ON CAPITAL DEPENDENT DYNAMICS OF KNOWLEDGE*

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(Received August 16, 2006)

We investigate the dynamics of growth models in terms of dynamical system theory. We analyse some forms of knowledge and its influence on economic growth. We assume that the rate of change of knowledge depends on both the rate of change of physical and human capital. First, we study a model with constant savings. The model with optimised behaviour of households is also considered. We show that the model where the rate of change of knowledge depends only on the rate of change of physical capital can be reduced to the form of the two-dimensional autonomous dynamical system. All possible evolutional paths and the stability of solutions in the phase space are discussed in details. We obtain that the rate of growth of capital, consumption and output are greater in the case of capital dependent rate of change of knowledge.

PACS numbers: 89.65.–s, 01.75.+m

1. Introduction

Physics gives us knowledge which allows to understand processes occurring in nature. This knowledge has a crucial significance for the development of the technological civilisation. And here appears the strict relation between physics and economics. In economics the knowledge and technology is an important factor of "wealth of nations". The wealth is accumulated in the process of economic growth. The contemporary theories of economic growth are based on the seminal paper of Solow [7]. The main assumption of the

* Presented at the 2nd Polish Symposium on Econo- and Sociophysics, Kraków, Poland, April 21–22, 2006.

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Solow model is introducing three inputs which are used to produce output. The first is capital which covers only the physical capital, *i.e.* equipment, machines, buildings. The specific feature of capital is that it is produced by using capital and labour. The second one is labour which is measured in terms of the numbers of workers and the amount of time of their work. The third input is the technology or knowledge how to put together the capital and labour in order to produce the output (new consumption or capital goods). In the Solow model the knowledge has the exogeneous character. The different concepts of knowledge give rise to many approaches to modelling the economic growth [5]. In turn this diversity enable us to include knowledge as an endogenous variable in different ways. For example, there is the research and development sector which produces knowledge [6]. Another approach is to treat the knowledge as the human capital produced in the education sector [3].

In this paper it is proposed to investigate the influence of the change of physical and human capital on the change of knowledge. We assume that the rate of knowledge growth is proportional to the weighed sum of the rates of physical and human capital growth. Both types of capital can be interpreted in different ways. We can assume that physical and human capital have some positive externalities on the technological progress. They can also be used directly in research and development, as scientific equipment, for example supercomputers, satellites or the knowledge and experience of scientists and engineers.

We also study the dynamics of the optimal growth with the technological progress influenced by the rate of growth of physical capital. We compare this model with the model of optimal growth with exogenous knowledge. We show that the economy with endogenous knowledge grows faster than the economy with exogenous knowledge.

2. Capital dependent model of growth of knowledge

We consider the economy where output Y is produced by using physical capital K, human capital H, labour L, and knowledge A as inputs

$$Y(t) = F(K(t), H(t), A(t)L(t)) \,. \tag{1}$$

This production function has constant returns to scale in $K(t)$, $H(t)$, and $A(t)L(t)$. Labour and knowledge enter multiplicatively to the production function, and $A(t)L(t)$ is also called effective labour. We assume that labour increases in the constant rate n

$$\frac{\dot{L}}{L} = n \,, \tag{2}$$

where a dot means the differentiation with respect to time.

The neoclassical model of economic growth is based on simplified assumption that knowledge grows with the constant rate. There are some propositions of relaxing this assumption. We also propose some alternative modification of exponential growth of knowledge. Our idea is to consider the more general assumption by including the capital. We assume that apart from the exogenous growth of knowledge both physical and human capital can influence on the rate of growth of knowledge. We assume that these processes are additive and proportional to rates of growth of these capitals

$$\frac{\dot{A}}{A} = g + \mu\frac{\dot{K}}{K} + \nu\frac{\dot{H}}{H} \tag{3}$$

or

$$A = A_0 e^{gt} K^{\mu} H^{\nu} \,. \tag{4}$$

For $\mu = \nu = 0$ we obtain the constant rate of growth of knowledge. The interpretation of the above assumption can be following. The physical capital is necessary in research of scientific and industrial laboratories. It is especially important in contemporary science.

