ON VALUE AT RISK FOR FOREIGN EXCHANGE RATES — THE COPULA APPROACH*

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The aim of this paper is to determine the Value at Risk (VaR) of the portfolio consisting of long positions in foreign currencies on an emerging market. Basing on empirical data we restrict ourselves to the case when the tail parts of distributions of logarithmic returns of these assets follow the power laws and the lower tail of associated copula C follows the power law of degree 1. We will illustrate the practical usefulness of this approach by the analysis of the exchange rates of EUR and CHF at the Polish forex market.

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1. Introduction

The present paper is a continuation of [1]. In the previous paper we dealt with the purely asymptotic estimations, whereas now our goal is to provide some estimates valid for quite a substantial part of the tail.

We shall deal with the following simple case. An investor operating on an emerging market, has in his portfolio two currencies which are highly dependent, for example euros (EUR) and Swiss franks (CHF). Let R_1 and R_2 be their rates of returns at the end of the investment. Let w_i be the part of the capital invested in the i-th currency, $w_1 + w_2 = 1$, $w_1, w_2 > 0$. So the final value of the investment equals

$$W_1 = W_0\,(1+R)\,, \qquad R = w_1R_1 + w_2R_2\,.$$

Our aim is to estimate the risk of keeping the portfolio. As a measure of risk we shall consider "Value at Risk" (VaR), which last years became one of the most popular measures of risk in the "practical" quantitative finance

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(see for example [2–9]). Roughly speaking the idea is to determine the biggest amount one can lose on certain confidence level α.

If the distribution functions of R_1 and R_2 are continuous then, for the confidence level $1-\alpha$, VaR is determined by the condition

$$P(W_0 - W_1 \leq VaR_{1-\alpha}) = 1 - \alpha\,.$$

Note that if Q_α denotes the α quantile of the rate of return R, then we can denote VaR in the following way

$$VaR_{1-\alpha} = -W_0 Q_\alpha\,.$$

We shall base on the Sklar theorem, which elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate margins (see [10–12]). We describe the joint distribution of rates of return R_1 and R_2 with the help of a copula C

$$P(R_1 \leq x_1, R_2 \leq x_2) = C(F_1(x_1), F_2(x_2))\,,$$

where F_i is a distribution function of R_i. Note, that C is the joint distribution function of the random variables $F_1(R_1)$ i $F_2(R_2)$.

We recall that a function

$$C : [0,1]^2 \longrightarrow [0,1]\,,$$

is called a *copula* if

$$C(0,y) = C(x,0) = 0\,, \qquad C(1,y) = y\,,\ C(x,1) = x\,,$$

$$x_1 < x_2\,, y_1 < y_2 \Rightarrow C(x_1,y_1) + C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) \geq 0\,.$$

2. Basic empirical observations

2.1. Copulas

The copulas of financial returns have a specific property, namely they have uniform tails.

We recall that a copula C has a uniform lower tail if for sufficiently small q_i

$$C(q_1,q_2) \approx L(q_1,q_2)\,,$$

where L is a nonzero function, which is positive homogeneous of degree 1 (compare [1, 13–16])

$$\forall t \geq 0 \quad L(tq_1, tq_2) = tL(q_1,q_2)\,.$$

For the daily exchange rates EUR and CHF in Polish złoty (PLN) we can observe this phenomenon even for 10% part of the tails. On the scatter diagram below (Fig. 1) we plot the ranks of daily returns of EUR and CHF (from January 1995 to April 2006, 2858 returns). One can observe that there is more points at the lower and upper corners than average. In Fig. 2 we enlarge the lower corner.

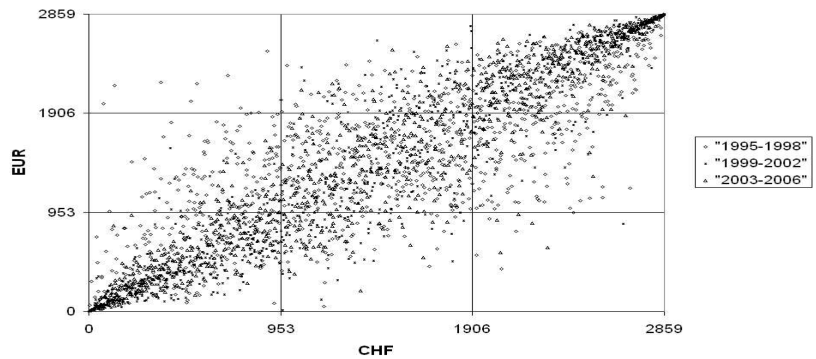

Fig. 1. Plot of ranks.

