [1], micro-differential operators [2] and algebraic geometry [3].

The *sine*-Gordon equation (SGE)

$$\phi_{tt} - a\phi_{xx} = b\sin(\lambda\phi)\,, \tag{1}$$

is one of the basic nonlinear equations both in mathematics and modern physics. In mathematics it appears as an equation for the surfaces of constant negative curvature ($a = \lambda = -b = 1$) and was already known to F. Minding and E. Beltrami. Its physical applications are related with the description of dislocations in solid state physics [4], motion of Bloch magnetic walls in magnetic crystals [5], magnetic flux propagation in superconductors [6] and so on [7]. In these applications, the SGE gives the simplest nonlinear description of the phenomena under consideration. More adequate models correspond to SGE (1) nonlocal generalizations.

All known nonlocal generalizations of SGE could be divided into two groups: (1) where the kinetic or (2) the dynamic term is under nonlocal generalization. To the first group belong various generalizations where the local operator $\partial_{xx}\phi$ is replaced by the integro-differential operator $L[\phi]$ [8]:

$$\phi_{tt} - L[\phi] = b\,\sin(\lambda\phi)\,. \tag{2}$$

In particular, to the family of the evolution Eq. (2) belong various interesting examples of nonlocal Josephson electrodynamics. These examples were introduced in [9–14], where one of the basic model equations is

$$\phi_{tt} - \hat{H}\phi_x + \sin\phi = 0\,, \tag{3}$$

where $\hat{H}$ is the Hilbert transform (see Appendix). The evolution Eq. (3) was an object of study in a series of papers [11, 12, 15–18].

To the first group belongs also the nonlocal generalization of SGE proposed in [19]:

$$\phi_{tt} - D_x^\alpha\phi + \sin\phi = 0\,, \tag{4}$$

where D_x^α is the Riesz partial fractional derivative (see Appendix). For this equation, a family of breather-like solutions (*i.e.* solutions that are localized in space and periodic in time) has been found numerically, and it has been shown that these entities are quite robust and can be generated in the course of evolution of initial states of a rather different shape.

Another type of nonlocal generalization of SGE was proposed in [20, 21]:

$$\phi_{xx} - \phi_{tt} = 2\cos\left[\frac{\phi(x,t)}{2}\right]\int f(x-y)\,\sin\left[\frac{\phi(y,t)}{2}\right]dy\,, \tag{5}$$

where $f(x) = 1/(x^4 + \sigma^4)$ or Gauss-type. It is shown that small amplitude solitons of the nonlocal SGE can create coupled states. The effect is due to a change of the dispersion originated by nonlocal nonlinearity. The evolution Eq. (5) in the general case could be generalized in the form

$$\phi_{xx} - \phi_{tt} = F[\phi]\,, \tag{6}$$

where $F[\phi]$ is a nonlinear and nonlocal function of $\phi(x,t)$.

In the current paper, a new type of nonlocal generalization of the Frenkel–Kontorova dislocation model is suggested. The corresponding evolution equation is the nonlocal SGE. Its limit cases, symmetries and exact analytical solutions are obtained. This type of the nonlocal SGE equation is shown to possess one-solitonic solutions which are a nonlocal deformation of the corresponding classical solutions of the SGE equation. Exact analytical solutions of this equation and its Lagrangian integrability and geometrical approach are considered.

2. Nonlocal generalization of the Frenkel–Kontorova dislocation model

Let us consider a one-dimensional lattice with atoms in the integer numbers of the real x-axis. The influence of the underlying layer of atoms is approximated by the potential energy

$$U(x) = \frac{af_0}{2\pi}\left[1 - \cos\left(\frac{2\pi x}{a}\right)\right], \tag{7}$$

where x is the coordinate, a is the lattice constant, and f_0 is a constant of the shape of the potential. Let the displacement of the n-th atom from the position of equilibrium be

$$y_n(t) = x_n(t) - na\,, \tag{8}$$

where $x_n(t)$ is the coordinate of the n-th atom (see Fig. 1). The influence of the neighboring atoms in the same layer is usually expressed by elastic forces: $k(x_{n+1} - x_n) - k(x_n - x_{n-1})$, where k is the coefficient of elasticity. Taking into account expressions (7) and (8), the equation of motion of the n-th atom of the mass m is

$$m(y_n)_{tt} = -f_0 \sin\left(\frac{2\pi y_n}{a}\right) + k(y_{n+1} - 2y_n + y_{n-1})\,. \tag{9}$$

