VIRIAL THEOREM FOR FOUR-DIMENSIONAL SUPERSYMMETRIC YANG–MILLS QUANTUM MECHANICS WITH SU(2) GAUGE GROUP*

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Supersymmetric Yang–Mills quantum mechanics (SYMQM) in four dimensions for SU(2) gauge group is considered. In this work a two-fermionic sector with the angular momentum $j = 0$ in discussed. Energy levels from discrete and continuous spectra are calculated. To distinguish localized states from non-localized ones the virial theorem is applied.

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1. Introduction

In this work we consider supersymmetric Yang–Mills quantum mechanics (SYMQM) [1,2]. Theory for SU($N \to \infty$) gauge group in $D = 10$ dimensions is related to M-theory and allows researches on the BFSS hypothesis [3]. For a smaller number of dimensions and a smaller number of colours, like in our case where $D = 4$ and $N = 2$ respectively, such mechanics pose excellent theoretical laboratory [4] to test amazing properties of supersymmetry, *i.e.* coexistence of discrete and continuous spectrum [5], action of supersymmetric generators or unique features of SUSY vacuum states [2, 6–8]. Moreover SYMQM without fermions describe glueballs which are also considered in many non-supersymmetric theories [4, 9, 10]. Here, in order to calculate the energy spectrum of our model the method proposed by van Baal in Ref. [8] is used. In this work eigenvalues as well as eigenfunctions of Hamiltonian of the model are computed. Finally, to distinguish the localized states form the non-localized ones the virial theorem is applied.

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2. Hamiltonian

The zero-volume Hamiltonian [8,9], *i.e.* in the long wave approximation [2,6], reads

$$H = \underbrace{\underbrace{-\frac{1}{2}\sum_{i,a}\left(\frac{\partial}{\partial\hat{c}_i^a}\right)^2}_{H_T} + \underbrace{\frac{1}{2}\sum_{i,a}\left(\hat{B}_i^a\right)^2}_{H_V}}_{H_B} \underbrace{- i\sum_{i,a,b,d}\varepsilon_{abd}\bar{\lambda}^a\bar{\sigma}^i\lambda^b\hat{c}_i^d}_{H_F}, \tag{2.1}$$

where in the bosonic part, *i.e.* in H_B, H_T is the kinetic Hamiltonian, H_V with

$$\hat{B}_i^a = -\frac{1}{2}\sum_{i,j,k,a,b,d}\varepsilon_{ijk}\varepsilon_{abd}\hat{c}_j^b\hat{c}_k^d, \tag{2.2}$$

corresponds to a bosonic potential while the fermionic part is denoted by H_F, where $\sigma^j = \tau^j$ are Pauli matrices. Bosonic variables, $\hat{c}_i^a$, have colour indices $a = 1,2,3$ and spatial indices $i = 1,2,3$. Except the colour indices the anticommuting Weyl spinors, λ_a^α, have spinor indices $\alpha = 1,2$. Thus, in this system the maximum number of fermions is $3 \times 2 = 6$.

3. The cut Fock space

The operators of the number of fermionic quanta, n_F, and the total angular momentum of the system, j commute with the Hamiltonian. Therefore, solving the eigenproblem of the Hamiltonian we can separately consider the sectors with fixed values of n_F and j. The system has particle-hole symmetry so it is enough to consider sectors with $n_F = 0,1,2,3$. Unfortunately, the operator of number of bosonic quanta, n_B, does not commute with H. Since the model has an infinite number of bosons solving it numerically we have to cut somewhere the Fock space off. A good choice for this cut-off is a maximal number of bosonic quanta in the system, $B \geq n_B$.