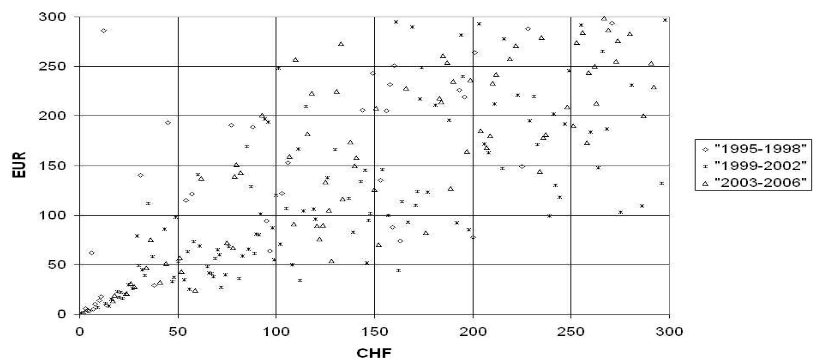

Fig. 2. Plot of ranks (lower corner).

To check the homogeneity of the lower tail of the copula we count the number of pairs of ranks in squares having the origin as a lower vertex

$$W(n) = \sharp\{(x_i, y_i) \;:\; x_i \le n,\, y_i \le n\,,\}$$

and number of pairs in these squares under and over the diagonal

$$W_+(n) = \sharp\{(x_i, y_i) \;:\; x_i < y_i \le n\}\,,$$

$$W_-(n) = \sharp\{(x_i, y_i) \;:\; y_i < x_i \le n\}.$$

In Fig. 3 we show the graphs of these functions. They are close to linear.

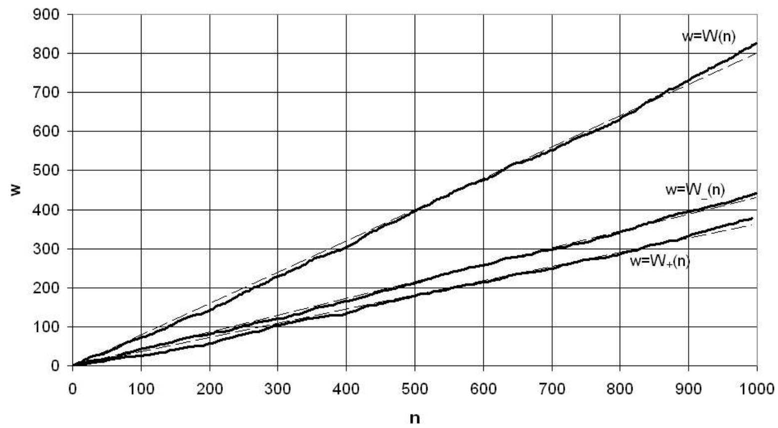

Fig. 3. Graph of $W(n)$.

2.2. Univariate tails

The daily log-returns of exchange rates ($ln(1 + R_i)$) have the power-like tails (compare [17] §9.3, [18] §2.3.1 or [19–22]).

For sufficiently small r ($-1 < r \ll 0$)

$$F_i(r) \approx a_i \, (b_i - \ln(1 + r))^{-\gamma_i} \, , \quad i = 1, 2 \, .$$

For the daily exchange rates EUR and CHF in Polish złoty (PLN) such approximation is valid even for 10% part of the lower tails. In Fig. 4 we plot the logarithms of minus log-returns against the logarithms of probability.

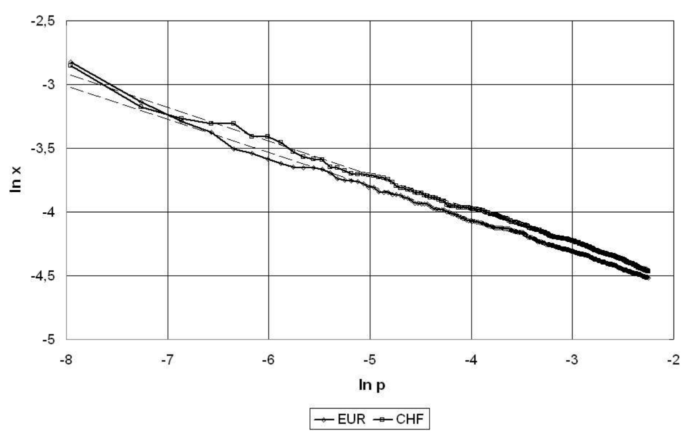

Fig. 4. Plot of logarithms.

