

THE ASYMPTOTIC DEPENDENCE OF ELLIPTIC RANDOM VARIABLES*

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In this paper, we try to answer the question, whether for bivariate elliptic random variable $X = (X_1, X_2)$ the marginal random variables X_1 and X_2 are asymptotically dependent. We show, that for some special form of the characteristic generator of X the answer is positive.

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1. Motivation

In order to give an answer to the question, “What is the origin of the interest of the asymptotic dependence of elliptic random variables?” one has to go back several dozen years.

Already in the years 1950’s and 1960’s researchers discovered the non-normal behaviour of financial market data. In the early 1990’s an understanding of the methodology underlying financial or insurance extremes became very important. Traditional statistics mostly concern the laws governing averages. But when we look at the largest, (respectively the smallest), elements in a sample, the assumption of normality seems not to be reasonable in the number of applications, particularly in finance and insurance. And heavy-tailed distributions have a chance to be more appropriate.

Why? Let X_1, X_2 be insurance claims due to flood disasters (X_1) and wind storms (X_2). Last year events taught us that very often the extreme values of X_1 are accompanied by extreme values of X_2 . In mathematical language it means, that X_1 and X_2 are asymptotically dependent. Traditional models based on multidimensional normal probability law give rise to quite opposite conclusion. Therefore, in modelling of extreme events more and more often the researchers use the wider class of distributions, which includes the normal distribution as a special case.

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Let us now consider a simple example from finance market. In the figure below you see daily log-returns of stock indices DAX and CAC (horizontal axis X_1 -DAX, vertical axis X_2 -CAC). The data cover the period from 1990 to 2004 (about 4000 data). The scatter plot assumes a shape of “elliptic cloud”. And the level sets of the probability density of the random vector (X_1, X_2) are ellipses. This empirical observation suggests, that the family of elliptic distributions should be taken under consideration. Furthermore, we can ask how often we observe the situation, when the daily log-returns of the both indices take the extreme values.

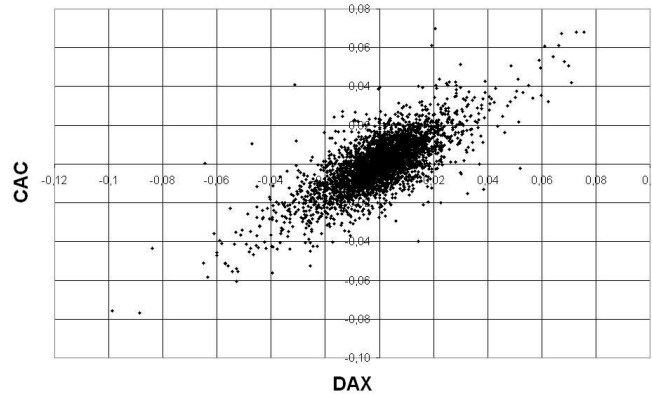


Fig. 1. Log-returns of stock indices DAX and CAC.

Let $W(j)$ be the quantity of observations $(x_{1,k}, x_{2,k})$ such that $(x_{1,k} > x_{1,j}, x_{2,k} > x_{2,j})$, where $x_{i,j}$ is the j -th order statistics of the random variable X_i , $i = 1, 2$.

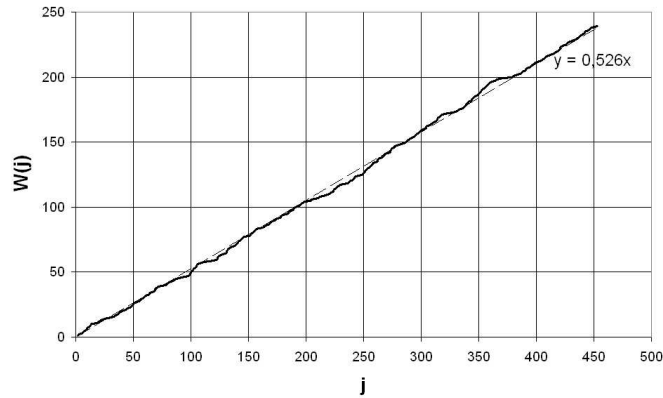


Fig. 2. Graph of $W(j)$.

