SOLVING SOME GAUGE SYSTEMS AT INFINITE N*

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After summarizing briefly some numerical results for four-dimensional supersymmetric SU(2) Yang–Mills quantum mechanics, we review a recent study of systems with an infinite number of colours. We study in detail a particular supersymmetric matrix model which exhibits a phase transition, strong-weak duality, and a rich structure of supersymmetric vacua. In the planar and strong coupling limits, this field theoretical system is equivalent to a one-dimensional XXZ Heisenberg chain and, at the same time, to a gas of q-bosons. This not only reveals a hidden supersymmetry in these well-studied models; it also maps the intricate pattern of our supersymmetic vacua into that of the now-popular ground states of the XXZ chain.

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1. Introduction

This lecture reviews a recent progress in studying the large N limit of simple supersymmetric quantum mechanical systems which result from the dimensional reduction of field theories with gauge symmetry. Models of this type have been studied for a long time [1–3] and have many different applications, depending on the space-time dimension, D, of the unreduced theory [4,5] (*cf.* Table I). For $D = 2$ they can be often solved analytically [1,4,6,7], providing quantitative realization of supersymmetry. In four space-time dimensions one of them is nothing but the small volume limit of the Yang–Mills gluodynamics revealing the spectrum of zero volume glueballas as a special case [8–11]. Finally, for $D = 10$ they make contact with the M-theory via the BFSS hypothesis [12].

Although we shall be mainly concerned with the large N limit, we would like to give one example of the $N = 2$ model.

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TABLE I

A "road map" of a parameter space of supersymmetric Yang–Mills quantum mechanics. Black dots denote models well understood/solved, while open circles mark ones under study. A checkmark labels parameters of a system discussed in this lecture.

D	2	4	...	10
N				
2	●	●		○
3	●			
.	○			
.	○			
.				
∞	✓			M

1.1. Four-dimensional supersymmetric Yang–Mills quantum mechanics at finite N

The system has three bosons and two fermions, both in the adjoint representation of SU(2). The Hamiltonian reads [1]

$$H = \frac{1}{2}p_a^i p_a^i + \frac{g^2}{4}\epsilon_{abc}\epsilon_{ade}x_b^i x_c^j x_d^i x_e^j + \frac{ig}{2}\epsilon_{abc}\psi_a^\dagger \Gamma^k \psi_b x_c^k \,,$$
$$i = 1,2,3 \quad a = 1,2,3 \,.$$

The spectrum is obtained numerically by diagonalizing H in the gauge invariant eigen-basis of the occupation numbers of all degrees of freedom $n_a^i = a^{\dagger i}_a a_a^i, \eta_a^i = f^{\dagger i}_a f_a^i$ [3]

$$|\{n_a^i, \eta_c^j\}\rangle = \sum_{\text{contractions}} a^{\dagger i}_c a^{\dagger j}_d a^{\dagger k}_e f^{\dagger m}_b f^{\dagger n}_a \ldots |0\rangle \,.$$

The Hilbert space was cut by restricting the gauge invariant total number of bosonic quanta

$$B = \sum_{b,i} a^{\dagger i}_b a_b^i < B_{\max} \,.$$

It turned out that the physical (*i.e.* convergent with the cutoff eigenenergies) could be computed for sizes of bases well within a reach of a reasonably fast PC. The spectrum obtained in this way is shown in Fig. 1 [5]. It reveals
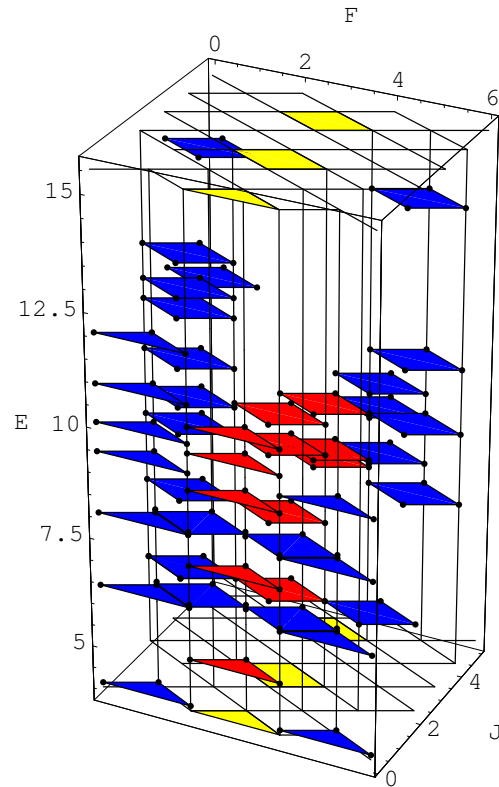
Fig. 1. The spectrum, and its supersymmetry structure, of the $D = 4$ supersymmetric Yang–Mills quantum mechanics.

a series of dynamical supermultiplets with degenerate eigenenergies. The eigenstates within a supermultiplet are indeed the supersymmetric images of one another. These states are localized and they form a discrete spectrum. In addition there is also a continuum. It occupies central sectors of the figure, *i.e.* ones with the conserved fermion number $F = 2, 3, 4$. Supermultiplets of non-localized states occur at every second value of the angular momentum. Moreover, in these sectors the discrete and continuous spectra coexists at the same energies. This unusual feature of gauge interactions has never been observed so directly before. There are two supersymmetric vacua with fermion numbers $F = 2$ and 4. This is nothing but the zero-volume manifestation of the existence of the $\lambda\lambda$ condensate in the space extended theories [13]. The condensate assumes two different (in fact opposite for SU(2)) values, again in agreement with the non-trivial predictions of the unreduced theory [14, 15] .

