THE AVERAGE BEHAVIOUR OF FINANCIAL MARKET BY 2 SCALE HOMOGENISATION*

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The financial market is non predictable, as according to the Bachelier, the mathematical expectation of the speculator is zero. Nevertheless, we observe in the price fluctuations the two distinct scales, short and long time. Behaviour of a market in long terms, such as year intervals, is different from that in short terms. A diffusion equation with a time dependent diffusion coefficient that describes the fluctuations of the financial market, is subject to a two-scale homogenisation, and long term characteristics of the market such as mean behaviour of price and variance, are obtained. We indicate also that introduction of convolution into diffusion equation permits to obtain L-stable behaviour of finance.

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1. Introduction

The prices on stock market are formed as a result of superposition of large number of different reasons and can be assumed to be governed by probability laws. The fluctuations of prices on stock market resemble an errating walk, as it was indicated yet in 1900 by Bachelier [1], when he derived the diffusion equation from a condition that speculators should receive no information from the past prices. The difference of action prices $x = x(t) \equiv p(t+\Delta t) - p(t)$, observed at two time moments t and $t+\Delta t$, plays in this diffusion equation role of independent spatial variable. Hence, the motion of prices on the financial market is similar to the Brownian movement, discovered by the biologist Robert Brown [2] and analysed by Einstein [3,4] and Smoluchowski [5–7], *cf.* also [8]. Bachelier's observation did not find large recognition at his life, but now is a basis of greater part of models of

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prices, especially the Black–Scholes model [9], *cf.* also [10]. Later Samuelson [11] indicated that instead of a simple difference (1) it is more proper to consider differences of the respective logarithms $x = \ln p(t + \Delta t) - \ln p(t)$.

However, as it was indicated by Mandelbrot [12], despite the fundamental importance of Bachelier's random walk of the price changes (one cannot imagine an advanced textbook on finances without the Brownian motion description as its starting point), the empirical samples of successive differences of stock price changes gathered from 1890 year, are not normally distributed: they are usually too peaked to be Gaussian and do not have finite variance. The distribution of price changes is leptokurtic, since the sample kurtosis is much greater than 3, the value for a normal distribution. Mandelbrot regarded that the price changes belong to the stable family of distributions, known as L-stable or Lévy–Pareto distributions. Mandelbrot and Wallis [13] distinguished two non-Gaussian kinds of events observed in the economic world: isolated catastrophic events, the Noah effect which refers to abrupt and discontinuous changes in speculative time series and regular alternations of good and bad series, termed the Joseph effect.

Besides those effects with stochastic non-Gaussian origin, another type of departure from normal distribution is observed when the irregular random behaviour of stock price changes is superposed on another regular periodic pattern. There is a definite evidence of periodic behaviour of price changes corresponding to intervals of a day, week, quarter and year, according to the rhythm of human activity. Osborne [14] indicates, for example, that there is a reproducible burst of trading at the beginning and the end of trading day. While diurnal cycle is almost obvious, a somewhat more subtle statistical analysis (χ^2 test) reveals a week periodicity in the daily across-the-market dispersion of stock price changes. This price dispersion is a maximum at middle of week, what can be interpreted that traders tend to forget the market business over a long week-end and make up their minds at the beginning of new week.

At the beginning of the present paper, we outline some properties of diffusion equation with nonhomogeneous coefficient (dependent on time t or price changes x) and describe its solutions as the Gauss and Lévy types. We also propose to use a two scale homogenisation method to describe an average behaviour of a financial market in a long time or in averaged market in the case in which the diffusion coefficient depends on stock price change.

2. Diffusion in 1 dimension

Movement of Brownian particle is described by a distribution function $f = f(x, t)$ satisfying the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right). \tag{1}$$

Function f gives the probability density of finding Brownian particle at position x at time t, and D denotes the coefficient of diffusion. According to the Einstein fluctuational dissipative relation $D \sim T/\eta$, the coefficient D is proportional to a quotient of the absolute temperature T and viscosity η, and if η does not depend on T, it is simply proportional to T.

The form of Eq. (1) admits dependence of the coefficient D on x which may be realized *e.g.* by dependence of T on x. If D is a function of time $D = D(t)$ only, or if it is constant, the following form is obtained

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}. \tag{2}$$

Depending on interpretation, the coefficient D denotes either the thermal diffusivity (quotient of the heat conductivity and proper heat) or the diffusion coefficient. The last meaning is used below.

According to Bachelier's observation, the time independent variable t in diffusion equation is measured by successive number of stock transactions and the independent variable x, denotes stock action price change. The coefficient D varies according to a market temperature, *cf.* [15].