The graph of the function $W(j)$ shows us, that X_1 and X_2 are asymptotically dependent. So the joint random variable (X_1, X_2) could not be normally distributed.

2. Preliminaries

To begin with, we recall the basic definitions.

Let F be an univariate distribution function and F^{-1} its generalised inverse

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} \quad \text{for all } u \in (0, 1).$$

DEFINITION 2.1. Let (X_1, X_2) be a random vector with marginal distribution functions F_1 and F_2 .

The coefficient of upper tail dependence of (X_1, X_2) is defined to be

$$\lambda_U(X_1, X_2) = \lim_{u \rightarrow 1} P(X_1 > F_1^{-1}(u) | X_2 > F_2^{-1}(u)),$$

provided that the limit $\lambda_U \in [0, 1]$ exists.

If $\lambda_U = 0$, then we say that X_1 and X_2 are asymptotically independent.

Otherwise (that is $\lambda_U > 0$ or λ_U does not exists), we say that they are asymptotically dependent.

For a pair of random variables upper tail dependence is a measure of joint extremes. That is they measure the probability that one component is at an extreme of size given that the other is at the same extreme, relative to the marginal distributions.

LEMMA 2.1. If two continuously distributed random variables X_1, X_2 are independent, then they are asymptotically independent.

Proof.

$$\lim_{u \rightarrow 1} P(X_1 > F_1^{-1}(u) | X_2 > F_2^{-1}(u)) = \lim_{u \rightarrow 1} P(X_1 > F_1^{-1}(u)) = 0.$$

Note that the bivariate normal distribution has the same property. The tail behaviour can be also described in a ‘‘symmetric way’’.

DEFINITION 2.2. The bivariate random variable $X = (X_1, X_2)$ is said to be regularly varying with index $\beta > 0$, if for all $y > 0$ and for every angle $[\alpha_0, \alpha_1]$

$$\lim_{t \rightarrow \infty} \frac{P(|X| > ty, \arg X \in [\alpha_0, \alpha_1])}{P(|X| > t)} = y^{-\beta} M(\alpha_0, \alpha_1).$$

where $M(\alpha_0, \alpha_1)$ is a certain measure on the interval $[0, 2\pi)$.

DEFINITION 2.3. If X is a bivariate random variable and, for some $\mu \in \mathbb{R}^2$, some 2×2 nonnegative definite symmetric matrix Σ and some function $\psi : [0, \infty) \rightarrow \mathbb{R}$, the characteristic function is of the form

$$\varphi(t) = \exp(it^* \mu) \psi(t^* \Sigma t),$$

then we say that X has an elliptical distribution with parameters μ, Σ and ψ , and we write $X \sim E_2(\mu, \Sigma, \psi)$.

The function ψ is referred to as the characteristic generator of X .

REMARK 2.1. The following widespread used distributions prove to be elliptic:

1. the normal distribution

$$\psi(t^2) = \exp\left(\frac{-t^2}{2}\right),$$

2. some α -stable with characteristic generator of the form

$$\psi(t^2) = \exp\left(\frac{-|t|^\alpha}{2}\right), \quad 0 < \alpha < 2,$$

3. T -Student distribution.

3. Result

THEOREM 3.1. Let

$$\phi(t) = \psi(t^* \Sigma t), \quad t \in \mathbb{R}^2,$$

be a characteristic function of a bivariate elliptically distributed random variable $X = (X_1, X_2)$.

1. Σ is a positive definite symmetric matrix,

2. $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is such that:

$$\psi(r^2) = \psi_0(r^2) + r^\beta \psi_1(r^\gamma), \quad 0 < \gamma \leq 2,$$

$$\beta \in \mathbb{R}_+ \setminus 2\mathbb{N},$$

$$\psi_0, \psi_1 \in \mathcal{C}^\infty(\mathbb{R}_+) \wedge \psi_1(0) \neq 0 \wedge \psi_0(0) = 1,$$

$$\forall 0 \leq k \leq 4 + [\beta] \quad \lim_{t \rightarrow \infty} t^{k+\frac{1}{2}} \psi^{(k)}(t^2) = 0,$$

then the marginal random variables X_1 and X_2 are asymptotically dependent.