2. The large N limit and the planar calculus

Above approach, even though quite successful, is naturally limited to the low number of colours. For higher N the Hilbert spaces become too large. Fortunately *at infinite N* Fock bases simplify enormously allowing again to reach quantitative results [16–20]. This simplification is usually phrased in terms of the Feynman diagrams and topological expansion [21]. However it can be also formulated in the way suitable for the Hamiltonian formalism [16, 17]. To this end introduce matrix creation and annihilation (c/a) operators.

$$a_{ik} = \sqrt{2}a^a T_{ik}^a, \quad f_{ik} = \sqrt{2}f^a T_{ik}^a, \quad i, k = 1, \ldots, N.$$

The physical basis can be conveniently generated by acting with the gauge invariant building blocks (bricks)

$$(a^\dagger a^\dagger), (a^\dagger a^\dagger a^\dagger), (a^\dagger a^\dagger a^\dagger a^\dagger), \ldots, (a^{\dagger^N}), \quad (.) \equiv \mathrm{Tr}\,[.]$$

and their products, on the empty Fock state $|0\rangle$. Similarly for the fermionic sectors one employs bricks with mixed fermionic and bosonic operators. It turns out that in the large N limit only the single trace operators are relevant. All products of traces are either non-leading or do not provide new information (see [22] for recent results on that point). This observation culminates in the set of simple rules to calculate explicitly matrix elements of various Hamiltonians. One basically applies the Wick theorem employing the commutation rules

$$[a_{ik}, a^\dagger{}_{jl}] = \delta_{il}\delta_{kj}$$

and identifying the leading contributions. Details of such "planar calculus" have been presented in [16, 17, 19]. Here we summarize only two examples.

A normalized state with n gluons in the $F = 0$ sector reads

$$|n\rangle = \frac{1}{\mathcal{N}_n}\mathrm{Tr}\,[(a^\dagger)^n]|0\rangle. \tag{1}$$

The normalization factor

$$\begin{aligned}
\mathcal{N}_n^2 &= \langle 0|\mathrm{Tr}\,[a^n]\mathrm{Tr}\,[(a^\dagger)^n]|0\rangle \\
&= \langle 0|(12)(23)\ldots(n1)[1'2'][2'3']\ldots[n'1']|0\rangle, \\
(12) &\equiv a_{i_1 i_2}, \quad [12] \equiv a^\dagger{}_{i_1 i_2}.
\end{aligned}$$

receives the maximal contribution only when the adjacent creation and annihilation operators are contracted. This gives

$$\mathcal{N}_n^2 = nN^n.$$

Similarly a matrix element of a typical term in a generic Hamiltonian

$$H_{n+2,n} = g^2 \langle n+2| \mathrm{Tr}\, \left[a^\dagger a^\dagger a^\dagger a \right] |n\rangle$$

can be explicitly calculated

$$H_{n+2,n} = g^2 N \sqrt{n(n+2)}\,,$$

and shown to depend only on the 't Hooft coupling $\lambda = g^2 N$.

   The rest of this lecture will be devoted to a rather simple system which turns out to be surprisingly rich. The model may be considered as a distant cousin of the $D = 1 + 1$ supersymmetric Yang–Mills theory reduced to the one point in space.

3. A simple supersymmetric system

   Consider one fermion and one boson with the following supersymmetry generators and the Hamiltonian [16].

$$
\begin{aligned}
Q &= \sqrt{2}\mathrm{Tr}\, \left[f a^\dagger (1 + g a^\dagger) \right]\,, \\
Q^\dagger &= \sqrt{2}\mathrm{Tr}\, \left[f^\dagger (1 + g a) a \right]\,, \\
H &= \{Q, Q^\dagger\} = H_B + H_F\,. \tag{2}
\end{aligned}
$$

Explicitly

$$
\begin{aligned}
H_B &= a^\dagger a + g \left(a^{\dagger 2} a + a^\dagger a^2 \right) + g^2 a^{\dagger 2} a^2, \\
H_F &= f^\dagger f + g \left(f^\dagger f (a^\dagger + a) + f^\dagger (a^\dagger + a) f \right) \\
&+ g^2 \left(f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a \right).
\end{aligned}
$$

This Hamiltonian conserves the gauge invariant fermion number $F = \mathrm{Tr}\,[f^\dagger f]$ and can be diagonalized in sectors with well defined F. For any finite N calculation of matrix elements of H quickly becomes cumbersome. At infinite N, however, the planar rules illustrated above give simple and compact expressions.

3.1. Hamiltonian matrix at large N

   The gauge invariant basis in the $F = 0$ sector can be chosen as (1) with $n = 0, 1, 2, 3, \ldots$. Only H_B contributes in this case. The matrix elements in the planar approximation read

$$
\begin{aligned}
\langle 0, n | H | 0, n \rangle &= (1 + \lambda(1 - \delta_{n1}))n\,, \\
\langle 0, n+1 | H | 0, n \rangle &= \langle 0, n | H | 0, n+1 \rangle = \sqrt{\lambda}\sqrt{n(n+1)}\,. \tag{3}
\end{aligned}
$$

In the sector with one fermion the basis is

$$|n\rangle = \frac{1}{\mathcal{N}_n}\mathrm{Tr}\,[(f^\dagger a^\dagger)^n]|0\rangle, \quad n = 0, 1, 2, \dots .$$

Now, and for higher F, both H_B and H_F contribute. The Hamiltonian matrix is again simple

$$
\begin{aligned}
\langle 1, n|H|1, n\rangle &= (1 + \lambda)(n + 1) + \lambda, \\
\langle 1, n + 1|H|1, n\rangle &= \langle 1, n|H_2|1, n + 1\rangle = \sqrt{\lambda}(2 + n).
\end{aligned}
\tag{4}
$$

The system reveals many interesting features already in these two lower sectors [16], therefore we postpone the discussion of higher F's and turn to the physics of ensembles with at most one fermion.

3.2. Numerical results

The Hamiltonian matrix is sparse but infinite with rows and columns labeled by the gauge invariant number of bosons $B = \mathrm{Tr}\,[a^\dagger a]$. To obtain, the spectrum numerically we introduce the cutoff, $B_{\max}$, limiting the number of bosonic quanta

$$B < B_{\max},$$

and increase the cutoff until the spectrum converges. Results are shown in Fig. 2 for few values of the 't Hooft coupling. The convergence with $B_{\max}$ is satisfactory for $\lambda \neq 1$ and is faster for lower eigenvalues. There, the limiting, "infinite volume", results can be easily recovered. The less trivial way so see this is to check for the supersymmetry which is broken by the cutoff and should be recovered only at infinite $B_{\max}$. Fig. 3 shows first few energy levels in four lower fermionic sectors. The degeneracies between bosonic and fermionic partners of the $F = (0, 1)$ supermultiplets are excellent. Moreover, there is also an unbalanced supersymmetric vacuum state with zero eigenenergy[1]. These results provide also the non-trivial test of our planar rules: the planar approximation does not break supersymmetry.