Let us apply the dynamical systems methods [2] to the model of growth with capital dependent growth of rate of knowledge.

$$\dot{k} = (1-\mu)s_k k^{\alpha} h^{\beta} - \nu s_h k^{\alpha+1} h^{\beta-1} - [(1-\mu-\nu)\delta + n + g]k \,, \tag{5a}$$

$$\dot{h} = (1-\nu)s_h k^{\alpha} h^{\beta} - \mu s_k k^{\alpha-1} h^{\beta+1} - [(1-\mu-\nu)\delta + n + g]h \,. \tag{5b}$$

System (3) have at least two critical points in finite domain of phase space. In Fig. 1 we choose for presentation $\delta = 0.007$, $\mu = 0.2$, $\nu = 0.2$, $\alpha = 0.35$,

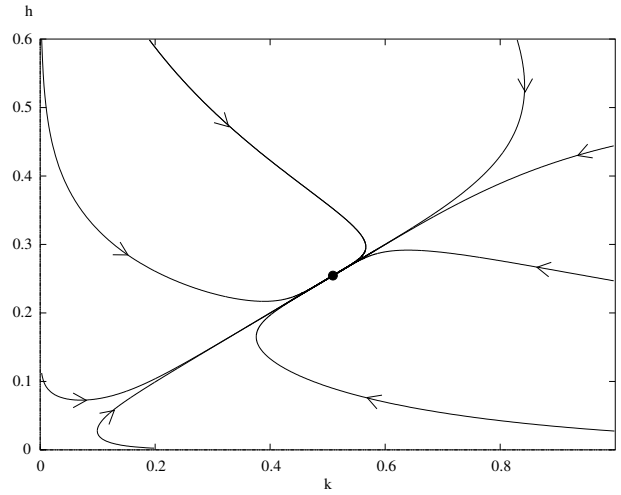

Fig. 1. The phase portrait for system (3).

$\beta = 0.4$, $n = 0.02$, $g = 0.04$. The critical point located at the origin is a saddle while the second one is a stable node. For different values of the parameters the node is located on the line $k \propto h$.

The comparison of the phase portraits for $\mu \neq 0$, $\nu \neq 0$ with $\mu = \nu = 0$ gives that while they are topologically equivalent the node for the latter case is located for higher k, h.

3. Optimisation in the model with endogenous knowledge

At first we avoid to explore the nature of knowledge and take the simplest assumption that knowledge has the exogenous character and grows in the constant rate g,

$$\frac{\dot{A}}{A} = g\,. \tag{6}$$

The capital accumulation comes from output which is not consumed. Taking into account the capital depreciation δ, capital change is given by

$$\dot{K} = F(K(t), A(t)L(t)) - C(t) - \delta K(t)\,. \tag{7}$$

It is convenient to use the variables in units of effective labour AL (denoted in small letters). In this case we obtain

$$\dot{k} = f(k(t)) - c - (g + n + \delta)k(t)\,. \tag{8}$$

In the original Solow model the savings are a fixed share of product. However, the households to choose between saving and consumption in their lifetime [4]. It means that the infinitely living households select such a level of consumption over time to maximise their utility function

$$U = \int_0^\infty e^{-\rho t} u(C(t))dt\,, \tag{9}$$

where ρ is a discount rate.

To solve the maximisation problem we use the Pontryagin Maximum Principle [1]. As the result we obtain the system of two differential equations

$$\dot{k} = k^\alpha - c - (\delta + g + n)k\,, \tag{10a}$$

$$\dot{c} = \frac{c}{\sigma}\left(\alpha k^{\alpha-1} - \delta - g - n - \rho\right)\,. \tag{10b}$$

To obtain this system we assume the Cobb–Douglas production function

$$f(k) = k^\alpha\,,$$

as well as the constant-relative-risk-aversion (CRRA) utility function

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma}\,, \tag{11}$$

which is characterised by the constant elasticity of substitution between consumption in any two moments of time.

