SELF-CONSISTENT GAUSSIAN MODEL OF NONPERTURBATIVE QCD VACUUM*

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(Received October 13, 2006)

We show that the minimal Gaussian model of nonlocal vacuum quark and quark–gluon condensates in QCD generates the non-transversity of vector current correlators. We suggest the improved Gaussian model of the nonperturbative QCD vacuum, which respects QCD equation of motion and minimizes the revealed gauge-invariance breakdown. We obtain the refined values of pion distribution amplitude (DA) conformal moments $\langle\xi^{2N}\rangle_\pi$ $(N = 1, \ldots, 5)$ using the improved QCD vacuum model, including the inverse moment $\langle x^{-1}\rangle_\pi$, being inaccessible if one uses the standard QCD sum rules. We construct the allowed region for Gegenbauer coefficients a_2 and a_4 of the pion DA for two values of the QCD vacuum nonlocality parameter, $\lambda_q^2 = 0.4$ and $0.5\,\mathrm{GeV}^2$.

PACS numbers: 11.15.Bt, 12.38.Bx, 12.38.Cy

1. Basics of nonlocal vacuum condensates in QCD

In order to analyze meson distribution amplitudes (DAs) and form factors the generalization of the standard QCD sum rules (SRs) approach has been suggested in [1–3]. This generalization is based on the notion of the nonlocal vacuum condensates (NLC) of quark and gluon fields in the nonperturbative QCD vacuum. The effects of QCD vacuum nonlocality appears to be very important in the pion DA analysis [4–7]. In what follows we consider the main objects of this approach and its application to the estimation of 2-point correlators in QCD.

We use, as usual in QCD Sum Rules (SR) approach, the fixed-point gauge $x^\mu A_\mu(x) = 0$. For this reason all Fock–Schwinger strings $\mathcal{E}(0,x) \equiv \mathcal{P}\exp\left[\int_0^x \hat{A}_\mu(z)\,dz^\mu\right] = 1$ if the integration path is a straight line going

* Presented at the XLVI Cracow School of Theoretical Physics, Zakopane, Poland May 27–June 5, 2006.

from 0 to x. We employ for the scalar and vector condensates the same minimal model as in [6, 7]

$$\begin{aligned} \langle\bar{q}(0)q(z)\rangle &= \langle\bar{q}q\rangle\, e^{-|z^2|\lambda_q^2/8}\,, \\ \langle\bar{q}(0)\gamma_\mu q(z)\rangle &= \frac{i\,z_\mu\,z^2}{4}\,\frac{2\alpha_s\pi\langle\bar{q}q\rangle^2}{81}\,e^{-|z^2|\lambda_q^2/8}\,. \end{aligned} \tag{1}$$

The nonlocality parameter $\lambda_q^2 = \langle k^2\rangle$ characterizes the average momentum of quarks in the QCD vacuum and has been estimated in QCD SRs [8,9] and on the lattice [10,11]: $\lambda_q^2 = 0.45\pm0.1$ GeV^2. For the quark–gluon–antiquark condensates

$$\begin{aligned} \left\langle\bar{q}(0)\gamma_\mu\left(-g\widehat{A}_\nu(y)\right)q(x)\right\rangle &= (y_\mu x_\nu - g_{\mu\nu}(yx))\overline{M}_1\left(x^2,y^2,(y-x)^2\right) \\ &\quad + (y_\mu y_\nu - g_{\mu\nu}y^2)\overline{M}_2\left(x^2,y^2,(y-x)^2\right)\,, \\ \left\langle\bar{q}(0)\gamma_5\gamma_\mu\left(-g\widehat{A}_\nu(y)\right)q(x)\right\rangle &= i\varepsilon_{\mu\nu yx}\overline{M}_3\left(x^2,y^2,(y-x)^2\right)\,, \end{aligned}$$

with $A_{1,2,3} = A_0\left(-\frac{3}{2},2,\frac{3}{2}\right)$

$$\overline{M}_i\left(x^2,y^2,z^2\right) = A_i\iiint\limits_0^{\infty} d\alpha\,d\beta\,d\gamma\,f_i(\alpha,\beta,\gamma)\,e^{\left(\alpha x^2+\beta y^2+\gamma z^2\right)/4}\,.$$