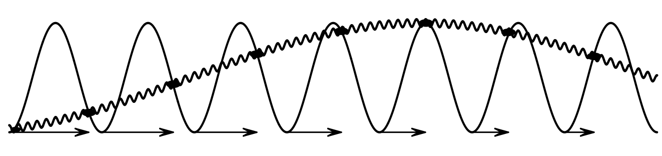

Fig. 1. Schematic picture of dislocation. Two layers of the crystal lattice. The position of the atoms of the bottom layer is marked by circles (o). The influence on the atoms of the upper layer (•) is reduced to the effect of potential energy (7). The interaction of atoms of the upper layer could be as usual described by spring interaction. Arrays show the corresponding displacements.

If the corresponding limits exist and are finite,

$$\lim_{a\to 0}\frac{am}{2\pi f_0}=t_0^2\,,\qquad \lim_{a\to 0}\frac{a^3k}{2\pi f_0}=x_0^2\,,\tag{10}$$

we may introduce dimensionless space and time variables

$$x'=\frac{x}{x_0}\,,\qquad t'=\frac{t}{t_0}\tag{11}$$

and derivatives of the function $\varphi(x',t)$:

$$\lim_{a\to 0}\left(\frac{2\pi y_n}{a}\right)=\lim_{a\to 0}\left(\frac{2\pi y_n/x_0}{a/x_0}\right)=\varphi_{x'}(x',t)\,,\tag{12}$$

$$\lim_{a\to 0}\,x_0^2\,\frac{2\pi}{a^3}\left(y_{n+1}-2y_n+y_{n-1}\right)=\varphi_{x'x'}(x',t)\,.\tag{13}$$

In the text below, the prime mark ($'$) is omitted.

This allows us to express the equation of motion (9) in a dimensionless form:

$$(\varphi_x)_{tt}-(\varphi_x)_{xx}+\sin\varphi_x=0\,.\tag{14}$$

After substitution $\varphi_x\equiv\phi$ we arrive to the classical SGE (1).

Let us assume that atoms in the one-dimensional lattice sites undergo a chaotic but in the general case non-Gaussian perturbation. In this case the limit $\lim\limits_{a\to 0}(2\pi y_n/a)$ could not exist, but there exists the limit $\lim\limits_{a\to 0}(2\pi y_n/a^\alpha)$, where $0<\alpha\le 1$. Then, instead of (12) we may obtain

$$\lim_{a\to 0}\left(x_0^{\alpha-1}\frac{2\pi y_n}{a^\alpha}\right)=D_x^\alpha\,\phi(x,t)\,,\tag{15}$$

where D_x^α is a fractional derivative of the order α. From the geometrical point of view, the existence of the limit (15) means that at a small distance the geometry of the displacement looks like

$$\Delta y\sim C(\Delta x)^\alpha\,,\tag{16}$$

and the coefficient C is the above-mentioned value of the fractional derivative. This property of the perturbation function $y_n(t)$ and the corresponding $\phi(x,t)$ is closely related to the property of the scale invariance,

$$A(x)\to\lambda^\Delta A(\lambda x)\,,\tag{17}$$

where Δ is the index of the scale invariant function $A(x)$, or in other words, to the property of fractal sets [22].