We are especially interested in the sector where $n_F = 2$ and $j = 0$. In this sector the localized and non-localized states coexist. Moreover, it contains the supersymmetric vacuum state. To find the energy spectrum we use van Baal approach [8,11]. For $j = 0$ and $n_F = 2$ the Hamiltonian can be rewritten in terms of three bosonic variables

$$r^2 = \sum_{j,a}(\hat{c}_j^a)^2, \quad u = r^{-4}\sum_{j,a}(\hat{B}_j^a)^2, \quad v = r^{-3}\det\hat{c}, \tag{3.1}$$

or equivalently in (x_1, x_2, x_3) where

$$r^2 = \sum_j x_j^2, \quad u = r^{-4}\sum_{i>j}x_i^2x_j^2, \quad v = r^{-3}\prod_j x_j. \tag{3.2}$$

For example the bosonic potential has a form

$$H_V = u\frac{r^4}{2} = \frac{(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2)}{2}\,. \tag{3.3}$$

The minimum of H_V is localized in six valleys along the x_i-axis. The other parts of the Hamiltonian have much more complicated structure [8, 12].

We construct the Fock space acting $\hat{c}_j^b$ and $\lambda_{\dot\alpha}^a$ on the empty state [4, 8] :

$$|n\rangle = \sum_{\substack{\text{contractions}\\ \{a_1,\ldots,a_r\}}} \hat{c}_{k_1}^{a_1}\ldots\hat{c}_{k_m}^{a_m}\bar\lambda_{a_{m+1}}^{\dot\alpha}\ldots\bar\lambda_{a_r}^{\dot\beta}|0\rangle\,, \tag{3.4}$$

where sum goes over gauge and rotation invariant combinations of the operators. For $n_F = 2$ we have two independent ways of the fermionic action:

$$\mathcal{I}_j{}^a = -2i\sum_{c,b,\dot\alpha,\dot\beta}\varepsilon_{c,b,a}\bar\lambda_{\dot\alpha}^c(\bar\sigma^{j0})^{\dot\alpha}{}_{\dot\beta}\bar\lambda^{b\dot\beta}|0\rangle\,,\qquad \mathcal{J}^{ab} = -\sum_{c,b,\dot\alpha,\dot\beta}\bar\lambda_{\dot\alpha}^a\bar\lambda_{\dot\beta}^b\epsilon^{\dot\alpha\dot\beta}|0\rangle\,, \tag{3.5}$$

where $\bar\sigma^{j0} = \frac{1}{2}\tau_j$ and $\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = -i\tau_2$ lowers spinor indices. Making contractions of (3.5) to bosons we obtain six independent invariants

$$\begin{aligned}
|e_1(u,v)\rangle &= \textstyle\sum_{j,a}\hat{c}_j^a/\hat{r}\mathcal{I}^j{}_a\,, & |e_4(u,v)\rangle &= \textstyle\sum_{a,b}\delta^{ab}\mathcal{J}_{ab}\,,\\
|e_2(u,v)\rangle &= \textstyle\sum_{j,a}\hat{B}_j^a/\hat{r}^2\mathcal{I}^j{}_a\,, & |e_5(u,v)\rangle &= \textstyle\sum_{a,b}\hat{c}_j^a\hat{c}_j^b/\hat{r}^2\mathcal{J}_{ab}\,,\\
|e_3(u,v)\rangle &= \textstyle\sum_{j,a}\hat{c}_j^b\hat{c}_k^b\hat{c}_k^a/\hat{r}^3\mathcal{I}^j{}_a\,, & |e_6(u,v)\rangle &= \textstyle\sum_{a,b}\hat{c}_j^b\hat{c}_j^d\hat{c}_k^d\hat{c}_k^b/\hat{r}^4\mathcal{J}_{ab}\,.
\end{aligned} \tag{3.6}$$

Other basis vector can be obtained acting with invariant combinations of bosonic variables, *i.e.* (r, u, v), on these six vectors. This gives following basis vectors

$$|n\rangle = \sum_{m=1}^{6} h_m^n(r,u,v)|e_m(u,v)\rangle\,, \tag{3.7}$$

where $h_m^n(r, u, v)$ are arbitrary functions. Following Refs. [8, 12] they are chosen as eigenfunctions of the harmonic oscillator. In this basis the matrix of the kinetic Hamiltonian, H_T, is tridiagonal.