The minimal model of nonlocal QCD vacuum suggests the following Ansatz

$$f_i\left(\alpha,\beta,\gamma\right) = \delta\left(\alpha-\Lambda\right)\,\delta\left(\beta-\Lambda\right)\,\delta\left(\gamma-\Lambda\right)\,,$$

with $\Lambda = \lambda_q^2/2$. However, this model does not obey the QCD equations of motion and gauge invariance of 2-point correlator of vector currents. In order to fulfill QCD equations of motion exactly and minimize non-transversity of V–V correlator we suggest the improved model of QCD vacuum with

$$f_i^{\text{imp}}\left(\alpha,\beta,\gamma\right) = (1+X_i\partial_x+Y_i\partial_y+Y_i\partial_z)\,\delta\left(\alpha-x\Lambda\right)\delta\left(\beta-y\Lambda\right)\delta\left(\gamma-z\Lambda\right)\,, \tag{2}$$

where $z=y$ and $\Lambda = \lambda_q^2/2$. From QCD equations of motion one can obtain the followings conditions [12] for this model

$$x+y=1\,,\qquad 12\,\left(X_2+Y_2\right)-9\,\left(X_1+Y_1\right)=1\,. \tag{3}$$

Vacuum condensates of 4-quarks operators are usually transformed to the product of two scalar quark condensates by means of the Hypothesis of Vacuum Dominance (HVD)[1]

$$\langle\bar{q}(0)Aq(y)\bar{q}(z)Bq(x)\rangle \cong \left(\frac{-\mathrm{Tr}\,AB}{16\,N_c^2}\right)\langle\bar{q}(0)q(x)\rangle\,\langle\bar{q}(0)q(z-y)\rangle\,.$$

[1] For shortness we consider operators A and B, which include also color matrixes t^a and t^b.

2. Operator product expansion of vector current correlator

Consider now the correlator

$$\Pi^N_{\mu\nu} = i \int d^4x \, e^{iqx} \left\langle 0 \right| T \left[J^N_\mu(0) J^+_\nu(x) \right] \left| 0 \right\rangle ,$$

of two vector currents corresponding to charged ρ meson

$$J^+_\nu(x) = \bar{u}(x)\gamma_\nu d(x) , \qquad J_\mu(0) = \bar{d}(0)\gamma_\mu u(0) .$$

In the first current we have the composite operator $(-in\nabla_0)^N$. Its action on the quark field is defined as

$$(-in\nabla_0)^N u(0) \equiv (-in\nabla_y)^N u(y)\Big|_{y=0} ,$$

where n is an arbitrary light-like vector, $n^2 = 0$, such that $nq \neq 0$. For shortness, we will write below $\Pi^N_{\mu\nu}$ in place of $\Pi^N_{\mu\nu}(q)$. All $O(\alpha_s \langle\bar{\psi}\psi\rangle^2)$ terms in $\Pi^N_{\mu\nu}$ are generated by bilocal vector, quark–gluon–antiquark (see Fig. 1), and 4-quarks condensates (see Fig. 2):

$$\Pi^N_{\mu\nu} = \Delta_{2\mathrm{V}}\Pi^N_{\mu\nu} + \Delta_{\bar{q}Aq}\Pi^N_{\mu\nu} + \Delta_{4\mathrm{Q}_1}\Pi^N_{\mu\nu} + \Delta_{4\mathrm{Q}_2}\Pi^N_{\mu\nu} + (\mathrm{M.C.}) . \qquad (4)$$

M.C. means terms due to mirror-conjugated diagrams: for example, in Fig. 1 they correspond to diagrams, in which NonLocal Condensate (NLC) are inserted in the bottom line instead of the top one.