Thus, in the case of a non-integer dimension of the real interval $x \in D$ of the axis x, $D \in \mathbb{R}$, we have to use the limit (15), and the equation of motion in this case takes the form

$$(D_x^\alpha \phi)_{tt} - (D_x^\alpha \phi)_{xx} + \sin(D_x^\alpha \phi) = 0\,, \tag{18}$$

or, using the properties of the fractional derivative (see Appendix), we obtain

$$D_x^\alpha \left[\phi_{tt} - \phi_{xx} + D_x^{-\alpha} \sin(D_x^\alpha \phi)\right] = 0\,. \tag{19}$$

Among the all possible solutions of this equation let us consider a set of functions $\phi(x,t)$ for which

$$\phi_{tt} - \phi_{xx} + D_x^{-\alpha} \sin(D_x^\alpha \phi) = 0\,, \tag{20}$$

and the boundary condition obeys the relation $D_x^\alpha \phi(x, t_0) = 0$. In this case, the inverse operator $D_x^{-\alpha} \equiv I_x^\alpha$ exists and for $\alpha \neq 1$ equals

$${}^R D_x^{-\alpha} \equiv {}^R I_x^\alpha = -2\cos\left(\frac{\alpha\pi}{2}\right)\left(I_+^\alpha + I_-^\alpha\right)\,. \tag{21}$$

Note here that according to the semigroup character of the fractional derivatives $D_x^\alpha D_x^{-\alpha} = 1$, but

$$D_x^{-\alpha} D_x^\alpha f(x) = f(x) - D_x^{\alpha-1} f(x) \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\,, \tag{22}$$

where a is the left border of the interval $x \in D \subset \mathbb{R}$.

Thus, we may consider the evolution equation (20) as a nonlocal deformation of the classical Frenkel–Kontorova model, which coincides with the SGE. The next problem is analysis of the evolution equation (20).

3. Symmetries, solutions, Lagrangian, *etc*.

Let us consider a special type of the nonlocal SGE (NSGE):

$$\phi_{tt} - a\phi_{xx} = bD_x^{-\alpha} \sin(\lambda D_x^\alpha \phi)\,, \tag{23}$$

where a, b and λ are constants, $\phi = \phi(x,t) \in C^2(D) \subset \mathbb{R}, x \in \Omega \subset \mathbb{R}$, $\dim \Omega = \alpha (0 < \alpha \leq 1), t \in \mathbb{R}, (x,t) \in D = \Omega \times \mathbb{R}, \dim D = 1 + \alpha$, and D_x^α means a space fractional Riesz derivative of the order α (see Appendix). This equation is a simple generalization of the above obtained evolution Eq. (20). In our classification, its belongs to the second group of the possible nonlocal generalizations of SGE (6), where the term of potential interaction is modified.

At the first sight NSGE (23) looks very complicated, but actually it is an equivalent transformation of the interaction term. Indeed, in the case of linear dependence this term does not change.

In the case of small values of the parameter α, the infinitesimal form of Eq. (23) is as follows:

$$\phi_{tt} - a\phi_{xx} = b \sin \lambda\phi + \alpha L[\phi] \,, \tag{24}$$

where $L[\phi]$ is a local perturbation of the classical SGE, when at $\alpha \to 0$ the NSGE turns into the ordinary SGE (1).

In the case of small amplitudes $|\lambda D_x^\alpha \phi| \ll 1$, the NSGE turns into the linear Klein–Gordon equation with the "mass" term $\lambda b\phi$.

If $\phi(x,t)$ is a solution of the SGE, then the function

$$\phi_1(x,t) = \frac{2\pi n}{\lambda} \pm \phi(C_1 \pm x, C_2 \pm t) \,, \quad n = 0, \pm 1, \pm 2, \ldots \,, \tag{25}$$

where C_1, C_2 are arbitrary constants, is also an exact solution of SGE. The signs in expression (25) could be chosen arbitrarily. Unfortunately, this does not hold for NSGE solutions, but would be useful for generating new solutions of NSGE by the known solution of SGE.