2.1. Fick's equations

Let $f = f(x, t)$ be the probability density of finding a Brownian (B) particle at point x and at time t, and let $j = j(x, t)$ be a stream of B particles. The continuity (or balance) equation describes conservation of the number of B particles

$$\frac{\partial f}{\partial t} + \frac{\partial j}{\partial x} = 0. \tag{3}$$

The transport relation, known as the first law of Fick reads

$$j = -D\frac{\partial f}{\partial x}, \tag{4}$$

where D denotes the diffusion coefficient, and we admit in general $D = D(x, t)$. The first Fick's law extended for the case of presence of external forces F has the form

$$j = -D\left(\frac{\partial f}{\partial x} - \frac{F}{T}f\right), \tag{5}$$

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where T is a temperature. From the mass balance and the first Fick law, the second Fick law — it is the diffusion equation (1) — can be derived.

2.2. Steady diffusion in temperature gradient

Let a diffusion in a slab $0 \leq x \leq \ell$ be stationary $j = J_0 = $ constant. In presence of an external force F, when the concentration within the diffusion volume does not change with respect to time ($j = $ constant), the Fick first law has a form

$$-D \left(\frac{\partial f}{\partial x} - \frac{F}{T} f \right) = J_0 \,.$$

In special case, when $J_0 = 0$ and the ends of the slab are kept at different temperatures $T(x = 0) = T_0, T(x = \ell) = T_\ell$, what gives a linear temperature distribution $T = Ax + B$, we obtain

$$f = C(Ax + B)^{F/A}. \tag{6}$$

Here $A = (T_\ell - T_0)/\ell$ and $B = T_0$, while the constant C normalises the distribution $\int_0^\ell f dx = 1$. We observe that even in such simple case the distribution f in slab is no longer gibbsian.

3. Time dependent coefficient of diffusion

In this case the diffusion equation has the form (2). If $f(x,0) = \delta(x)$, the solution of (2) is, *cf.* [16],

$$f(x,t) = \frac{1}{2\sqrt{\pi \int_0^t D dt}} \exp \left\{ -\frac{x^2}{4 \int_0^t D dt} \right\}. \tag{7}$$

The variance of this distribution is

$$\sigma^2 = \sigma^2(t) \equiv 2 \int_0^t D(\tau) d\tau \,. \tag{8}$$

Hence

$$f(x,t) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-x^2/2\sigma^2}. \tag{9}$$

If the diffusion coefficient D does not depend on t and is constant, we have for the dispersion (standard deviation)

$$\sigma = \sqrt{2Dt} \tag{10}$$

the classical result for the Gaussian distribution in one-dimensional process.

3.1. Periodic time dependence of the diffusion coefficient

As it was said it is observed a periodic oscillation of the across-the-market dispersion of price change for time intervals (day, week, and so on). As the dispersion is proportional to the diffusion coefficient D, it means that D is a periodic function of time.

Therefore, let $D(t)$ be a function of time with period T. For $t = nT$, with a whole number n, we have

$$\int\limits_0^t D dt = \int\limits_0^T D dt + \int\limits_T^{2T} D dt + \cdots + \int\limits_{(n-1)T}^{nT} D dt = n \int\limits_0^T D dt. \tag{11}$$

Hence, according to (8)

$$\frac{1}{2}\sigma^2 = \int\limits_0^t D dt = nT \frac{1}{T} \int\limits_0^T D dt. \tag{12}$$

Denoting the mean value of D over the period T by

$$\overline{D} = \frac{1}{T} \int\limits_0^T D dt \tag{13}$$

and introducing time $t' = nT$ counted in new units $[T]$ we obtain

$$f(x, t') = \frac{1}{2\sqrt{\pi \overline{D} t'}} e^{-x^2/4\overline{D}t'}. \tag{14}$$

We observe in more coarse time units the classical Brownian movement formula is recovered.

3.2. 2 scale time homogenisation of the Brownian motion of stock prices

To the analogous result we arrive applying more general method of asymptotic homogenisation, *cf.* [17, 18]. We introduce two time variables t and τ measured in different scales, it is in different units of time. The time t is measured by a slow clock and time τ by a fast (more accurate) clock. We have

$$\tau = \frac{t}{\varepsilon}, \tag{15}$$

where the scale parameter ε is positive ($\varepsilon > 0$) and small. For example, if $[t]$ = day (the duration of a session $\equiv$ 6 hours) and $[\tau]$= hour, then $\varepsilon = $ hour/day $\approx 1/6$.