3. Main results

Our aim is to show how to estimate Value at Risk of the portfolio ($VaR(W)$) in terms of Values at Risk of portfolios of the same initial value W_0 but consisting only of one currency ($VaR(S_1)$ and $VaR(S_2)$). The first estimate requires only that the tail part of the copula is homogeneous of degree 1.

Theorem 3.1 *If for $q_1, q_2 < \alpha_*$ $C(q) = L(q)$, where L is homogeneous of degree 1, then for $\alpha < \alpha_*$*

$$VaR_{1-L(1,1)\alpha}(W) \geq w_1 VaR_{1-\alpha}(S_1) + w_2 VaR_{1-\alpha}(S_2)\,.$$

The second estimate requires also some properties of lower tails of the marginal distribution.

Theorem 3.2 *If for $q_1, q_2 < \alpha_*$ $C(q) = L(q)$, where L is homogeneous of degree 1, and for $-1 < t \leq F_i^{-1}(\alpha_*)$*

$$F_i(t) = a_i\,(b_i - \ln(1+t))^{-\gamma_i}\,, \quad a_i > 0\,, \quad \gamma_i > 1\,, \quad i = 1, 2\,,$$

then for $\alpha < \alpha_*$

$$VaR_{1-\alpha}(W) \leq w_1 VaR_{1-\alpha}(S_1) + w_2 VaR_{1-\alpha}(S_2)\,.$$

In Figs. 5 and 6 we show the plot of the empirical VaR of the portfolio ($w_1 = 0,4$ EUR, $w_2 = 0,6$ CHF) and the estimates based on the theoretical VaR's for both currencies. We put $W_0 = 1$.

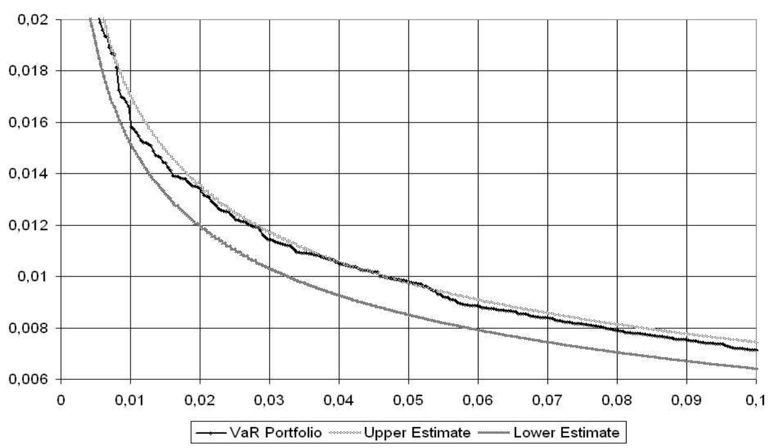

Fig. 5. $VaR(W)$.

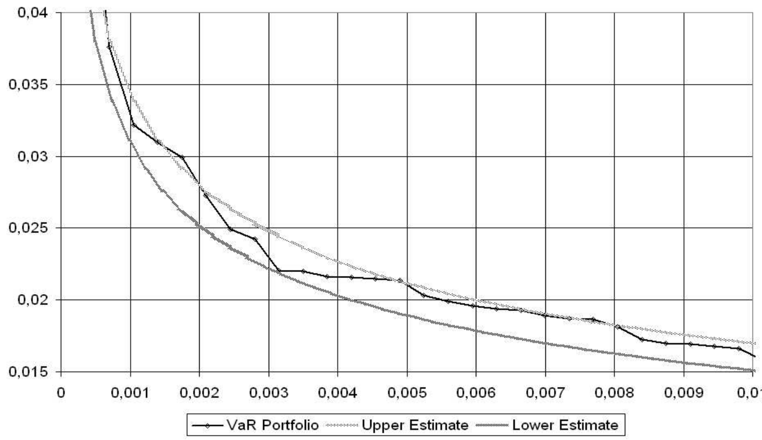

Fig. 6. $VaR(W)$ (the extreme part).

4. Proofs

$$P(R \leq r) = P(w_1 R_1 + w_2 R_2 \leq r) = \mu_C(V_r) \approx \mu_L(V_r)\,,$$

where

$$V_r = \left\{ q : w_1 F_1^{-1}(q_1) + w_2 F_2^{-1}(q_2) \leq r \right\}\,.$$

Note that the domain V_r is a generalized trapezoid.

$$V_r = \{ q : 0 \leq q_1 \leq q_*,\;\; 0 \leq q_2 \leq \varphi_r(q_1) \}\,,$$

where $q_* = F_1(\frac{r+w_2}{w_1})$ and $\varphi_r(q_1) = F_2\left(\frac{r - w_1 F_1^{-1}(q_1)}{w_2}\right)$.