3.1. The Lagrangian

It could be verified that the NSGE (23) has the following Lagrangian form:

$$L = \int\limits_{-\infty}^{+\infty} \left\{ \frac{1}{2} \left[(D_x^\alpha \phi_t)^2 - (D_x^{1+\alpha} \phi)^2 \right] + \frac{b}{\lambda} \left[1 - \cos\left(\lambda D_x^\alpha \phi\right)\right] \right\} dx \,. \tag{26}$$

Thus, the equation of motion (23) could be derived by using the modified Noether theorem. For instance, the energy-momentum tensor T_{ik} in the metric η_{ik} $(x^i = (x,t) \subset D;\ i,k = 0,1)$

$$T_{ik} = (D_{x^i}^\alpha \phi)(D_{x^l}^\alpha \phi)\eta_{lk} - \eta_{ik}\mathcal{L} \,, \tag{27}$$

where $\mathcal{L}$ is the Lagrangian density in expression (26).

The existence of the energy-momentum tensor T_{ik} (27) and equation of motion (7) means the vanishing of the divergence of tensor T_i^k in the whole area D

$$\frac{\partial T_i^k}{\partial x^k} = 0 \,, \tag{28}$$

and the existence of the related conserved quantities. Then, for instance, the condition (12) means the existence of conservation of momentum P_i:

$$P_i = \lambda \int\limits_{\infty}^{+\infty} T_i^0 \, dx \Big|_{t=\mathrm{const.}} . \tag{29}$$

Indeed, for the field functions $\phi(x) \sim |x|^{-(1+\varepsilon)}$ for $x \to \pm\infty$ from the condition (28) and Stockes theorem it follows that

$$\int\limits_C T_i^k \, dl_k = 0 \,, \tag{30}$$

where C is a rectangle with the generating lines $x^0 = x_1^0$ and $x^0 = t^0$. Thus,

$$P_i(t_1) = \int\limits_{-\infty}^{+\infty} T_i^k \, dS_k \Big|_{t=t_1} = \int\limits_{-\infty}^{+\infty} T_i^k \, dS_k \Big|_{t=t_1} = P_i(t_2) \,. \tag{31}$$

If the variable x in the general case corresponds to a generalized coordinate (*e.g.*, an angle), the other conserved quantity in the case of symmetric T^{ik} is the angular momentum M^{ik}:

$$M^{ik} = \int (x^i \, dP^k - x^k \, dP^i) \,. \tag{32}$$

Indeed, this is the case when the divergence of the density of M^{ik} vanishes. Note here that in the case of one space variable the angular momentum has only one nonzero component M^{01}. In the general case,

$$\frac{\partial}{\partial x^l} \left(x^i T^{kl} - x^k T^{il} \right) = T^{ik} - T^{ik} = 0 \,. \tag{33}$$

Thus, in the nonlocal case of our NSGE we may introduce the corresponding nonlocal generalizations of the classical conserved quantities.

3.2. The travelling wave solution

The NSGE has the travelling wave solution — a nonlocal generalization of one-solitonic solution.

(a) Let $b\lambda(\mu^2 - ak^2) > 0$, then

$$\phi(x,t) = \frac{4}{\lambda} \, D_x^{-\alpha} \mathrm{arctg} \left\{ \exp \left[\pm \frac{b\lambda(kx + \mu t + \theta_0)}{\sqrt{b\lambda(\mu^2 - ak^2)}} \right] \right\} , \tag{34}$$

where k, μ, θ_0 are arbitrary constants.

(b) Let $b\lambda(\mu^2 - ak^2) < 0$, then

$$\phi(x,t) = \frac{4}{\lambda}\, D_x^{-\alpha} \mathrm{arctg}\left\{\exp\left[\pm\frac{b\lambda(kx+\mu t+\theta_0)}{\sqrt{b\lambda(ak^2-\mu^2)}}\right]\right\}, \tag{35}$$

where k, μ, θ_0, like above, are arbitrary constants. Thus, the influence of nonlocality leads to the space shape deformation of the solution.