We are interested in the quantities corresponding to the non-transversal structure of $\Pi^N_{\mu\nu}$:

$$\Delta_k \Pi^N_{\mathrm{L}}(M^2) \equiv \frac{M^4}{2A_0} \, \widehat{B}_{-q^2 \to M^2} \, \frac{\Delta_k \Pi^N_{\mu\nu} \, n^\mu q^\nu}{nq}$$

with k = 2V, $\bar{q}Aq$, 4Q$_1$ and 4Q$_2$. Here, in close analogy with QCD SR approach, we work with Borelized quantities, which are obtained after Borel

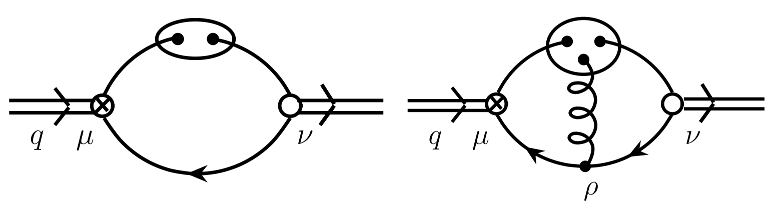

Fig. 1. Vector quark–quark ($\Delta_{2\mathrm{V}}\Pi^N_{\mu\nu}$, left) and quark–gluon–antiquark ($\Delta_{\bar{q}Aq}\Pi^N_{\mu\nu}$, right) condensates contributions to the correlator $\Pi^N_{\mu\nu}$.

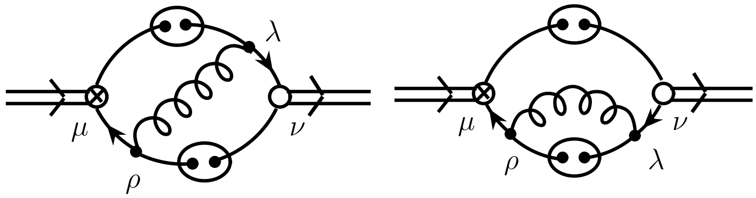

Fig. 2. Four-quark condensates contributions to the correlator $\Pi^N_{\mu\nu}$: $\Delta_{4\mathrm{Q}_1}\Pi^N_{\mu\nu}$ (left) and $\Delta_{4\mathrm{Q}_2}\Pi^N_{\mu\nu}$ (right).

transformation $\widehat{B}_{-q^2\to M^2}$. Having in mind further applications for meson DAs, we calculate the corresponding contributions ($k = 2\mathrm{V}$, $\bar{q}Aq$, $4\mathrm{Q}_1$, $4\mathrm{Q}_2$):

$$\Delta_k \Pi^N_{\mathrm{T}}(M^2) \equiv \frac{M^6}{2A_0}\,\widehat{B}_{-q^2\to M^2}\frac{\Delta_k \Pi^N_{\mu\nu}\,n^\mu n^\nu}{nq^2}\,.$$

The functions $\Delta_k \Pi^N_{\mathrm{L}}(M^2)$ and $\Delta_k \Pi^N_{\mathrm{T}}(M^2)$ are given in explicit form in [12].

3. Analysis of Gaussian models

Vector current conservation in QCD claims for the transversity (with respect to the index ν) of the sum of contributions all quark condensates:

$$\Delta\Pi^N_{\mathrm{L}} \equiv \Delta_{2V}\Pi^N_{\mathrm{L}} + \Delta_{\bar{q}Aq}\Pi^N_{\mathrm{L}} + \Delta_{4\mathrm{Q}_1}\Pi^N_{\mathrm{L}} + \Delta_{4\mathrm{Q}_2}\Pi^N_{\mathrm{L}} + (\mathrm{M.C.}) = 0\,.$$

