SPHERICALLY SYMMETRIC SOLUTIONS OF THE EINSTEIN–BACH EQUATIONS AND A CONSISTENT SPIN-2 FIELD THEORY*

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We briefly present a relationship between General Relativity coupled to certain spin-0 and spin-2 field theories and higher derivatives metric theories of gravity. In a special case, described by the Einstein–Bach equations, the spin-0 field drops out from the theory and we obtain a consistent spin-two field theory interacting gravitationally, which overcomes a well known inconsistency of the theory for a linear spin-two field coupled to the Einstein's gravity. Then we discuss basic properties of static spherically symmetric solutions of the Einstein–Bach equations.

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1. Introduction

In General Relativity (GR) among physically relevant solutions we have static spherically symmetric spacetimes, when the metric (in the canonical coordinates) is given by

$$g = g_{\mu\nu}dx^\mu dx^\nu = -A(r)dt^2 + B(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (1)$$

In vacuum, the Einstein–Hilbert action $I_{\text{E–H}} = \kappa\int d^4x\sqrt{-\det(g_{\mu\nu})}\,R$, where R is a Ricci scalar of the metric g, implies the most important Schwarzschild solution

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2M}{r} .$$

Adding to the Einstein–Hilbert Lagrangian the Maxwell term $-1/4F_{\mu\nu}F^{\mu\nu}$ and the cosmological constant Λ we extend the 1-parameter Schwarzschild class to a 3-parameter class of metrics given by

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2M}{r} + \frac{e^2}{r^2} + \frac{\Lambda}{2}r^2 .$$

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In the case of $\Lambda = 0 \neq e$ this is called the Reissner–Nordström solution, when $\Lambda \neq 0 = e$ this is the Köttler solution. The above spacetimes were discovered in first two years since GR had been established.

In general, when we are to describe a matter field interacting with gravity we usually add to the Einstein–Hilbert term the action of the given matter field in a form just as in a flat spacetime but instead of standard partial derivatives we put covariant ones. This rule is called the minimal coupling principle. However, such a procedure fails for a linear spin-two field, which may exist in the Minkowski spacetime [1].

It has been noticed by various authors [2, 3] that the inconsistency of a theory for a linear massive spin-two field interacting with Einstein's gravity can be overcome by the nonlinear field generated in higher-derivative gravity upon reduction to a second-order theory.

In this paper we sketch briefly the main aspects of the Lagrangian field theory for the massive spin-two field which arises in the appropriate Legendre transformation of a Lagrangian quadratically depending on Ricci tensor. For a systematic exposition of the theory we refer to [4]. Furthermore, our goal is to clear up some basic properties of static spherically symmetric solutions of the Einstein–Bach equations, four-dimensional Lorentzian manifolds arising from the action principle for the following Lagrangian density

$$\mathcal{L} = R + \frac{1}{m^2}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right) = R + \frac{1}{2m^2}\left(L_{\mathrm{GB}} - C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}\right) , \quad (2)$$

where $R_{\mu\nu}$, $R_{\alpha\beta\mu\nu}$, $C_{\alpha\beta\mu\nu}$ are correspondingly Ricci, Riemann and Weyl tensors of the metric g and $L_{\mathrm{GB}} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, the Gauss–Bonnet term, is a total divergence in four dimensional spaces, *i.e.* it does not contribute to the field equations.

Since $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ — the square of the Weyl tensor is the Lagrangian density of the Weyl Conformal Theory (WCT), and its field equations are just the Bach equations we justify our terminology.

2. Dynamical equivalence of Jordan, Helmholtz–Jordan and Einstein frames

2.1. Jordan frame

By Jordan Frame (JF) we will mean a higher derivative metric theory of gravity, where a Lagrangian density is given by a quadric polynomial of the metric $\tilde{g}$ and its Ricci tensor

$$\mathcal{L} = \tilde{R} + a\tilde{R}^2 + b\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} . \quad (3)$$

The coefficients a and b, as well as $1/m^2$ have dimension $[length]^2$. There are possible more general Lagrangians, however, our choice is quite sufficient for conceptual and practical purposes. From the action principle we obtain a system of fourth order partial differential equations for a Lorentzian metric $\tilde{g}$ on a four dimensional manifold M. The important idea is that $(M, \tilde{g})$ need not be considered as a real spacetime with a physical metric, but rather a unifying tensor field on an abstract space.