The slowing of the cutoff dependence around $\lambda = 1$ is a characteristic signature of a phase transition. At this point the system looses the energy gap and the spectrum becomes continuous. Any finite number of low-lying levels collapses to zero at infinite cutoff, but the cutoff dependence becomes characteristic of that for the continuous spectrum.

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[1] An apparent lack of the degeneracy between some states with higher F will be explained later.
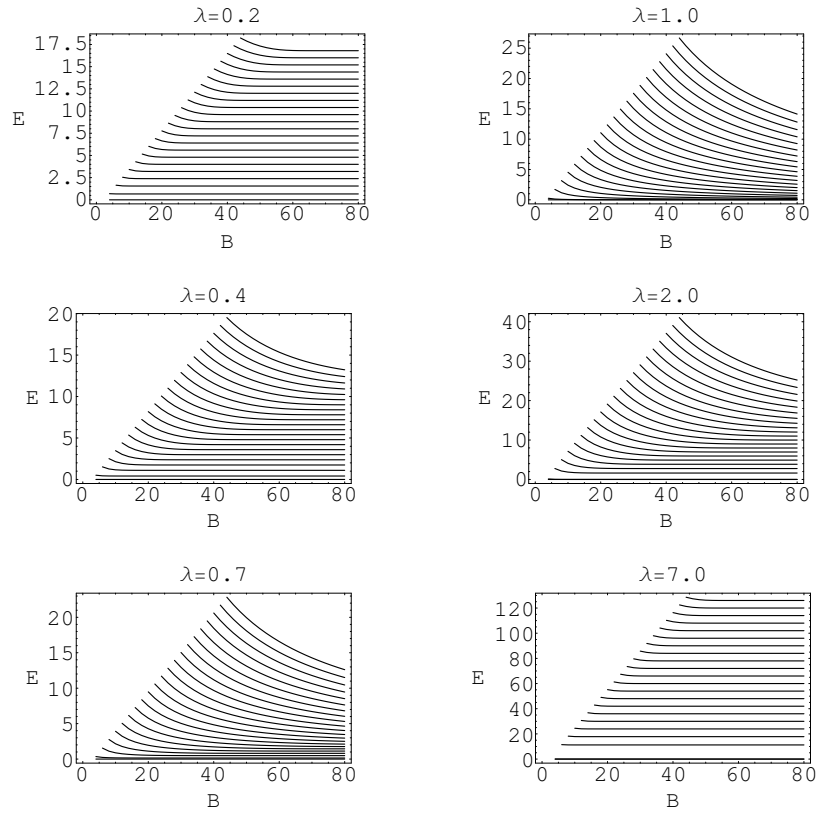
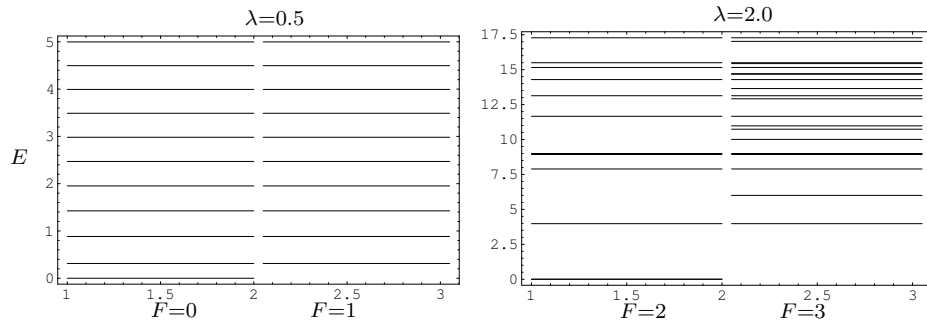Fig. 2. The cutoff dependence of the spectra of H, in the $F = 0$ sector, and for a range of λ's.

Fig. 3. Low lying bosonic and fermionic levels in the first four fermionic sectors.

Another interesting feature of this phase transition is the rearrangement of members of supermultiplets. This is shown in Fig. 4 where the λ dependence of the first four eigenenergies from both sectors is displayed for few values of $B_{\max}$. Away from the critical region supersymmetry is quickly
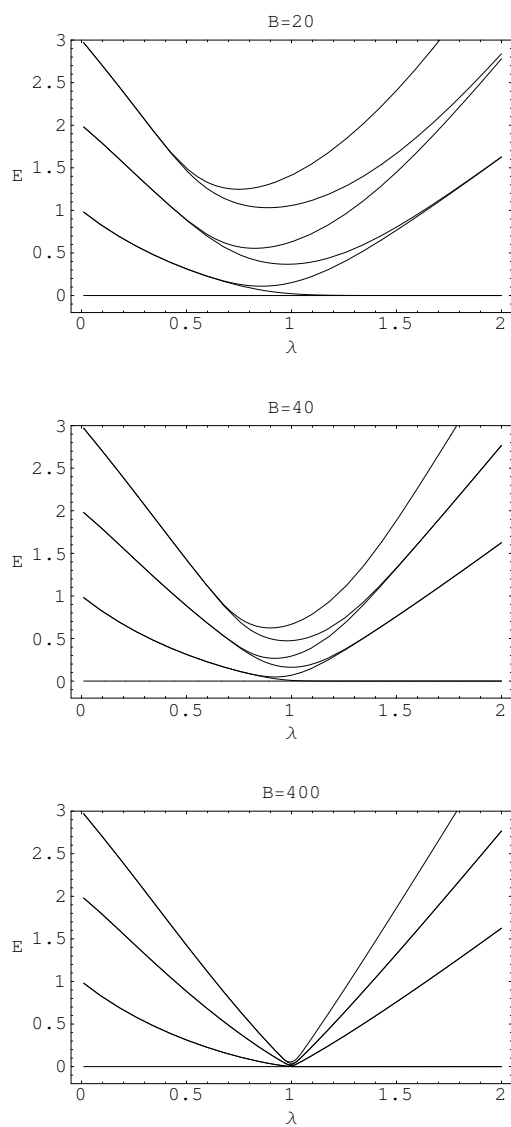

Fig. 4. Lowest bosonic and fermionic levels, as functions of λ, for different cutoffs.

restored and the $F = 0$ and $F = 1$ levels are un-distinguishable even for low cutoffs. Close to the criticality partners do not have the same energies since SUSY is broken at finite cutoff. Interestingly however, they *rearrange* while the 't Hooft coupling passes its critical value. In particular, the *new* vacuum state appears when λ moves from the low to the large coupling phases. This also implies (and was readily found) that the Witten index (restricted to the $F = 0, 1$ sectors) jumps by one unit across the phase transition point.