Note here that since we study Gaussian model based on delta-Ansatz (1), (2), the sum $\Delta\Pi^N_{\mathrm{L}}$ cannot be equal zero exactly. The reason is very simple — we insert the Gaussian behavior by hands. So, the only thing we can hope to realize, is to minimize $|\Delta\Pi^N_{\mathrm{L}}|$ by the special choice of the Ansatz's parameters. More precisely, we are interested in minimization of conformal moments $\Delta\langle\xi^{2N}\rangle_{\mathrm{L}}$

$$\Delta\langle\xi^{2N}\rangle \equiv \int\limits_0^1 (2x-1)^{2N}\Delta\Pi(x)\,dx = \sum_{k=0}^{2N}(-2)^{2N-k}\binom{2N}{k}\Delta\Pi^{2N-k}\,,$$

which are used in the QCD SR analysis of meson DAs.

Consider now the set of available parameters $\{X_i\}$ in our improved model. We apply delta Ansatz (1) with one parameter X_v, relating nonlocalities in vector and scalar quark condensates: $\lambda^2_{\mathrm{V}} = X_v\,\lambda^2_q$. For the quark–gluon–quark condensate we use Ansatz (2) with $\Lambda = X_v\lambda^2_q/2$. There are parameters, which are not independent due to Eq. (3), derived from the QCD equation of motion. After taking into account all mentioned relations we have the following 7 parameters: $x, X_1, X_2, X_3, Y_1, Y_3$ and X_v. In order to find these values of our parameters $\{X_i\}$ we introduce the optimization function, which is considered in [12]. Minimization of the optimization function gives us the following set of parameters

$$\begin{aligned} X_1 &= +0.082\,, & Y_1 = Z_1 = -2.243\,, && x = 0.788\,, && X_v = 1.00\,,\\ X_2 &= -1.298\,, & Y_2 = Z_2 = -0.239\,, && y = 1-x = 0.212\,, &&\\ X_3 &= +1.775\,, & Y_3 = Z_3 = -3.166\,, && z = 1-x = 0.212\,. && \qquad (5) \end{aligned}$$

We will consider this set as the basic parameter set of the improved Gaussian model.

To illustrate the quality of the improved Ansatz we show in Fig. 3 plots of the functions $\Delta\langle\xi^{2N}\rangle_{\rm L}(\Delta)$ with $N = 0, 2, 5$ (solid lines) in comparison with corresponding quantities for the minimal Ansatz (dashed lines). As is clearly seen from this comparisons, the improved Ansatz (5) strongly suppresses the absolute values of non-transverse conformal moments $\Delta\langle\xi^{2N}\rangle_{\rm L}$, *i.e.* takes the vector correlator transversity into account much better.

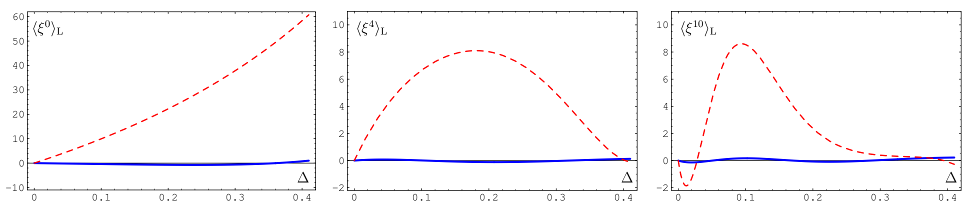

Fig. 3. We show functions $\Delta\langle\xi^{2N}\rangle_{\rm L}(\Delta)$ with $N = 0, 4, 10$ for the improved NLC model (5) (solid line) in comparison with ones, corresponding to the minimal NLC model (dashed line).