4. Diagonalization of the Hamiltonian matrix

The Hamiltonian (2.1) in the basis (3.7) gives the eigenequation

$$\sum_n H^{n'n}v_k^n = E_k v_k^{n'}\,, \tag{4.1}$$

where $H^{n'n} = \langle n'|H|n\rangle$ is the Hamiltonian matrix [8, 12] while E_k are its eigenvalues. The matrix for cut-off $B \leq 11$ is plotted in Fig. 1. Intensity of gray scale shows the amplitude of the matrix elements. We can see five broad branches of the bosonic Hamiltonian, which corresponds to change of n_B by $0, \pm 2, \pm 4$ and two other narrower branches of the fermionic part, H_F, where this change equals ± 1. For $B = 60$ this matrix has 10416×10416 elements and its diagonalization is much more time consuming[1].

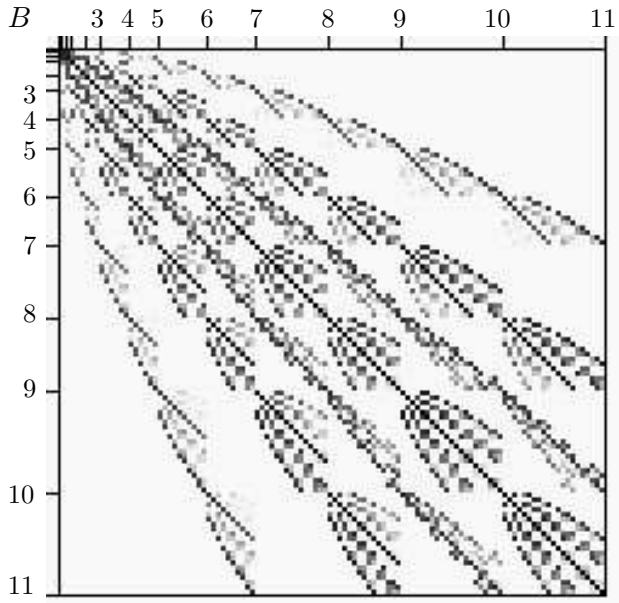


Fig. 1. The 112×112 Hamiltonian matrix in the base (3.7) for cut-off $B \leq 11$.

The dependence of the energy, E_k, on the cut-off is shown in Fig. 2. We can see two different behaviours of the energy levels. The first levels converge rapidly. They correspond to localized states. The other ones never converge but they slowly fall down as $E(B) \sim 1/B$. Cut-off analyses of free particle systems gives conjecture that the latter levels gives the continuous spectrum at $B \to \infty$ [13].

The eigenfunctions of H read

$$|\Phi_k(r,u,v)\rangle = \sum_n v_k^n |n\rangle = \sum_n v_k^n \sum_{m=1}^{6} h_m^n(r,u,v)|e_m\rangle\,. \tag{4.2}$$

[1] The evaluation time grows exponentially with B. To speed up calculation the van Baal's Mathematica code [8] was rewritten to C++.