Finally, the system has the exact strong-weak duality

$$b\left(E_n^{(F=0)}\left(\frac{1}{b}\right) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_{n+1}^{(F=0)}(b) - b^2\right),$$

$$b\left(E_n^{(F=1)}\left(\frac{1}{b}\right) - \frac{1}{b^2}\right) = \frac{1}{b}\left(E_n^{(F=1)}(b) - b^2\right).$$

It follows directly from the matrix representation (4) for $F = 1$. However it is not obvious in the $F = 0$ sector, *cf.* Eq. (3), but it is a direct consequence of supersymmetry.

3.3. Analytic solution

All above results have been subsequently derived analytically [16]. For example the second vacuum state has the form $b \equiv \sqrt{\lambda}$

$$|0\rangle_2 = \sum_{n=1}^{\infty} \left(\frac{-1}{b}\right)^n \frac{1}{\sqrt{n}} |0, n\rangle. \tag{5}$$

It is indeed annihilated by $Q^\dagger$, and exists only for $b > 1$.

Surprisingly, one can find analytically the complete spectrum and construct all eigenstates in the $F = 0$ sector. To this end it is convenient to introduce another, not orthonormal basis,

$$|B_n\rangle = \sqrt{n}|n\rangle + b\sqrt{n+1}|n+1\rangle.$$

This basis is convenient since the action of the Hamiltonian on $|B_n\rangle$ is so simple that the generating function for the expansion of the eigenstates in that basis

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \leftrightarrow \quad |\psi\rangle = \sum_{n=0}^{\infty} c_n |B_n\rangle$$

can be constructed. The solution reads

$$f(x) = \frac{1}{\alpha} \frac{1}{x + 1/b} F\left(1, \alpha; 1 + \alpha; \frac{x+b}{x+1/b}\right), \quad b < 1,$$

$$f(x) = \frac{1}{1-\alpha} \frac{1}{x+b} F\left(1, 1-\alpha; 2-\alpha; \frac{x+1/b}{x+b}\right), \quad b > 1,$$
$$E = \alpha(b^2 - 1),$$

and together with the quantization condition

$$f(0) = 0, \tag{6}$$

reproduces numerical eigenvalues obtained earlier in the infinite cutoff limit.

As a one check, set $\alpha = 0$ in the generating function, for $b > 1$, to obtain

$$f_0(x) = \frac{1}{1+bx} \log \frac{b+x}{b-1/b}, \quad b > 1, \tag{7}$$

which indeed reproduces the second ground state (5).

4. Higher fermionic sectors: $F = 2, 3$

Single trace states with two fermions

$$|n_1, n_2\rangle = \frac{1}{\mathcal{N}_{n_1 n_2}} \text{Tr}\left[a^{\dagger n_1} f^\dagger a^{\dagger n_2} f^\dagger\right] |0\rangle,$$

are labeled by two integers whose ordering is important modulo a cyclic permutation. Pauli principle eliminates states with $n_1 = n_2$, hence we can always take $n_2 < n_1$. The planar rules give for the Hamiltonian matrix in this sector [19]

$$\langle n_1, n_2 | H | n_1, n_2\rangle = (n_1 + n_2 + 2)(1 + b^2)$$
$$-b^2(2 - \delta_{n_1,0}) - 2b^2 \delta_{n_2, n_1+1},$$
$$\langle n_1 + 1, n_2 | H | n_1, n_2\rangle = b(n_1 + 2) = \langle n_1, n_2 | H | n_1 + 1, n_2\rangle,$$
$$\langle n_1, n_2 + 1 | H | n_1, n_2\rangle = b(n_2 + 2) = \langle n_1, n_2 | H | n_1, n_2 + 1\rangle.$$
$$\langle n_1 + 1, n_2 - 1 | H | n_1, n_2\rangle = \langle n_1, n_2 | H | n_1 + 1, n_2 - 1\rangle$$
$$= 2b^2(1 - \delta_{n_2, n_1+1}).$$

Planar states in the three fermion sector are labeled by three integers modulo cyclic translations

$$|n_1, n_2, n_3\rangle = \frac{1}{\mathcal{N}_{n_1 n_2 n_3}} \text{Tr}[a^{\dagger n_1} f^\dagger a^{\dagger n_2} f^\dagger a^{\dagger n_3} f^\dagger]|0\rangle,$$
$$0 \leq n_1, \quad n_1 \leq n_2, \quad n_1 \leq n_3.$$

Again the Hamiltonian matrix was explicitly calculated

$$\langle n_1, n_2, n_3 | H | n_1, n_2, n_3\rangle = (n_1 + n_2 + n_3 + 3)(1 + b^2)$$
$$-b^2(3 - \delta_{n_1,0} - \delta_{n_2,0} - \delta_{n_3,0}),$$

$$\langle n_1 + 1, n_2, n_3 | H | n_1, n_2, n_3 \rangle = b(n_1 + 2)\Delta \;=\; \langle n_1, n_2, n_3 | H | n_1 + 1, n_2, n_3 \rangle ,$$
$$\text{plus cyclic} ,$$

$$\langle n_1 + 1, n_2 - 1, n_3 | H | n_1, n_2, n_3 \rangle = b^2 \Delta \;=\; \langle n_1, n_2, n_3 | H | n_1 + 1, n_2 - 1, n_3 \rangle ,$$
$$\text{plus cyclic} ,$$

where $\Delta = 1/\sqrt{3}$ if $n_1 = n_2 = n_3$, and $\Delta = \sqrt{3}$ if the final state is of this form, otherwise $\Delta = 1$.