Let us consider the dynamics of system (8). For simplification we put $b_1 = \delta + g + n$ and find three critical points:
the unstable node

$$k_1 = c_1 = 0\,, \tag{12}$$

the stable node

$$k_2 = b_1^{1/(\alpha-1)}\,, \qquad c_2 = 0\,, \tag{13}$$

and the saddle

$$k_3 = \left(\frac{b_1+\rho}{\alpha}\right)^{1/(\alpha-1)}\,, \tag{14a}$$

$$c_3 = \left(\frac{b_1+\rho}{\alpha}\right)^{\alpha/(\alpha-1)} - b_1\left(\frac{b_1+\rho}{\alpha}\right)^{1/(\alpha-1)}\,. \tag{14b}$$

Two first points have no economic concern because they represent economies without consumption. Only the third critical point, the saddle, is relevant in our discussion. Households choose such a level of consumption which is optimal for a given amount of capital. And it always lies on one of two trajectories moving to the saddle point. Once the economy reach the saddle point it enters the balanced growth path where all quantities per a unit of

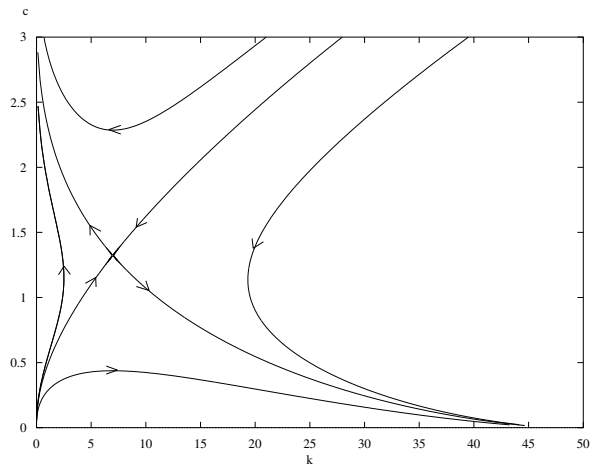

Fig. 2. The phase portrait of system (10).

effective labour are constant. However, capital, consumption, and output as well as their counterparts *per capita* (per unit of labour alone) increase in time. The phase portrait of this system is shown in Fig. 2. The bold lines denote two trajectories which lead to the saddle.

Let us return now to the endogenous technological progress. However, we consider that only the rate of growth of physical capital has influence on knowledge. Then equation (3) assumes the form

$$\frac{\dot{A}}{A} = g + \mu \frac{\dot{K}}{K} \,. \tag{15}$$

We assume that some part of technological progress has the exogenous character. There is also the additional term which describes the influence of change in capital stock on the knowledge growth. The proportionality parameter μ belongs to $[0, 1)$. For $\mu = 0$ we have the model with the exogenous knowledge analysed in the previous section. This additional component could be interpreted as the capital equipment used in research and development.

Assuming the form of the production function and the utility function as in the previous section, the optimisation procedure gives us the following two-dimensional dynamical system

$$\dot{k} = (1-\mu)k^{\alpha} - (1-\mu)c - [(1-\mu)\delta + g + n]k \,, \tag{16a}$$

$$\dot{c} = \frac{c}{\sigma}\left[\alpha(1-\mu)k^{\alpha-1} - (1-\mu)\delta - g - n - \rho\right] \,. \tag{16b}$$

When we put $\mu = 0$ we obtain system (10). We use this feature to compare the dynamics of system (16) with system (10).