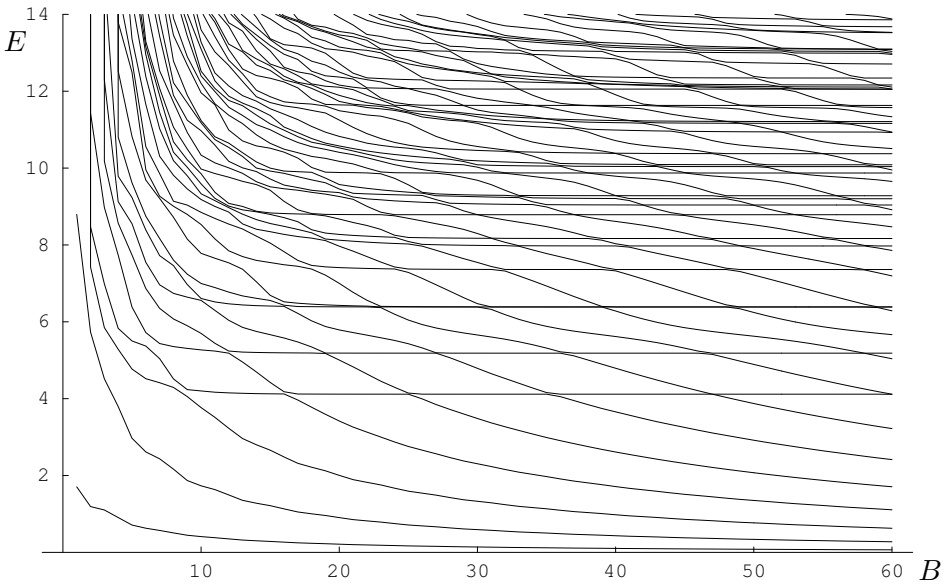


Fig. 2. Energy spectrum as a function of cut-off $B \geq n_B$.

Examples of such wavefunctions were presented in [12]. Indeed, the rapidly converging states are localized in the centre of the x-coordinate system. On the other hand, the non-localized ones penetrate the system along the potential valleys [12]. For these states realization of the $B \to \infty$ limit is more complicated and it is performed with fixed energy E and changing k, *i.e.*

$$|\,\Phi^{(E)}(r,u,v)\rangle = \lim_{B\to\infty} |\,\Phi_{k(E_K=E,B)}(r,u,v)\,\rangle. \tag{4.3}$$

Given the explicit form of the energy eigenstates using (4.2) we can calculate averages of other operators

$$\langle\mathcal{O}\rangle_k = \langle\,\Phi_k(r,u,v)\rangle|\mathcal{O}|\,\Phi_k(r,u,v)\rangle\,. \tag{4.4}$$

The only technical problem is to rewrite the operators in terms of (r,u,v).

5. Virial theorem

In order to distinguish the localized states from the non-localized ones we apply the virial theorem. One can derive it from the Heisenberg equation

$$\frac{d\hat{F}}{dt} = \frac{\partial\hat{F}}{\partial t} + \frac{1}{i\hbar}[\hat{F},\hat{H}]\,. \tag{5.1}$$

For a motion in a compact space the virial, $\vec{x}\cdot\vec{p}$, is a limited physical variable. Therefore, its average does not change with time:

$$0 = \frac{d}{dt}\langle \vec{x}\cdot\vec{p}\rangle = \frac{1}{i\hbar}\langle [\vec{x}\cdot\vec{p}, \hat{H}]\rangle \,, \tag{5.2}$$

where the second equality follows from (5.1).

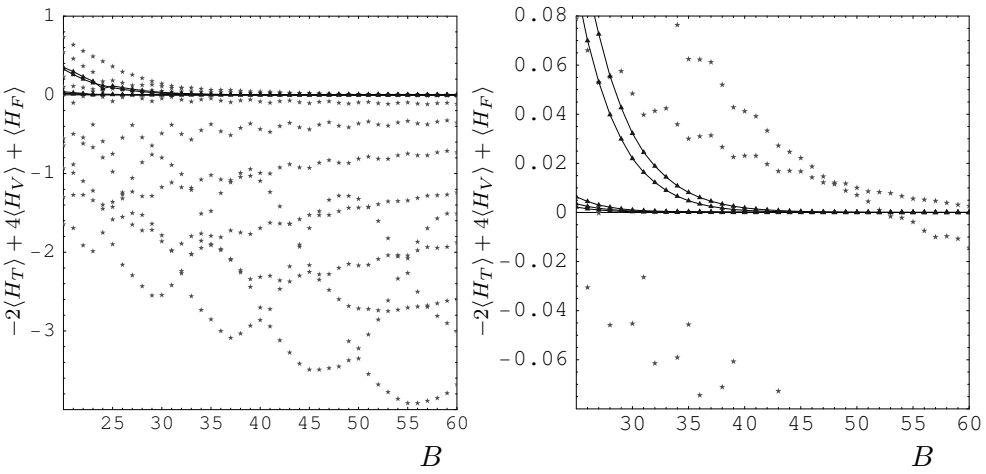


Fig. 3. The test function (5.6) for the bound states as a function of cut-off $B \geq n_B$ for the lowest fifteen energies. Here to guide the eye joined *triangles* correspond to states from discrete spectrum while *stars* are related to states with continuous spectrum.