Numerical computation of the spectrum proceeds as before. There is again a phase transition causing the critical slowing down around $\lambda_{\rm c} = 1$. Away from it, the eigenenergies converge satisfactorily, with the infinite volume results shown in Fig. 3. The spectrum in higher fermionic sectors is essentially different from the $F = 0, 1$ case. We again find dynamical $F = (2, 3)$ supermultiplets, however not all states with three fermions have their superpartners in the $F = 2$ sector. Instead, they are degenerate with ones from the $F = 4$ sector. This pattern continues now for all F as follows from counting the number of states: it is an increasing function of F. Another novel feature is that the spectrum is rather irregular, while the eigenenergies are almost equidistant for $F = 0, 1$. There is again the rearrangement of members of supermultiplets across the phase transition. Interestingly however, the *two* new SUSY vacua with $F = 2$ appear in the strong coupling phase, while there is none at weak coupling.

Similarly to the $F = 0$ case, the two vacua can be constructed analytically. To this end consider the "extreme" strong coupling limit of the Hamiltonian (2) [19]

$$H_{\rm SC} = \lim_{\lambda \to \infty} \frac{1}{\lambda} H = {\rm Tr}\left(f^\dagger f \right) + \frac{1}{N}\left[{\rm Tr}\left(a^{\dagger\,2} a^2 \right) + {\rm Tr}\left(a^\dagger f^\dagger a f \right) + {\rm Tr}\left(f^\dagger a^\dagger f a \right) \right]. \tag{8}$$

Surprisingly, $H_{\rm SC}$ conserves the number of bosonic quanta as well and proves very useful in mapping the structure of the model in *all* fermionic sectors (*cf.* the following section). Coming back to $F = 2$, one can identify the finite dimensional $(F = 2, B)$ sectors with zero eigenvalues, and construct the corresponding eigenstates. They read

$$|F = 2, v\rangle_1^\infty \;=\; |0, 1\rangle , \tag{9}$$
$$|F = 2, v\rangle_2^\infty \;=\; |0, 3\rangle - 2|1, 2\rangle , \tag{10}$$

the superscript referring to the infinite value of the 't Hooft coupling. These states can then be used to construct the two vacua at finite $\lambda > 1$

$$|F = 2, v\rangle_1^\lambda \; = \; \sum_{n=1}^{\infty} (-\sqrt{\lambda})^{-n} |0, n\rangle \,,$$

$$|F = 2, v\rangle_2^\lambda \; = \; (1 + Q_s H_s^{-1} Q^\dagger{}_w)^{-1} |F = 2, v\rangle_2^\infty \,, \qquad (11)$$

with $Q_s = g \mathrm{Tr} \,[f a^{\dagger 2}]$, $H_s = \{Q_s, Q^\dagger{}_s\}$, $Q^\dagger{}_w = \mathrm{Tr} \,[f^\dagger a]$ [19].

The restoration of supersymmetry can also bee seen on the level of eigenstates. Members of supermultiplets transform among themselves by SUSY charges Q and $Q^\dagger$, which become the well defined matrices in the planar bases. To see SUSY in this way define the following supersymmetry fractions

$$q_{mn} \equiv \sqrt{\frac{2}{E_m + E_n}} \langle F + 1, E_m | Q^\dagger | F, E_n \rangle \,, \qquad (12)$$

which are the coordinates of the supersymmetric images of eigenstates in the corresponding fermionic sectors. Fig. 5 shows that SUSY fractions indeed quickly stabilize with increasing $B_{\max}$.
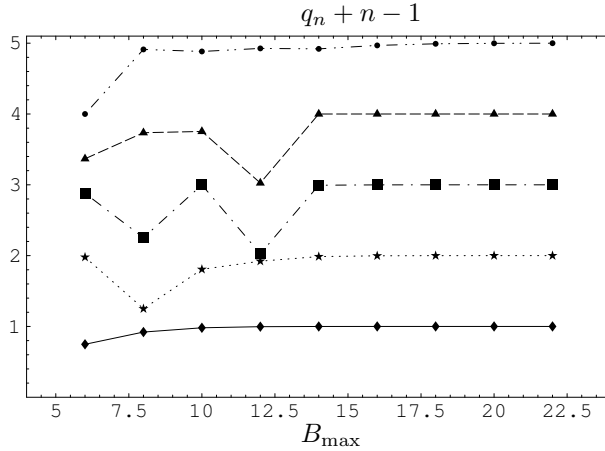

Fig. 5. First five supersymmetry fractions.

Supersymmetry fractions are also useful in defining the restricted Witten index which smoothly interpolates, at finite cutoff, between the two phases. The straightforward restriction of the sum

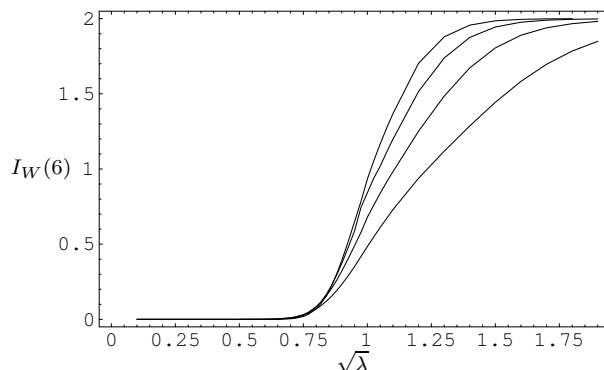$$W(T, \lambda) = \sum_i (-1)^{F_i} e^{-T E_i}$$

Fig. 6. Behavior of the restricted Witten index, at $T = 6$, around the phase transition.

to the $F = 2, 3$ eigenstates does not suffice since some of the states with three fermions remain unbalanced. Instead one can define

$$W_R(T, \lambda) = \sum_i \left(e^{-TE_i} - e^{-T\bar{E}_i} \right), \quad \bar{E}_i = \frac{\sum_f E_f |q_{fi}|^2}{\sum_f |q_{fi}|^2}$$

i.e. we take as the energy of the supersymmetric partner the energy weighted by supersymmetric fractions. This definition enforces summation only over complete supermultiplets away from the critical region, while provides the smooth smearing among possible candidates around λ_c. The index defined this way changes smoothly at finite cutoff, and varies by two units as expected (*cf.* Fig. 6) [19].