4. Integrability

The classical SGE belongs to the family of integrable evolutionary equations. Is it possible to proof the same for the NSGE?

Let us consider the SO$(2,1)$ linear integrable system,

$$\varPhi_t = U\varPhi\,, \qquad \varPhi_x = V\varPhi\,, \tag{36}$$

where U and V take values in the Lie algebra so$(2,1)$. This means that U and V may be of the following two types:

$$(i): U = \begin{pmatrix} 0 & C & B \\ -C & 0 & A \\ B & A & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & F & E \\ -F & 0 & D \\ E & D & 0 \end{pmatrix}, \tag{37}$$

$$(ii): U = \begin{pmatrix} 0 & C & B \\ C & 0 & A \\ B & -A & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & F & E \\ F & 0 & D \\ E & -D & 0 \end{pmatrix}, \tag{38}$$

where the coefficients A, B, C, D, E and F are suitable functions of ϕ and their (non)local derivatives.

For the case *(i)*, the integrable condition for system (36) in the case of local functions and their derivatives is the Gauss equation of the imbedding of the pseudo-sphere $S^{1,1} \subset \mathbb{R}^{2,1}$, and for the case *(ii)* it is the Gauss equation of the imbedding of the hyperplane $H^2 \subset \mathbb{R}^{2,1}$. Since the SGE corresponds to the case of $H^2 \subset \mathbb{R}^{2,1}$, let us consider the case *(ii)*.

Let l, m, n be an orthonormal frame of $\mathbb{R}^{2,1}$, and $-l^2 = m^2 = n^2 = 1$. The condition $l^2 = -1$ is the equation for $H^2 \subset \mathbb{R}^{2,1}$.

As follows from the linear system (36), the integrability condition is

$$U_x - V_t + [U, V] = 0\,. \tag{39}$$

For the case *(ii)* it can be written in the following form:

$$\begin{aligned} &\left(\frac{E_t - B_x}{CE - BF}\, B + \frac{F_t - C_x}{CE - BF}\, C\right)_x \\ &- \left(\frac{E_t - B_x}{CE - BF}\, E + \frac{F_t - C_x}{CE - BF}\, F\right)_t + CE - BF = 0\,. \end{aligned} \tag{40}$$

The nonlinear partial differential equation which admits an SO$(2,1)$ linear integrable system ($CE - BF \neq 0$) is (40). Moreover, Eq. (40) is the Gauss equation for $H^2 \subset \mathbb{R}^{2,1}$, when B, C, E, F are arbitrary functions of ϕ and their local derivatives. The basic difference between the local and nonlocal cases is dependence of the coefficients A, B, C, D, E and F on the possible nonlocal derivatives.

Let $B = F = 0$, and $C = \sqrt{\lambda b} \cos(\lambda D_x^\alpha \phi/2)$, $E = \sqrt{\lambda b} \sin(\lambda D_x^\alpha \phi/2)$. From Eq. (40) we get the NSGE:

$$\phi_{tt} - \phi_{xx} = b D_x^{-\alpha} \sin(\lambda D_x^\alpha \phi) \,. \tag{41}$$

Thus, the NSGE belongs to the family of integrable nonlinear and nonlocal evolution equations.

5. The geometrical approach

The classical SGE describes the surface of a constant negative curvature imbedded in D-dimensional space. Regarding the Tschebyscheff coordinates, the first and second fundamental forms of the surface are

$$\mathrm{I} \; = \; ds^2 = \cos^2\frac{\phi}{2}\, dt^2 + \sin^2\frac{\phi}{2}\, dx^2 \,, \tag{42}$$

$$\mathrm{II} \; = \; -d\vec{r} \cdot d\vec{n} = \cos\frac{\phi}{2}\, \sin\frac{\phi}{2}\, (dt^2 - dx^2) \,. \tag{43}$$

It is easy to verify that the Gauss curvature K of such surface is

$$K = \frac{\mathrm{det}Q}{\mathrm{det}G} = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = -1 \,, \tag{44}$$

where Q and G are matrices of the second and first fundamental forms in expressions (42) and (43). The mean curvature $H = \mathrm{Sp}\,(G^{-1}Q)$:

$$H = \frac{g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}}{g_{11}g_{22} - g_{12}^2} = -2\,\mathrm{ctg}\,\phi \,. \tag{45}$$

The quantities K and H (44) and (45) express the geometrical contents of the SGE.