In the case of Hamiltonian $H = T + V$ where T is the kinetic energy and the potential energy scales as $V(\alpha\vec{x}) = \alpha^n V(\vec{x})$ one obtains from (5.2):

$$-2\langle T\rangle + n\langle V\rangle = 0 \,. \tag{5.3}$$

In our case

$$\vec{x}\cdot\vec{p} = -ir\frac{\partial}{\partial r} \,, \tag{5.4}$$

and the consecutive parts of the Hamiltonian scale as

$$H_T(\alpha r) = \alpha^{-2} H_T(r) \quad H_V(\alpha r) = \alpha^4 H_V(r) \quad \text{and} \quad H_F(\alpha r) = \alpha H_F(r) \,. \tag{5.5}$$

Therefore for the bound states the test function

$$f \equiv -2\langle H_T\rangle + 4\langle H_V\rangle + \langle H_F\rangle = 0 \,. \tag{5.6}$$

Applying (4.4) the averages $\langle H_X\rangle$ are calculated over the eigenstates of the Hamiltonian (4.2) as

$$E_X = \langle H_X \rangle_k = \langle \Phi_k(r,u,v) | H_X | \Phi_k(r,u,v) \rangle \,. \tag{5.7}$$

Substituting the obtained values of the averages to (5.6) we can check the relation (5.6) for the bound states. We plot the test function (5.6) as a function of the cut-off $f(B)$ in Fig. 3. One can easily see that for the localized states the relation (5.6) is fulfilled with a very good approximation even for not so high B.

For the states with the continuous spectrum, condition (5.6) is not satisfied. This is seen in Fig. 3 where we compare the test functions for the bound states and the states related to the continuous spectrum as a function of cut-off $B \geq n_B$. The test function (5.6) related to the bound states are nearly equal to zero while the test function for the states which form the continuous spectrum seem to grow with B.

6. Summary

In this work we discuss supersymmetric Yang–Mills quantum mechanics (SYMQM) [1, 2] in four dimensions for SU(2) gauge group. Investigation of such models allow to understand complicated properties of the supersymmetric theories, *i.e.* coexistence of localized and non-localized states, non-triviality of the supersymmetric vacuum, which are common for different supersymmetric theories.

We focus on the sector with the number of fermionic quanta $n_F = 2$ and the total angular momentum $j = 0$. This sector possess both discrete and continuous spectrum [5], and the supersymmetric vacuum state [4, 8, 12]. In order to find the energy spectrum we use a method proposed by van Baal in Ref. [8] where the cut-off of our Fock space, *i.e.* B, is defined as a maximal number of bosonic quanta, n_B. To confirm localization of states the virial theorem is used (5.2).

The result of the Hamiltonian spectrum agree with the previous works [4, 8, 14]. However, here the calculations have been performed for the very high cut-off $B = 60$. For this cut-off not only the spectra of bound states converge but also the corresponding eigenstates [12].

Our calculation shows that the quantum virial theorem is applicable for the systems with more complicated potential, *e.g.* one considered in this work which consists of two different parts. Moreover, the virial theorem can be used to determinate localization of the states where it is not possible to calculate the Hamiltonian eigenstates and where the averages of operators can be computed in a different way. Thus in some cases the virial method as the localization test of states is easier and more applicable that direct calculation.

In the future using this method one can calculate matrix representations for other operators (4.4) and test various properties and laws for this model. Furthermore, using similar methods one can also try to solve the models in more dimensions and for different gauge groups [15].

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