5. Arbitrary *F*

A simple generalization of the $F = 2, 3$ cases for arbitrary number of fermions reads [18]

$$|n\rangle = |n_1, n_2, \ldots, n_F\rangle = \frac{1}{\mathcal{N}_{\{n\}}} \text{Tr} \left(a^{\dagger n_1} f^\dagger a^{\dagger n_2} f^\dagger \ldots a^{\dagger n_F} f^\dagger \right) |0\rangle. \quad (13)$$

So the states of the basis may be labeled by F bosonic occupation numbers $\{n_1, n_2, \ldots n_F\}$ (configurations) modulo cyclic shifts. Some states are excluded by the Pauli principle. For example

$$\{n, n\}, \qquad \text{or} \qquad \{2, 1, 1, 2, 1, 1\}$$

are not allowed, since they change sign while retaining their identity after the suitable number of cyclic shifts. When calculating normalizations and

matrix elements one should keep track of symmetry factors defined as the number of cyclic shifts which bring a state back to itself. For example the state $|2, 1, 2, 1, 2, 1\rangle$ has the symmetry factor $d = 3$. Alternatively one can label planar states with the periodic binary strings (necklaces) with, *e.g.*, zeros/ones corresponding to the bosonic/fermionic creation operators (for a more complete discussion see [18]).

Detailed calculation of spectra for arbitrary fermion number is now in progress. However many properties, *e.g.* the supersymmetry structure, has been inferred from the strong coupling limit of the model.

TABLE II

Sizes of gauge invariant bases in the (F, B) sectors.

11	1	1	6	26	91	273	728	1768	3978	8398	**16796**
10	1	1	5	22	73	201	497	1144	2438	4862	9226
9	1	1	5	19	55	143	335	715	**1430**	2704	**4862**
8	1	1	4	15	42	99	212	429	809	1430	2424
7	1	1	4	12	30	66	**132**	247	**429**	715	1144
6	1	1	3	10	22	42	76	132	217	335	497
5	1	1	3	7	**14**	26	**42**	66	99	143	201
4	1	1	2	5	9	14	20	30	43	55	70
3	1	1	**2**	4	**5**	7	10	12	15	19	22
2	1	1	1	2	3	3	3	4	5	5	5
1	**1**	1	**1**	1	1	1	1	1	1	1	1
0	1	1	0	1	0	1	0	1	0	1	0
B											
F	0	1	2	3	4	5	6	7	8	9	10

As already mentioned, the strong coupling Hamiltonian (8) conserves both fermionic and bosonic numbers of quanta. Hence the Hilbert space splits into the finite dimensional sectors, *cf.* Table II, and the Hamiltonian in each sector becomes a finite matrix. Still the H_{SC} has the exact supersymmetry with the strong-coupling charges

$$Q_{\text{SC}} = \frac{1}{\sqrt{N}} \text{Tr} \left[f a^{\dagger 2} \right], \quad Q^{\dagger}_{\text{SC}} = \frac{1}{\sqrt{N}} \text{Tr} \left[f a^{\dagger 2} \right], \tag{14}$$

acting along the "diagonals" $2F + B = $ const. of Table II. Diagonalizing few of these matrices we have indeed found expected degeneracies[2]. Interestingly, we have also found experimentally [18, 20] that the zero energy eigenstates are located *only* in the sectors with

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[2] Obviously no cutoff was required this time.

$$B = F \pm 1, \quad F \text{ even} . \tag{15}$$

which form the regular "magic" staircase, shown in the boldface, in Table II.

This observation, when combined with the weak coupling (harmonic oscillator) limit, explains the structure of SUSY vacua for all F and at any value of the 't Hooft coupling. Namely, for any even F there are two supersymmetric vacua for any λ in the strong coupling phase. On the other hand, there is only one SUSY vacuum in the weak coupling phase and it has $F = 0$.

The question *why* the supersymmetric vacua are located in the magic sectors (15) will be answered in the next section.

6. Two equivalencies with statistical systems

Interestingly, our strong-coupling model is equivalent to the two well known and nontrivial statistical systems [20].

6.1. A gas of q-bosons

Consider a one dimensional, periodic lattice with size F. At each lattice site put a bosonic degree of freedom described by its c/a operators $a^\dagger_i, a_i$. The strong coupling Hamiltonian (8) is equivalent to the following one expressed solely in terms of bosonic variables

$$H = B + \sum_{i=1}^{F} \delta_{N_i,0} + \sum_{i=1}^{F} b_i b^\dagger_{i+1} + b_i b^\dagger_{i-1} , \tag{16}$$

where $N_i = a^\dagger_i a_i$, $B = N_1 + N_2 + \ldots + N_F$, and the action of $b_i, b^\dagger_i$ is

$$b^\dagger |n\rangle = |n+1\rangle, \quad b|n\rangle = |n-1\rangle, \quad b|0\rangle \equiv 0 \tag{17}$$

that is, they create and annihilate one quantum *without* the usual $\sqrt{n}$ factors. They satisfy the following commutation rules

$$[b, b^\dagger] = \delta_{N,0} \tag{18}$$

with the same non-linear (in b's) δ operator as in the Hamiltonian. Using the planar rules discussed earlier, one can show that the action of the first two terms of (8) on the planar basis is exactly the same as the action of the first two terms of (16) in the eigenbasis of N_i [20]. The last two terms of (8) are the same as the hopping terms of (16). Since the planar states acquire a phase $(-1)^{F-1}$ upon cyclic shifts, the planar system has the same spectrum

as (16) in the eigen-sector of the lattice shifts, U, with $\lambda_U = (-1)^{F-1}$. This equivalence was cross-checked numerically for $3 \leq F, B \leq 7$.

The c/a operators are known in the literature [23, 24]. Their algebra is the special case of the q-deformed harmonic oscillator algebra

$$[b, b^\dagger] = q^{-2N} \tag{19}$$

with $q \to \infty$. Transitions (17) without the square roots are often referred to as *assisted* (*i.e.* independent of the occupancy) transitions. The system of q-bosons described by the Hamiltonian (16) is, to our knowledge, regarded as non-soluble. In view of the second equivalence, discussed below, it turns out to be in fact soluble.