The classical way to derive the SGE is the substitution of the Christoffel connection coefficients Γ_{ij}^k, which are determined by the coefficients of the first fundamental form g_{ij},

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g^{il}}{\partial \xi^j} + \frac{\partial g^{jl}}{\partial \xi^i} - \frac{\partial g^{ij}}{\partial \xi^l} \right) , \tag{46}$$

into the Gauss equation:

$$\frac{\partial \Gamma^l_{ij}}{\partial \xi^k} - \frac{\partial \Gamma^l_{ik}}{\partial \xi^j} + \Gamma^s_{ij}\Gamma^l_{ks} - \Gamma^s_{ik}\Gamma^l_{js} = b_{ij}b^l_k - b_{ik}b^l_j \,, \tag{47}$$

where $\xi^i = (t, x)$.

Here we can consider one simplification. In the case of the first fundamental form $ds^2 = A^2dt^2 + B^2dx^2$, the Gauss curvature could be obtained from the expression (see *e.g.* [23])

$$K = -\frac{1}{AB}\left[\left(\frac{A_t}{B}\right)_t + \left(\frac{B_x}{A}\right)_x\right] . \tag{48}$$

We can see, that the substitution of $A = \cos\phi/2$ and $B = \sin\phi/2$ into (42) leads to the classical SGE:

$$\phi_{tt} - \phi_{xx} = -K\sin\phi \,. \tag{49}$$

In the case of the nonlocal value of the coefficients, *i.e.* in the case when there exists the fractional derivative D^α_x and

$$A = \cos(D^\alpha_x\phi/2)\,, \quad B = \sin(D^\alpha_x\phi/2)\,, \tag{50}$$

by substituting (50) into (48) we obtain the nonlocal generalization of the SGE:

$$\phi_{tt} - \phi_{xx} = -KD^{-\alpha}_x\sin(D^\alpha_x\phi)\,. \tag{51}$$

Together with the coefficients of the second fundamental form,

$$b_{11} = -b_{22} = \cos(D^\alpha_x\phi/2)\,\sin(D^\alpha_x\phi/2)\,, \tag{52}$$

according to Eq. (44), we can obtain the value of the Gauss curvature $K = -1$.

Thus, for the first and second fundamental forms,

$$\mathrm{I} = \cos^2\left(\frac{D^\alpha_x\phi}{2}\right)dt^2 + \sin^2\left(\frac{D^\alpha_x\phi}{2}\right)dx^2\,, \tag{53}$$

$$\mathrm{II} = \cos\left(\frac{D^\alpha_x\phi}{2}\right)\sin\left(\frac{D^\alpha_x\phi}{2}\right)(dt^2 - dx^2)\,, \tag{54}$$

the surface of the constant negative curvature $K = -1$ imbedded in $(1+\alpha)$-dimensional space obeys the NSGE:

$$\phi_{tt} - \phi_{xx} = D^{-\alpha}_x\sin(D^\alpha_x\phi)\,. \tag{55}$$

6. Conclusions

Thus, the NSGE, like the ordinary SGE, has a Lagrangian form (26) and one-solitonic solutions (34), (35). Despite the nonlocal nature of the interaction term in the evolution equation, this model possesses nonlocal deformations of localized solutions.

The asymptotic form has slowly falling tails $\phi(x) \sim x^\alpha$, which converge to zero at $\alpha < 0$, as follows from explicit expressions of the solutions. At the same time the total value of the momenta $I[\phi] = \int_{-\infty}^{+\infty} \phi(x,0)\,dx$ diverges for any $\alpha > -1$. This means a nonlocal distribution of the momenta, energy and related quantities.