For simplification we denote $b_2 = (1-\mu)\delta + g + n$. System (16) has three critical points:
the unstable node

$$k_1 = c_1 = 0 \,, \tag{17}$$

the stable node

$$k_2 = \left(\frac{b_2}{1-\mu}\right)^{1/(\alpha-1)} \,, \quad c_2 = 0 \,, \tag{18}$$

and the saddle

$$k_3 = \left(\frac{b_2+\rho}{\alpha(1-\mu)}\right)^{1/(\alpha-1)} \,, \tag{19a}$$

$$c_3 = \left(\frac{b_2+\rho}{\alpha(1-\mu)}\right)^{\alpha/(\alpha-1)} - \frac{b_2}{1-\mu}\left(\frac{b_2+\rho}{\alpha(1-\mu)}\right)^{1/(\alpha-1)} \,. \tag{19b}$$

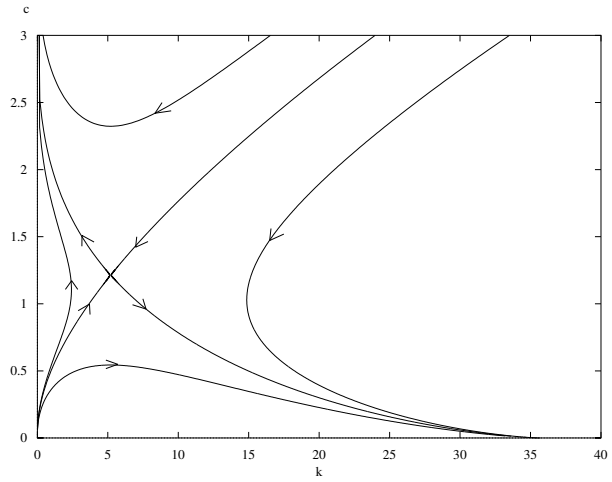

Fig. 3. The phase portrait of system (16).

The dynamics of system (16) is presented in Fig. 3. Comparing with Fig. 2 we can see that both phase portraits are topologically equivalent. The systems are structurally stable.

Two nodes represent unrealistic economies with zero level consumption. The households choose the optimal level of consumption for given capital stock. These points form two trajectories which approach the saddle solution. When the economy converges to the saddle point it reaches the balanced growth path. The capital, consumption, and output per unit of effective labour are constant. The dynamics of capital, consumption, output and capital, consumption, output per unit of labour depend on the parameters g, n, and μ.

Table I presents the rates of change of capital, consumption, output as well as their *per capita* counterparts in both considered models.

TABLE I

The rates of growth of capital K, consumption C, and output Y in the models with exogenous and endogenous growth of knowledge.

Variables	Rate of growth with exogenous knowledge	Rate of growth with endogenous knowledge
K, C, Y	$g+n$	$(g+n)/(1-\mu)$
$\frac{K}{L}$, $\frac{C}{L}$, $\frac{Y}{L}$	g	$(g+\mu n)/(1-\mu)$

We can compare the rate of growth of capital, consumption, and output in the models with endogenous and exogenous knowledge. The ratio of rates of growth for capital, consumption, and output is

$$R_X = \frac{\frac{g+n}{1-\mu}}{g+n} = \frac{1}{1-\mu}\,, \tag{20}$$

where X means K, C, and Y. The ratio of rates of growth of capital, consumption, output in these two models depends only on the parameter μ. The rate of growth of all the three variables is greater in the presence of endogenous knowledge. Fig. 4 shows how many times the rate of growth in the model with endogenous knowledge is greater than in the model with exogenous knowledge for different values of μ. For example, for $\mu = 0.2$ the rate of growth is 25% higher, and for $\mu = 0.5$ the rate of growth is 2 times higher, in the model with endogenous technological progress than in the model with exogenous technological progress.

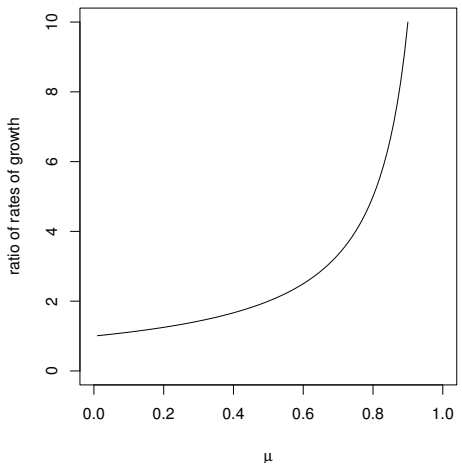

Fig. 4. The dependence of ratio of rates of growth of capital, consumption, output in the models with endogenous and exogenous knowledge on the parameter μ.