6.2. The XXZ Heisegberg chain

The second equivalence has been proved employing another representation of the planar states (13) [20]

$$\frac{1}{\mathcal{N}_n}|\text{Tr}\left[a^{\dagger m_1}(f^\dagger)^{n_1}a^{\dagger m_2}(f^\dagger)^{n_2}\ldots(f^\dagger)^{n_k}\right]|0\rangle \equiv |(0)^{m_1}(1)^{n_1}\ldots(0)^{m_r}(1)^{n_r}\rangle,$$
$$(r \geq 1, \ m_i, n_i > 0). \tag{20}$$

Where the states are labeled by the binary strings with, *e.g.* 0/1 corresponding to the bosonic/fermionic creation operators. In this basis, the action of the strong coupling Hamiltonian (8) is equivalent to that of the XXZ chain

$$H_{\text{XXZ}}^{(\Delta)} = -\frac{1}{2}\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\ \sigma_i^z\sigma_{i+1}^z\right)$$

for the particular values of the anisotropy parameter Δ. The detailed correspondence reads

$$H_{\text{SC}}(F, B) = \begin{cases} -H_{\text{XXZ}}^{(+1/2)} + \frac{3}{4}L\ , & F \text{ odd}, \\ +H_{\text{XXZ}}^{(-1/2)} + \frac{3}{4}L\ , & F \text{ even}, \ B \text{ odd}, \end{cases} \tag{21}$$

where the spin Hamiltonians are restricted to the translationally invariant sectors with the lattice size and the total spin fixed by (F, B):

$$L = F + B\ , \quad S^z = \sum_{i=1}^{L}s_i^z = \frac{1}{2}(F - B)\ . \tag{22}$$

Remarkably, this equivalence implies the existence of the magic staircase of the supersymmetric vacua discussed above. More than thirty years ago

Baxter has found that, for $\Delta = -1/2$, the ground states with $S_z = \pm 1/2$ have particularly simple eigenenergy $E_0 = -\frac{3}{4}L$ for infinite L [25]. Recently his findings have been extended by Riazumov and Stroganov to any finite, odd L [26]. In view of (21), (22) the Riazumov–Stroganov states are nothing but our strong coupling vacua [20]. Moreover, supersymmetry requires a host of degeneracies among the seemingly unrelated excitations of the XXZ chain. Amusingly, since the supersymmetry transformations change the lattice size, these excitations live on different lattices![3]

Finally, since the XXZ model is exactly soluble (*e.g.* by the Bethe Ansatz [28]), our supersymmetric system is also soluble at infinite coupling. As one application we give here the algebraic determination of the Bethe phase factors for the first three steps of the magic staircase. The eigenenergies of $H_{\mathrm{XXZ}}(\Delta)$ are [29]

$$E_{\mathrm{XXZ}}(\Delta) = -L\frac{\Delta}{2} + 2m\Delta - 2\sum_{j=1}^{m} \cos p_j, \qquad (23)$$

where the momenta $-\pi < p_j < \pi$ satisfy the following set of Bethe equations

$$e^{iLp_j} = (-1)^{m-1} \prod_{l=1}^{m} e^{i(p_j - p_l)} \frac{e^{ip_l} + e^{-ip_j} - 2\Delta}{e^{ip_j} + e^{-ip_l} - 2\Delta}, \quad j = 1, \ldots, m. \qquad (24)$$

With m denoting the number of down spins in a chain. For the supersymmetric model $m = B$ and (23) translate into

$$E_{\mathrm{SC}}(F, B) = F + 2\sum_{j=1}^{B} \cos p_j, \quad \text{for } F \text{ odd}, \ \Delta = +\frac{1}{2}, \qquad (25)$$

$$E_{\mathrm{SC}}(F, B) = F - 2\sum_{j=1}^{B} \cos p_j, \quad \text{for } F \text{ even}, \ \text{and} \ B \ \text{odd} \ \Delta = -\frac{1}{2}. \ (26)$$

Solving numerically Eqs. (24) we have found that the supersymmetric vacua occur at F even and B odd, and are always given by the yet simpler sub-ansatz of the Bethe Ansatz

$$p_1 = 0, \quad p_{2k+1} = -p_{2k}, \quad k = 1, \ldots, \frac{B-1}{2} \qquad (27)$$

which reduces the number of independent variables roughly by a factor of 2. Still, the Bethe equations can only be solved numerically in general. However, for the first three sectors the problem can be managed algebraically. We shall discuss it separately for each sector.

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[3] Supersymmetry of the, $\Delta = -1/2$, XXZ chain and other statistical systems, has been discussed in the literature, albeit in slightly different contexts (see, *e.g.* [27] and references therein). We thank Jan de Gier for bringing this to our attention.

6.2.1. $F = 4$, $B = 3$

This corresponds to $L = 7$, $m = 3$, and Eq. (27) implies

$$p_1 = 0, \quad p_2 = p, \quad p_3 = -p,\tag{28}$$

and (24) reduces to $(z = e^{ip})$

$$z^6 = 1.\tag{29}$$

The admissible solution, which gives $E_{\mathrm{SC}} = 0$, is

$$z = e^{i\frac{\pi}{3}}.\tag{30}$$

6.2.2. $F = 4$, $B = 5$

Now $L = 9$, $m = 5$, and with the aid of (27) Bethe equations reduce to $(x = e^{ip_2}, \ y = e^{ip_4})$

$$x^8 = \frac{xy + x + y}{x + y + 1} \frac{xy + x + 1}{xy + y + 1},$$

$$y^8 = \frac{xy + x + y}{x + y + 1} \frac{xy + y + 1}{xy + x + 1}.\tag{31}$$

Solving (31) is difficult in general, however one can easily find solutions with vanishing energy (26)

$$E_{\mathrm{SC}}(4,5) = 2 - 2\left(x + y + \frac{1}{x} + \frac{1}{y}\right) = 0.\tag{32}$$

Introducing two symmetric variables

$$s = x + y, \quad p = xy,\tag{33}$$

one obtains from (31), (32) the following two equations for s and p

$$\frac{s + p}{s + 1} = \pm p^4,\tag{34}$$

$$s + \frac{s}{p} = 1.\tag{35}$$

The admissible solution is on the negative branch and reads

$$x = \frac{1}{64}\left(16 + i\sqrt{2}\sqrt{15 + \sqrt{33}}(7 - \sqrt{33})\right.$$

$$\left. -4\sqrt{-16(3 + \sqrt{33}) - i2\sqrt{2}\sqrt{15 + \sqrt{33}}(9 + \sqrt{33})}\right),$$

$$y = \frac{1}{64}\left(16 + i\sqrt{2}\sqrt{15 + \sqrt{33}}(7 - \sqrt{33})\right.$$

$$\left. +4\sqrt{-16(3 + \sqrt{33}) - i2\sqrt{2}\sqrt{15 + \sqrt{33}}(9 + \sqrt{33})}\right).$$

$$(36)$$

This pair indeed satisfies both Bethe equations, together with (32), and therefore corresponds to our supersymmetric vacuum.