From the asymptotic and infinitesimal form of NSGE (23) follow the corresponding dispersion relations,

$$\omega^2 - ak^2 = \lambda b \qquad \text{and} \qquad \omega^2 - ak^2 = W(k)\,, \tag{56}$$

where $W(k)$ corresponds to the Fourier transform for the linearized part of the $b \sin \lambda\phi + \alpha L[\phi]$ according to Eq. (24) and which are the Klein–Gordon and SGE modified dispersion relations.

The variety of the physical origination of SGE (1) allows us to apply the obtained solutions not only to dislocation evolution in the modified Frenkel–Kontorova model [1], but also to the Josephson effect [6-11], magnetic crystals [2], semiconductors [3], *etc.* In the case of particle physics, worth noting is the interesting idea of mass generation possibility for the classical SGE in the case of $|\lambda D_x^\alpha \phi| \ll 1$.

Note here one important property. The continuation of the parameter $\alpha \in [0;2]$ does not mean a continuous transition of one evolution equation to another. Let us have an evolution equation in the form

$$\phi_{tt} - a\phi_{xx} = N_\alpha[\phi]\,, \tag{57}$$

where $N_\alpha[\phi]$ means the nonlocal operator on $\phi(x,t)$, and α is the parameter of nonlocality. The transformation of the operator $N_\alpha[\phi]$ for $\alpha \in [0;2]$ induces transformation of the automorphism groups G_0 and G_2 (group symmetries of Eq. (57) for $\alpha = 0$ and $\alpha = 2$) for the corresponding local evolution equations:

$$\begin{tikzcd} N_0[\phi] \arrow[r, "\alpha"] \arrow[d, "\mathrm{Aut}"'] & D^\alpha \arrow[r, "\alpha"] \arrow[d] & N_2[\phi] \arrow[d, "\mathrm{Aut}"] \\ G_0 \arrow[r, "\alpha"] & T^\alpha \arrow[r, "\alpha"] & G_2 \end{tikzcd}$$

where, in the general case, the operator of the fractional derivative D^α induces an action on the group of translation operators T^α.

In other words, the continuous deformation of the evolution equation by the nonlocal parameter α related with the space fractional derivative D_x^α is not continuous in the topological sense, *i.e.* this continuous deformation is not a diffeomorphysms in the group space of symmetries of the corresponding equation induced by the translation operator T^α.

This fact, which is a great drawback from the point of view of mathematics, but an advantage in phase transitions, induces the idea to apply fractional calculus in phase transition theory. Here we will have not only the asymptotic values of the symmetry groups like G_0 and G_2 as in the above diagram, but also a detailed kinetic description of all intermediate states $0 < \alpha < 2$.

At the microscopic level, the fractional character of the space derivative D_x^α is the result of the random motion of individual atoms. In the simplest usual diffusion-like model we have to use Laplacian operators $\partial_x \phi$ to model the atom motion where the key assumption is that the random motion is a stochastic Gaussian process.

The origin of non-Gaussian motion can be traced back to the existence of long-range correlations in the dynamics, or the presence of anomalously large particle displacements described by broad probability distributions.

Qualitatively, in the Frenkel–Kontorova model we have the perturbed wave motion, but in the fractional case these perturbations have a nonlocal nature. The influence of such nonlocal perturbations leads to the long-range correlations or the presence of anomalously large particle displacements in the dynamics of atoms.

Such a specific motion of atoms is the reason not only for the dynamics of defects in the solid body. Recently, a growing number of works have shown the existence of anomalous diffusion processes for which the mean square displacement $\langle [x - \langle x \rangle] \rangle \sim t^\gamma$ grows faster ($\gamma > 1$), in the case of superdiffusion, or slower ($\gamma < 1$), in the case of subdiffusion, than in the Gaussian diffusion process [27].