4. Pion distribution amplitude

The obtained QCD vacuum model allows us to calculate moments of the pion DA $\varphi_\pi(x,\mu^2)$ [13] more accurately.

$$\langle 0 \mid \bar{d}(z)\gamma^\mu\gamma_5 u(0) \mid \pi(P)\rangle\Big|_{z^2=0} = i f_\pi P^\mu \int\limits_0^1 dx\, e^{ix(zP)}\ \varphi_\pi(x,\mu^2)\,. \qquad (6)$$

The results of the analysis of $\langle\xi^{2N}\rangle_\pi$ in the NLC QCD sum rules are given in the Table I. One can see from this table that the values of the pion DA moments in the new Gaussian model of QCD vacuum are systematically different from those, corresponding to the minimal model. Allowed region for the Gegenbauer coefficients a_2 and a_4 are shown in Fig. 4. These coefficients

TABLE I

Pion DA moments $\langle \xi^N \rangle_\pi(\mu_0^2)$, determined at $\mu_0^2 = 1.35$ GeV^2.

Model	f_π (GeV)	$N=2$	$N=4$	$N=6$	$N=8$	$N=10$
Minimal [7]	0.137(8)	0.266(20)	0.115(11)	0.060(7)	0.036(5)	0.025(4)
Ansatz (5)	0.140(13)	0.290(29)	0.128(13)	0.067(7)	0.040(5)	0.025(4)

define the pion DA in a form of the expansion in Gegenbauer polynomials $C_{2n}^{3/2}(2x-1)$, being the eigenfunctions of the 1-loop ER-BL [14,15] evolution kernel:

$$\varphi_\pi\left(x;\mu^2 = 1.35 \text{ GeV}^2\right) = 6\,x\bar{x}\left[1 + a_2\,C_2^{3/2}(2x-1) + a_4\,C_4^{3/2}(2x-1)\right]. \quad (7)$$

In order to test the self-consistency of our procedure of DA restoration on the basis of information about its first five conformal moments, we use the same technique as in [6, 7]. Namely, we construct the special SR for the inverse moment $\langle x^{-1}\rangle_\pi$ and the result of its processing $\langle x^{-1}\rangle_\pi^{\text{SR}}$ is compared with the inverse moment obtained from representation (7):

$$\langle x^{-1}\rangle_\pi^{\text{DA}} \;=\; 3\;(1 + a_2 + a_4)\;.$$

For the value $\lambda_q^2 = 0.4$ GeV^2 we get following results:

$$\langle x^{-1}\rangle_\pi^{\text{DA}} = 3.25 \pm 0.20\,, \quad \langle x^{-1}\rangle_\pi^{\text{SR}} = 3.40 \pm 0.34\,,$$

and for the value $\lambda_q^2 = 0.5\,$GeV^2 — these:

$$\langle x^{-1}\rangle_\pi^{\text{DA}} = 3.08 \pm 0.15\,, \quad \langle x^{-1}\rangle_\pi^{\text{SR}} = 3.27 \pm 0.35\,.$$

The obtained inverse moments in both cases are in good mutual agreement. This confirms the self-consistency of the pion DA recovery procedure.

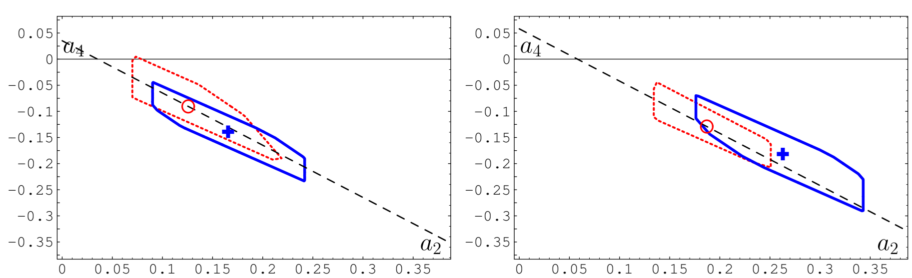

Fig. 4. Allowed values of the pion DA parameters a_2 and a_4 are bounded by the solid blue line. Region bounded by the dotted red line represents results obtained in the minimal model [7]. Left panel show the results for the value $\lambda_q^2 = 0.5\,$GeV^2, right panel — for the value $\lambda_q^2 = 0.4$ GeV^2. All values are normalized at $\mu^2 = 1.35$ GeV^2.