4. Concluding remarks

1. For all $\alpha \in (0, 2)$, if $X = (X_1, X_2)$ is elliptic and α -stable, then X_1 and X_2 are asymptotically dependent.
2. The result is also valid for the characteristic generator of the form $\psi(r^2) = \psi_0(r^2) + r^{\beta_1} \psi_1(r^{\gamma_1}) + \dots + r^{\beta_m} \psi_m(r^{\gamma_m})$, where $0 < \beta_1 < \beta_2 < \dots < \beta_m$.

5. Proof of the Theorem

LEMMA 5.1. *Let us have the same assumptions as in Theorem 3.1. Then the asymptotics of the probability density of bivariate random variable $X = (X_1, X_2)$ formulates as follows*

$$\begin{aligned} g(x) &= c\|x\|^{-2-\beta} + O(\|x\|^{-3-\beta}), \quad \|x\| \rightarrow \infty, \quad c = \text{const} > 0, \\ \|x\| &= \sqrt{x^* \Sigma x}, \quad x = (x_1, x_2). \end{aligned} \tag{1}$$

Proof.

$$\begin{aligned} \lambda(u) &= P(X_1 > F_1^{-1}(u) | X_2 > F_2^{-1}(u)) \\ &= \frac{P(X_1 > F_1^{-1}(u) \wedge X_2 > F_2^{-1}(u))}{P(X_2 > F_2^{-1}(u))}. \end{aligned}$$

Let us denote $F_j^{-1}(u) = as_j, j = 1, 2, s_1, s_2 = \text{const} > 0, a \ll 0,$

$$\begin{aligned} P(X_1 > F_1^{-1}(u) \wedge X_2 > F_2^{-1}(u)) &= P(X_1 > as_1 \wedge X_2 > as_2) \\ &= \int_{as_1}^{+\infty} dx_1 \int_{as_2}^{+\infty} dx_2 \int_{\mathbb{R}^2} e^{-ix^*t} \psi(t^* \Sigma t) dt. \end{aligned}$$

We calculate the asymptotics of the integral above, for $a \rightarrow +\infty.$
 $\Sigma > 0$ and symmetric $\implies \Sigma = A^*A$

$$g(x) = \int_{\mathbb{R}^2} e^{-ix^*t} \psi(t^* \Sigma t) dt = \int_{\mathbb{R}^2} e^{-ix^*A^{-1}w} \psi(w^*w) (\det \Sigma)^{-\frac{-1}{2}} dw,$$

after the change of the variables $t = A^{-1}w.$

Next we substitute $x = A^*y$ and obtain

$$g(A^*y) = \int_{\mathbb{R}^2} e^{-iy^*w} \psi(w^*w) (\det \Sigma)^{-\frac{-1}{2}} dw = (\det \Sigma)^{\frac{-1}{2}} G(y).$$

We change the variables a second time $w_1 = r \cos \varphi, w_2 = r \sin \varphi,$ and let us express y_1, y_2 in the form $y_1 = \|y\| \sin \alpha, y_2 = \|y\| \cos \alpha.$ Then

$$\begin{aligned} G(y) &= \int_0^{+\infty} r \psi(r^2) dr \int_0^{2\pi} e^{-ir\|y\| \sin(\varphi+\alpha)} d\varphi \\ &= \int_0^{+\infty} r \psi(r^2) dr \int_{\alpha}^{2\pi+\alpha} e^{-ir\|y\| \sin \varphi} d\varphi = 2\pi \int_0^{+\infty} r \psi(r^2) J_0(r\|y\|) dr, \end{aligned}$$

where J_0 is Bessel function

$$J_0(r\|y\|) = \frac{1}{\sqrt{\pi r\|y\|}} \left[\cos\left(r\|y\| - \frac{\pi}{4}\right) + O\left((r\|y\|)^{-1}\right) \right],$$

for $r\|y\| \rightarrow \infty$ i $|\arg r\|y\|| \leq \pi - \epsilon < \pi$, cf. [4]

$$\int_0^{+\infty} \frac{r\psi(r^2)}{\sqrt{\pi r\|y\|}} \cos\left(r\|y\| - \frac{\pi}{4}\right) dr = \frac{1}{\sqrt{\pi\|y\|}} \operatorname{Re} e^{-\frac{i\pi}{4}} \int_0^{+\infty} e^{ir\|y\|} r^{\frac{1}{2}} \psi(r^2) dr.$$