Accordingly, an important open problem is to understand the dynamics of such diffusion systems when the assumption of Gaussian diffusion fails. This problem has a particular relevance to plasma physics, perturbative transport experiments, in numerical simulation of three-dimensional turbulence, and problems of solid body physics [27].

Appendix A

To give an explicit expression for the Riesz pseudo-differential operator, we first introduce the Weyl fractional integrals $I_{\pm}^{\beta}$ of the order $\beta > 0$ [24,25]:

$$I_{\pm}^{\beta}\phi(x) = \begin{cases} \frac{1}{\Gamma(\beta)} \int\limits_{-\infty}^{x} (x-\xi)^{\beta-1}\phi(\xi)\, d\xi\,, \\ \frac{1}{\Gamma(\beta)} \int\limits_{x}^{+\infty} (\xi - x)^{\beta-1}\phi(\xi)\, d\xi\,. \end{cases} \tag{A.1}$$

Then the Weyl fractional derivatives could be introduced by the relations

$$D_{\pm}^{\alpha}\phi(x) = \begin{cases} \pm(\frac{d}{dx} I_{\pm}^{1-\alpha})\phi(\xi)\,, & 0 < \alpha < 1\,, \\ (\frac{d^2}{dx^2} I_{\pm}^{2-\alpha})\phi(x)\,, & 1 < \alpha < 2\,, \end{cases} \tag{A.2}$$

where $I_{\pm}^{\alpha}$ denotes the Weyl fractional integrals of the order $\alpha > 0$. When $\alpha = 0$, the Weyl fractional derivative degenerates into the identity operator

$$D_{\pm}^{0}\phi(x) = I\phi(x) = \phi(x)\,. \tag{A.3}$$

For the continuity of $D_{\pm}^{\alpha}\phi(x)$ with respect to α,

$$D_{\pm}^{1} = \pm\frac{d}{dx}\,, \qquad D_{\pm}^{2} = \frac{d^2}{dx^2}\,. \tag{A.4}$$

For an arbitrary α we have the definition

$$D_{\pm}^{\alpha}\phi(x) = \begin{cases} \frac{1}{\Gamma[\{\alpha\}]} \frac{d^{[\alpha]}}{dx^{[\alpha]}} \int\limits_{-\infty}^{x} \frac{\phi(t)\, dt}{(x-t)^{1+\{\alpha\}}}\,, \\ \frac{-1}{\Gamma[\{\alpha\}]} \frac{d^{[\alpha]}}{dx^{[\alpha]}} \int\limits_{x}^{+\infty} \frac{\phi(t)\, dt}{(t-x)^{1+\{\alpha\}}}\,, \end{cases} \tag{A.5}$$

where $\{\alpha\}$ and $[\alpha]$ are the fractional and integer parts of $\alpha > 0$.

The Riesz fractional derivative, denoted sometimes as $\partial^{\alpha}/\partial|x|^{\alpha}$, is defined as

$$^{R}D_{x}^{\alpha}\phi(x) = \begin{cases} -\frac{D_{+}^{\alpha}+D_{-}^{\alpha}}{2\cos(\alpha\pi/2)}\,\phi(x)\,, & \alpha \neq 1\,, \\ \left(\frac{d}{dx}\hat{H}\right)\phi(x)\,, & \alpha = 1\,, \end{cases} \tag{A.6}$$

where $\hat{H}$ is the Hilbert transformation

$$\hat{H}\phi(x) = \text{v.p.}\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\phi(\xi)}{x-\xi}\, d\xi\,, \tag{A.7}$$

and the integral is understood in the sense of the Cauchy principal value.

An important property of the Riesz fractional derivative ${}^{R}D^{\alpha}$ is that it is a Fourier multiplier operator with the symbol $|k|^{\alpha}$. For some application of the fractional calculus in the case of nonlinear and nonlocal integrable models see, *e.g.*, [26].

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