5. Conclusion

Here we considered the Gaussian model of the nonlocal vacuum quark and quark–gluon condensates in QCD. We analyzed the Lorenz structure of the correlator $\Pi_{\mu\nu}(q)$ of two vector quark currents and showed that in the minimal Gaussian model of the nonperturbative QCD vacuum [1, 6, 7], this correlator is non-transversal and nonlocal condensates do not satisfy QCD equations of motion. To ameliorate the situation we suggested the improved Gaussian model for nonlocal vacuum quark and quark–gluon condensates in QCD, Eqs. (1) and (2). This model satisfies QCD equations of motion for quark fields and the revealed breakdown of gauge invariance is minimized by the special choice of parameters, see Eqs. (5).

Using this improved model of the nonlocal QCD vacuum we analyzed QCD SRs for the pion DA. We revealed that in the new QCD vacuum model the NLC SRs produce again a 2-parameter "*bunch*" of admissible DAs. The allowed values of this bunch parameters a_2 and a_4 are shown in Fig. 4. These models are in a good agreement with the results of independent SR for the pion DA inverse moment, $\langle x^{-1}\rangle_{\pi}^{\text{SR}}$.

We emphasize here that obtained earlier in the minimal Gaussian model of QCD vacuum the BMS model [7], shown in Fig. 4 by symbol ○, is inside the allowed region dictated by the improved QCD vacuum model. This testifies to the heredity of both Gaussian models, the minimal one and the improved one. Moreover, that also means that all the characteristic features of the BMS bunch are valid also for the improved bunch: one can see in Fig. 5, that

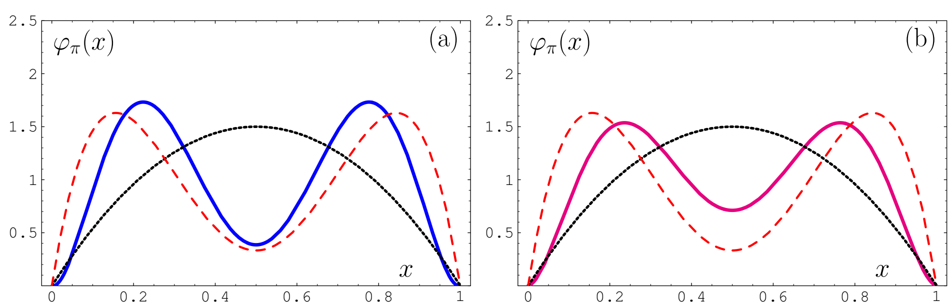

Fig. 5. Profiles of the pion DAs corresponding to the central points of the "bunches" for the value of the nonlocality parameter $\lambda_q^2 = 0.4\,\text{GeV}^2$. (a): The blue solid line represents the result obtained in the improved Gaussian model (symbol ✙ on the right part of Fig. 4). (b): The solid line represents the result obtained in the minimal Gaussian model (BMS model [7], symbol ○ on the right part of Fig. 4). For comparison we show here also the asymptotic DA (dotted line) and Chernyak–Zhitnitsky (CZ) DA [16] (dashed line).

in comparison with the CZ model [16] (dashed line, $a_2 = 0.52$ and $a_4 = 0$ at $\mu^2 = 1.35$ GeV^2) the NLC-dictated models are much more end-point suppressed, although are double-humped.

This results in completely different values of the inverse moment: $\langle x^{-1}\rangle_\pi^{\rm CZ} = 4.56$, whereas in our case $\langle x^{-1}\rangle_\pi = 3.24 \pm 0.20$.

We would like to thank A. Dorokhov, S. Mikhailov, N. Stefanis and O. Teryaev for stimulating discussions and useful remarks. This work was supported in part by the Bogoliubov–Infeld and the Heisenberg–Landau Programs (grants 2006), and the Russian Foundation for Fundamental Research (grant No. 06-02-16215).

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