Lemma 4.1 *If $rW_0 = -w_1 VaR_{1-\alpha}(S_1) - w_2 VaR_{1-\alpha}(S_2)$ then the square $[0,\alpha] \times [0,\alpha]$ is contained in V_r.*

Proof. If $q_i \leq \alpha$ then

$$F_i^{-1}(q_i) \leq F_i^{-1}(\alpha) = -\frac{VaR_{1-\alpha}(S_i)}{W_0}\,.$$

Therefore,

$$w_1 F_1^{-1}(q_1) + w_2 F_2^{-1}(q_2) \leq -w_1 \frac{VaR_{1-\alpha}(S_1)}{W_0} - w_2 \frac{VaR_{1-\alpha}(S_2)}{W_0} = r\,.$$

Now we are able to finish the proof of theorem 3.1.

Proof of theorem 3.1. (compare [23]).
Due to the homogeneity we get

$$\mu_C([0,\alpha]\times[0,\alpha]) = C(\alpha,\alpha) = L(1,1)\alpha\,.$$

Let $rW_0 = -w_1 VaR_{1-\alpha}(S_1) - w_2 VaR_{1-\alpha}(S_2)$. Since the square $[0,\alpha]\times[0,\alpha]$ is contained in V_r, we have

$$\mu_C(V_r) \geq L(1,1)\alpha\,.$$

Therefore, the $L(1,1)\alpha$ quantile of R is smaller than r. Thus

$$VaR_{1-L(1,1)\alpha}(W) \geq -rW_0 = w_1 VaR_{1-\alpha}(S_1) + w_2 VaR_{1-\alpha}(S_2)\,.$$

This finishes the proof of theorem 3.1.

Lemma 4.2 *Let $rW_0 = -w_1 VaR_{1-\alpha}(S_1) - w_2 VaR_{1-\alpha}(S_2)$. If*

$$F_i(t) = a_i\,(b_i - \ln(1+t))^{-\gamma_i}\,, \qquad a_i > 0\,,\;\; \gamma_i > 1\,,\quad i = 1,2\,,$$

then the function

$$\psi : [0, q_*) \longrightarrow [0,+\infty)\,, \qquad q_* = F_1\left(\frac{r+w_2}{w_1}\right)\,, \qquad \psi(q_1) = \frac{q_1}{\varphi_r(q_1)}\,,$$

has the following properties:

- *ψ is strictly convex and increasing;*
- *$\psi(0) = 0$, $\lim_{q\to q_*^-}\psi(q) = +\infty$, $\psi(\alpha) = 1$;*
- *$\lim_{q\to 0^+}\psi'(q) = F_2((r+w_1)w_2^{-1})^{-1}$, $\lim_{q\to q_*^-}\psi'(q) = +\infty$.*

Proof. We have

$$\begin{aligned}\psi(q_1) &= \frac{q_1}{F_2\left(\frac{r-w_1F_1^{-1}(q_1)}{w_2}\right)}\\ &= a_2^{-1}q_1\left(b_2+\ln(w_2)-\ln\left(1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)\right)\right)^{\gamma_2}.\end{aligned}$$

Hence

$$\psi(0) = 0\,, \qquad \psi(q_*^-) = \frac{q_*}{F_2(-1)} = +\infty\,, \qquad \psi(\alpha) = \frac{\alpha}{\alpha} = 1\,.$$

Furthermore,

$$\begin{aligned}
\psi'(q_1) &= a_2^{-1}\left(b_2+\ln(w_2)-\ln\left(1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)\right)\right)^{\gamma_2}\\
&\quad + a_2^{-1}\gamma_2\left(b_2+\ln(w_2)-\ln\left(1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)\right)\right)^{\gamma_2-1}\\
&\quad \times \frac{w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)}{1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)}\,\frac{1}{\gamma_1}\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}.
\end{aligned}$$

Hence

$$\psi'(0^+)=\frac{1}{F_2((r+w_1)w_2^{-1})}\,,\qquad \psi(q_*^-)=\frac{q_*}{F_2(-1)}=+\infty\,.$$

Moreover, the first derivative is always positive, hence ψ is strictly increasing. Also the second derivative is always positive (hence ψ is strictly convex). Indeed: the second component of the first derivative is a product of four positive factors, from which only the last one ($q_1^{-1/\gamma}$) has negative derivative but it is reduced by positive derivative of the first component.