We can also compare the rates of growth of *per capita* quantities in the models with endogenous and exogenous technological progress. The ratio of rates of growth is

$$R_{X/L} = \frac{\frac{g+\mu n}{1-\mu}}{g} = \frac{g+\mu n}{g(1-\mu)}\,. \tag{21}$$

The ratio of rates of growth of capital, consumption, output per unit of labour in these two models depends both on the parameter μ and g. Fig. 5

presents the ratio of rates of growth with respect to the parameter μ. For example assuming the same values of parameters g and n we can find that for $\mu = 1/3$ the rate of growth is 2 times higher, and for $\mu = 2/3$ the rate of growth is 5 times higher, in the model with endogenous technological progress than in the model model with exogenous technological progress. When $g > n$ ($g < n$) the ratio is lower (higher) for a given μ.

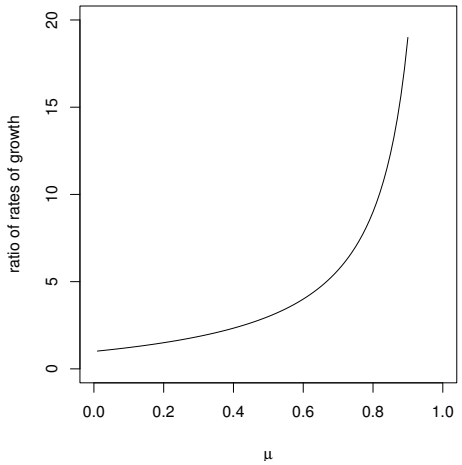

Fig. 5. The dependence of ratio of rates of growth of capital, consumption, output per unit of labour in the models with endogenous and exogenous knowledge on the parameter μ.

4. Conclusions

We investigate the dynamics of growth models in terms of dynamical system theory. We analyse some forms of knowledge and its influence on economic growth. We assume that the rate of change of knowledge depends on both the rate of change of physical and human capital. First, we study model with constant savings. The model with optimised behaviour of households is also considered. We show that the model where the rate of change of knowledge depends only on the rate of change of physical capital can be reduced to the form of the two-dimensional autonomous dynamical system. All possible evolutional paths and the stability of solutions in the phase space are discussed in details. We obtain that the rate of growth of capital, consumption and output are greater in the case of capital dependent rate of change of knowledge.

Our proposition of parameterisation of knowledge seems to be a unification of exogenous and endogenous factors. If we consider three different cases of endogenous ($g = 0$, and $\mu \neq 0$ or $\nu \neq 0$) exogenous ($g \neq 0$, and

$\mu = \nu = 0$) and mixed ($g \neq 0$, and $\mu \neq 0$ or $\nu \neq 0$), we find that the qualitative dynamics is the same for reasonable values of the rest parameters of the model. The only observable difference is different values of rates of change of the phase variables at the critical point. In other words the endogenous factors give additional contribution to the rate of change of the variables.

We presented the modification of the Ramsey model of optimal economic growth where knowledge growth depends on the rate of growth of physical capital. We compare this model with the optimal growth model with the constant rate of growth of knowledge.

We reduced the growth model with physical capital dependence of knowledge to two-dimensional dynamical system and investigated its solutions using the qualitative methods of dynamical systems. We presented the dynamics of the models on the phase portraits.

We calculated the rates of growth of capital, consumption, and output as well as their counterparts *per capita*. We compared these rates for both models and found how many times faster the model variables grows in the model with endogenous knowledge than in the model with exogenous knowledge.

It can be interpreted that physical capital growth add to the rate the knowledge growth some additional impact which makes the physical capital, consumption and output to grow faster.

The paper was supported by Marie Curie Host Fellowship MTKD-CT-2004-517186.

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