Now we compute the first term of asymptotics of the integral

$$F(\|y\|) = \int_0^{+\infty} e^{ir\|y\|} r^{\frac{1}{2}} \psi(r^2) dr, \quad \text{for } \|y\| \rightarrow \infty.$$

We assumed that the function $\psi(r^2)$ and its derivatives tend quickly to 0, as $r \rightarrow \infty$. Therefore, with the help of localisation rule and Erdelyi Lemma cf. [4] we obtain

$$a) \operatorname{Re} \left[e^{-\frac{i\pi}{4}} \int_0^b e^{ir\|y\|} r^{\frac{1}{2}} \psi_0(r^2) dr \right] = 0, \quad \text{the asymptotics is trivial,}$$

$$\begin{aligned} b) & e^{-\frac{i\pi}{4}} \int_0^b e^{ir\|y\|} r^{\frac{1}{2}+\beta} \psi_1(r^2) dr \\ &= e^{-\frac{i\pi}{4}} \psi_1(0) \Gamma\left(\frac{3}{2} + \beta\right) e^{\frac{i\pi(\frac{3}{2}+\beta)}{2}} \|y\|^{-\frac{3}{2}-\beta} + O\left(\|y\|^{-\frac{5}{2}-\beta}\right) \\ &= ie^{\frac{i\pi\beta}{2}} \psi_1(0) \Gamma\left(\frac{3}{2} + \beta\right) e^{\frac{i\pi(\frac{3}{2}+\beta)}{2}} \|y\|^{-\frac{3}{2}-\beta} + O\left(\|y\|^{-\frac{5}{2}-\beta}\right), \end{aligned}$$

$$\begin{aligned} & \operatorname{Re} \left[ie^{\frac{i\pi\beta}{2}} \psi_1(0) \Gamma\left(\frac{3}{2} + \beta\right) e^{\frac{i\pi(\frac{3}{2}+\beta)}{2}} \|y\|^{-\frac{3}{2}-\beta} \right] \\ &= -\sin \frac{\pi\beta}{2} \psi_1(0) \Gamma\left(\frac{3}{2} + \beta\right) \|y\|^{-\frac{3}{2}-\beta}. \end{aligned}$$

The expression above is not trivial, when β is not a natural even number. Hence the first term of the asymptotics of the integral $G(y)$ is given by a formula:

$$\begin{aligned} G(y) &= 2\sqrt{\pi} \left(-\sin \frac{\pi\beta}{2} \right) \psi_1(0) \Gamma\left(\frac{3}{2} + \beta\right) \|y\|^{-2-\beta} + O\left(\|y\|^{-3-\beta}\right) \\ &= c_1 \|y\|^{-2-\beta} + O\left(\|y\|^{-3-\beta}\right) \quad \text{for } \|y\| \rightarrow \infty. \end{aligned}$$

Thus

$$g(x) = c_1(\det \Sigma)^{\frac{-1}{2}} \|(A^{-1})^* \|^{\beta} \|x\|^{-2-\beta} + O(\|x\|^{-3-\beta}).$$

LEMMA 5.2. *Under the assumptions of Theorem 3.1 the bivariate random variable $X = (X_1, X_2)$ is regularly varying with the index β .*

Proof.

$$\begin{aligned} \frac{P(|X| > ty, \arg X \in [\alpha_0, \alpha_1])}{P(|X| > t)} &= \frac{\int_{ty}^{+\infty} \int_{\alpha_0}^{\alpha_1} rg(r, \theta) d\theta dr}{\int_t^{+\infty} \int_0^{2\pi} rg(r, \theta) d\theta dr} = \\ &= \frac{(\alpha_1 - \alpha_0) \int_{ty}^{+\infty} [r^{-1-\beta} + O(r^{-2-\beta})] dr}{2\pi \int_t^{+\infty} [r^{-1-\beta} + O(r^{-2-\beta})] dr} = \frac{\alpha_1 - \alpha_0}{2\pi} y^{-\beta}. \end{aligned}$$

Lemma 5.2 implies the thesis of Theorem 3.1 cf. [5].

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