$$\begin{aligned}
\psi''(q_1) &= a_2^{-1}\gamma_2\left(b_2+\ln(w_2)-\ln\left(1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)\right)\right)^{\gamma_2-1}\\
&\quad \times \frac{w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)}{1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)}\,\frac{1}{\gamma_1}\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}q_1^{-1}\\
&\quad + \ldots + \ldots + \ldots\\
&\quad + a_2^{-1}\gamma_2\left(b_2+\ln(w_2)-\ln\left(1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)\right)\right)^{\gamma_2-1}\\
&\quad \times \frac{w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)}{1+r-w_1\exp\left(b_1-\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\right)}\,\frac{1}{\gamma_1}\left(\frac{q_1}{a_1}\right)^{\frac{-1}{\gamma_1}}\frac{(-1)}{\gamma_1 q_1}\\
&\quad = \ldots + (\ldots)\times q_1^{-1}\left(1-\frac{1}{\gamma_1}\right).
\end{aligned}$$

Since γ_1 is greater then 1 the final result is positive.

Lemma 4.3 *Let $rW_0 = -w_1 VaR_{1-\alpha}(S_1) - w_2 VaR_{1-\alpha}(S_2)$. If the function $\psi(q_1) = \frac{q_1}{\varphi_r(q_1)}$ has properties listed in lemma 4.2 then $\mu_L(V_r) \leq \alpha$.*

Proof.

$$\begin{aligned}
\mu_L(V_r) &= \mu_L(\{q : w_1 F_1^{-1}(q_1) + w_2 F_2^{-1}(q_2) \leq r\}) \\
&= \mu_L(\{q : 0 \leq q_2 \leq \varphi_r(q_1), 0 \leq q_1 \leq q_*\}) \\
&= \int_0^{q_*} \int_0^{\varphi_r(q_1)} \frac{\partial^2 L}{\partial q_1 \partial q_2}(q_1, q_2) dq_2 dq_1 = \int_0^{q_*} \frac{\partial L}{\partial q_1}(q_1, \varphi_r(q_1)) dq_1 \,.
\end{aligned}$$

Since L is homogeneous of degree 1, its first derivative is homogeneous of degree 0. Thus

$$\begin{aligned}
\mu_L(V_r) &= \int_0^{q_*} \frac{\partial L}{\partial q_1} \left(\frac{q_1}{\varphi_r(q_1)}, 1 \right) dq_1 \\
&= L\left(\frac{q_1}{\varphi_r(q_1)}, 1 \right) \frac{1}{\left(\frac{q_1}{\varphi_r(q_1)} \right)'} \Bigg|_0^{q_*} + \int_0^{q_*} L\left(\frac{q_1}{\varphi_r(q_1)}, 1 \right) \frac{\left(\frac{q_1}{\varphi_r(q_1)} \right)''}{\left(\left(\frac{q_1}{\varphi_r(q_1)} \right)' \right)^2} dq_1 \,.
\end{aligned}$$

For every copula there is an upper bound $C(q_1, q_2) \leq \min(q_1, q_2)$ ([10]). Since L coincides with C in the lower corner, the same bound is valid for L. Therefore,

$$\begin{aligned}
\mu_L(V_r) &\leq \int_0^{q_*} \min\left(\frac{q_1}{\varphi_r(q_1)}, 1 \right) \frac{\left(\frac{q_1}{\varphi_r(q_1)} \right)''}{\left((\frac{q_1}{\varphi_r(q_1)})' \right)^2} dq_1 \\
&= \int_0^{\alpha} \frac{q_1}{\varphi_r(q_1)} \frac{\left(\frac{q_1}{\varphi_r(q_1)} \right)''}{\left((\frac{q_1}{\varphi_r(q_1)})' \right)^2} dq_1 + \int_{\alpha}^{q_*} \frac{\left(\frac{q_1}{\varphi_r(q_1)} \right)''}{\left((\frac{q_1}{\varphi_r(q_1)})' \right)^2} dq_1 \\
&= \left(q_1 - \frac{\frac{q_1}{\varphi_r(q_1)}}{(\frac{q_1}{\varphi_r(q_1)})'} \right) \Bigg|_0^{\alpha} + \frac{-1}{\left(\frac{q_1}{\varphi_r(q_1)} \right)'} \Bigg|_{\alpha}^{q_*} = \alpha - \frac{1}{\psi'(\alpha)} + \frac{1}{\psi'(\alpha)} = \alpha \,.
\end{aligned}$$

To finish the proof of theorem 3.2 one has to observe that if

$$\mu_C(V_r) = \mu_L(V_r) \leq \alpha \,,$$

then

$$VaR_{1-\alpha}(W) \leq -rW_0 = w_1 VaR_{1-\alpha}(S_1) + w_2 VaR_{1-\alpha}(S_2) \,.$$

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