6.2.3. $F = 6$, $B = 5$

This sector corresponds to $L = 11$, $m = 5$. Now the reduced Bethe equations are

$$x^{10} = \frac{xy + x + y}{x + y + 1}\frac{xy + x + 1}{xy + y + 1},$$

$$y^{10} = \frac{xy + x + y}{x + y + 1}\frac{xy + y + 1}{xy + x + 1}. \tag{37}$$

As previously, we look for the solutions which satisfy $E_{\text{SC}}(6,5) = 0$, that is

$$x + y + \frac{1}{x} + \frac{1}{y} = 2, \quad \text{or} \quad \frac{s}{p} = \frac{2}{1 + p}. \tag{38}$$

The second equation follows from the product of the Bethe equations

$$p^5 = \pm\frac{p + s}{1 + s}, \quad \text{or} \quad \frac{s}{p} = -\frac{1 \mp p^4}{1 \mp p^5}. \tag{39}$$

These equations reduce to the fourth order polynomial equation for p. Again there is only one admissible solution and it lies on the negative branch

$$x = \frac{1}{72}\left(36 + i\sqrt{2}\sqrt{11 + \sqrt{13}}(7 + \sqrt{13})\right.$$

$$\left. -6\sqrt{2}\sqrt{6(-3 + \sqrt{13}) + i\sqrt{2}\sqrt{11 + \sqrt{13}}(-5 + \sqrt{13})}\right),$$

$$y = \frac{1}{72}\left(36 + i\sqrt{2}\sqrt{11 + \sqrt{13}}(7 + \sqrt{13})\right.$$

$$\left. +6\sqrt{2}\sqrt{6(-3 + \sqrt{13}) + i\sqrt{2}\sqrt{11 + \sqrt{13}}(-5 + \sqrt{13})}\right). \quad (40)$$

As the last exercise let us show that these numbers are indeed unimodular which is not obvious at first sight. However it is easy to calculate

$$|x + y|^2 = \frac{1}{3}\left(7 + \sqrt{13}\right), \ |x - y|^2 = \frac{1}{3}\left(5 - \sqrt{13}\right), \ \text{therefore } |x|^2 + |y|^2 = 2\,. \quad (41)$$

Further, since $x = a + b$ and $y = a - b$, one can observe that

$$(ab^*)^2 = \frac{1}{72}\left(-11 + \sqrt{13}\right), \ \Rightarrow \ |x|^2 = |y|^2 \quad (42)$$

which together with (41) implies that x and y are indeed pure phase factors. As a byproduct one obtains

$$\mathrm{Re}\,(xy^*) \equiv \cos(p_2 - p_4) = \frac{1}{6}\left(1 + \sqrt{13}\right). \quad (43)$$

7. Discussion

The direct diagonalization of a Hamiltonian matrix is usually considered a viable tool, for finding a spectrum, only for finite matrices. It turns out however, that the approach works in many cases with the infinite dimensional Hilbert space as well. Although such Hilbert spaces appear already in many quantum mechanical systems with finite number of degrees of freedom, the true challenge for the Hamiltonian formalism is posed by the field theoretical problems. An interesting family of systems results from the dimensional reduction of the field theoretical models to a one point in space. As such, they again have the finite number of degrees of freedom, however they inherit many advanced features of the parent field theories, for example their symmetries including supersymmetry. The straightforward diagonalization proved quite successful in uncovering quite rich and nontrivial spectra of supersymmetric Yang–Mills quantum mechanics in various dimensions.

This lecture reviews the recent study of a system with the infinite number of degrees of freedom, namely a particular supersymmetric gauge quantum mechanics (also referred to as a matrix model) with the infinite number of colours. The model was conceived as the illustration of the planar calculus in the Fock space. However it was subsequently found that the system has an interesting physics which connects to many recently discussed issues. For example, the system undergoes the discontinuous phase transition in the

't Hooft coupling which is accompanied by the remarkable rearrangement of dynamical supermultiplets. It enjoys the strong-weak duality in the lowest (and simplest) fermionic sectors where the complete, exact spectrum can be found analytically. The full structure of intervening supermultiplets begins with two fermions and goes on ad infinitum. In each bosonic sector two new supersymmetric vacua appear in the strong coupling phase while there is only one in the weak coupling region. The behavior in the strong coupling phase has been found by studying the system at the infinite value of the 't Hooft coupling and then extending it to the whole strong coupling regime.

At infinite coupling the model reveals also its connections with the statistical physics, which proves, among other things, that a quantum mechanics at infinite N becomes a *bona fide* field theory. The supersymmetric planar model is exactly equivalent to the one-dimensional quantum XXZ Heisenberg chain, and at the same time, to the one-dimensional lattice gas of q-bosons. The strong coupling vacua found in the SUSY matrix model turn out to be nothing but simple ground states of the XXZ chain which were found more than thirty years ago. In addition, the XXZ chain appears to have the hidden supersymmetry which results in many degeneracies among various energy eigenstates. An unusual feature of these supersymmetry transformations is that they connect states propagating on different lattices.

Second equivalence is with a gas of q-bosons, with the infinite deformation parameter. The latter mapping holds exactly in all fermionic and bosonic sectors while the former does not work for even F and B.

The chain of correspondences discovered in [20] implies also that the specific, nonlinear ∞-bosonic Hamiltonian is in fact soluble via solubility of the XXZ chain. *Vice versa*: the same solubility implies that the supersymmetric matrix model is exactly soluble at the infinite value of the 't Hooft coupling. It is also quite conceivable that this solubility holds in the whole strong coupling phase [19].

Summarizing: planar calculus applied directly in the Fock space turned out be rather promising tool in studying some simple, but non-trivial, models. It allowed to find phenomena which bear a tantalizing similarity to ones found in the much more advanced systems [30–35]. It remains to be seen if this approach can be extended to the full, *i.e.* space extended field theories.

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