Instead of $f(x,t)$ we write $f(x,t,\tau)$ and observe that

$$\frac{\partial f(x,t,\tau)}{\partial t} = \frac{\partial f(x,t,\tau)}{\partial t} + \frac{\partial f(x,t,\tau)}{\partial \tau}\frac{1}{\varepsilon}\,.$$

We assume an Ansatz

$$f^{\varepsilon} = f^{(0)}(x,t,\tau) + \varepsilon f^{(1)}(x,t,\tau) + \varepsilon^2 f^{(2)}(x,t,\tau) + \cdots .$$

Then the diffusion equation (2) can be written in the form

$$\left(\frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}\right)\left(f^{(0)}(x,t,\tau) + \varepsilon f^{(1)}(x,t,\tau) + \varepsilon^2 f^{(2)}(x,t,\tau) + \cdots\right)$$
$$= D(\tau)\frac{\partial^2}{\partial x^2}\left(f^{(0)}(x,t,\tau) + \varepsilon f^{(1)}(x,t,\tau) + \varepsilon^2 f^{(2)}(x,t,\tau) + \cdots\right). \quad (16)$$

We compare expressions at the same powers of ε, and find consecutively:
At ε^{-1}

$$\frac{\partial f^{(0)}(x,t,\tau)}{\partial \tau} = 0$$

what means that $f^{(0)}$ does not depend on the quick variable τ

$$f^{(0)} = f^{(0)}(x,t)\,. \quad (17)$$

At ε^0 we have

$$\frac{\partial f^{(0)}}{\partial t} + \frac{\partial f^{(1)}}{\partial \tau} = D(\tau)\frac{\partial^2}{\partial x^2}f^{(0)}(x,t)\,. \quad (18)$$

We put

$$f^{(1)} = \chi(\tau)\frac{\partial f^{(0)}}{\partial t}\,, \quad (19)$$

where $\chi(\tau)$ is a periodic function such that

$$\int_0^T \chi(\tau)d\tau = 0\,. \quad (20)$$

After substitution (19) into (18) we get

$$\frac{\partial f^{(0)}}{\partial t}(1 + \chi(\tau)) = D(\tau)\frac{\partial^2 f^{(0)}}{\partial x^2}\,. \quad (21)$$

Taking of mean with respect to variable τ over period T gives

$$\frac{\partial f^{(0)}}{\partial t} = \left(\frac{1}{T}\int_0^T D(\tau)d\tau\right)\frac{\partial^2 f^{(0)}}{\partial x^2}$$

or

$$\frac{\partial f^{(0)}}{\partial t} = \overline{D}\,\frac{\partial^2 f^{(0)}}{\partial x^2}\,, \tag{22}$$

where definition (13) of the mean diffusion coefficient was used. The solution of the last equation with the initial condition $f(x,0) = \delta(x)$ is again given by (14), if only introduce t instead of t', according to the present meaning of time t as a slow variable.

4. Coefficient of diffusion dependent on price change

Consider Fick's first law with convolution, a more general than (4),

$$j(x,t) = -\int\limits_{-\infty}^{\infty} D(x-\xi)\frac{\partial f(\xi,t)}{\partial \xi}d\xi\,. \tag{23}$$

Then instead of (1) we have the following equation of diffusion

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left(\int\limits_{-\infty}^{\infty} D(x-\xi)\frac{\partial f(\xi,t)}{\partial \xi}d\xi\right). \tag{24}$$

To both sides of the equation we apply the Fourier transformation and get

$$\frac{\partial \tilde{f}(k,t)}{\partial t} = (ik)^2 \tilde{D}(k)\tilde{f}(k,t)\,, \tag{25}$$

where

$$\tilde{f}(k,t) = \int\limits_{-\infty}^{\infty} f(x,t)e^{ikt}dx \quad \text{and} \quad \tilde{D}(k) = \int\limits_{-\infty}^{\infty} D(x)e^{ikt}dx\,. \tag{26}$$

Solution of (25) reads
$$\tilde{f}(k,t) = e^{-(ik)^2 \tilde{D}(k)t} \tag{27}$$

or

$$\tilde{f}(k,t) = e^{-\gamma k^2}\,, \tag{28}$$

where

$$\gamma \equiv \tilde{D}(k)t\,. \tag{29}$$

The function $\tilde{f}(k,t)$ in form (28) is known as the characteristic function of Gauss stochastic process, *cf.* [19].

Assume that the random of diffusion coefficient $\tilde{D}$ is such that

$$\gamma = \gamma_0 k^{-\mu},\tag{30}$$

where γ_0 depends linearly on t but does not depend on k while μ is a positive constant. If

$$\alpha \equiv 2 - \mu\tag{31}$$

satisfies inequalities

$$0 < \alpha \le 2\tag{32}$$

we deal with the L-stable process, *cf.* [19].

5. Conclusions

Above we tried to find a compromise between the classical view on finance as a gaussian process and the modern view insisting on Lévy form of stock price changes. We have shown that:

1. In the case of periodically varying standard deviation of prices, the averaging over time period restitutes gaussian character of the process.

2. Introduction of convolution in the diffusion equation may lead to the Lévy distribution.

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