Jordan frame can be then reformulated as the Einstein gravity described by the metric field alone g, and the other fields contained in the multiplet acting as a "matter source" in the Einstein field equations.

$$G_{\mu\nu} = \kappa T_{\mu\nu}\,.$$

A tool providing a proper decomposition of the unifying field $\tilde{g}$ into a multiplet of gravitational fields is a specific Legendre transformation to the Hamilton picture. Although the Legendre transformation is essentially unique, the resulting multiplet can be given different physical interpretations and, therefore, JF can be transformed into frames including fields of definite spin in two ways. The first possibility is that the field $\tilde{g}$ remains the metric tensor, now carrying two degrees of freedom, while the other degrees of freedom (carried previously by its higher derivatives) are encoded into fields of definite spins: this is the Helmholtz–Jordan Frame (HJF). A second possibility is to introduce via an appropriate redefinition of the Legendre transformation a new spacetime metric g, while the symmetric tensor $\tilde{g}$ is decomposed into the metric g and a mixture of spin-2 and spin-0 fields, forming the massive, non-geometric components of the gravitational multiplet. This is the Einstein Frame (EF).

2.2. Helmholtz–Jordan frame

In order to provide general covariance of HJF and its dynamical equivalence to JF we should choose properly quantities, which will play a role of generalized velocities, since only a generally covariant theory may be a consistent theory of a spin-two field. We make Legendre transformations of the Lagrangian with respect to the two irreducible components of $\tilde{R}_{\mu\nu}$: its trace $\tilde{R}$ and the traceless part $\tilde{S}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu}$. Assuming $4a + b \neq 0$ and $b \neq 0$ we define a scalar and tensor canonical momentum via corresponding Legendre transformations:

$$\chi + 1 \equiv \frac{\partial \mathcal{L}}{\partial \tilde{R}}\,, \qquad \pi^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial \tilde{S}_{\mu\nu}}\,.$$

The new triplet of field variables $\{\tilde{g}_{\mu\nu}, \chi, \pi^{\mu\nu}\}$ defines the Helmholtz–Jordan Frame (HJF). Then we construct the Hamiltonian density and express it in terms of the metric an the canonical momenta

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \tilde{R}}\tilde{R} + \frac{\partial \mathcal{L}}{\partial \tilde{S}_{\mu\nu}}\tilde{S}_{\mu\nu} - \mathcal{L} = \frac{1}{4a+b}\chi^2 + \frac{1}{4b}\pi_{\mu\nu}\pi^{\mu\nu}$$

and the Helmholtz Lagrangian density

$$\begin{aligned}\mathcal{L}_{\mathrm{H}} &\equiv \frac{\partial \mathcal{L}}{\partial \tilde{R}}\tilde{R}(\tilde{g}, \partial\tilde{g}, \partial^2\tilde{g}) + \frac{\partial \mathcal{L}}{\partial \tilde{S}_{\mu\nu}}\tilde{S}_{\mu\nu}(\tilde{g}, \partial\tilde{g}, \partial^2\tilde{g}) - \mathcal{H}(\tilde{g}_{\mu\nu}, \chi, \pi^{\mu\nu}) \\ &= \tilde{R} + \chi\tilde{R} + \pi^{\mu\nu}\tilde{S}_{\mu\nu} - \frac{1}{4a+b}\chi^2 - \frac{1}{4b}\pi_{\mu\nu}\pi^{\mu\nu}\,.\end{aligned}$$

From the Helmholtz Lagrangian one can derive Hamilton equations of motion for $\tilde{g}_{\mu\nu}$ which are exactly Einstein equations $\tilde{G}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi)$, where

$$\tilde{T}_{\mu\nu} \equiv \frac{1}{\sqrt{-\tilde{g}}}\frac{\delta}{\delta \tilde{g}^{\mu\nu}}\left(\sqrt{-\tilde{g}}(\tilde{R} - \mathcal{L}_{\mathrm{H}})\right)\,.$$

Furthermore, we can derive propagation equations for the fields:

- the linear Klein–Gordon for scalar χ
- a quasi-linear second order for $\pi_{\mu\nu}$, which is the Klein–Gordon when linearized around the Minkowski space.

and the corresponding masses are real under so-called non-tachyon conditions

- $3a + b > 0\,, \quad m_\chi^2 = \dfrac{1}{2(3a+b)}\,,$
- $-b > 0\,, \quad m_\pi^2 = -\dfrac{1}{b}\,.$

Especially interesting case is

- when $3a + b = 0$ the only solution of the scalar field is $\chi = 0$. Then the Bianchi identity for the Einstein tensor $\tilde{G}_{\mu\nu}$ implies four constraints $\nabla_\nu \pi^{\mu\nu} = 0$. There is also an algebraic constraint $\tilde{g}_{\mu\nu}\tilde{\pi}^{\mu\nu} = 0$, and no other. This ensures a pure spin-two character of $\pi^{\mu\nu}$.

2.3. Einstein frame

We construct a true spacetime metric g from the metric tensor in Jordan frame,

$$g^{\mu\nu} \equiv \Big(-\det(\tilde{g}_{\alpha\beta})\Big)^{-\frac{1}{2}} \left|\det(\frac{\partial \mathcal{L}}{\partial \tilde{R}_{\alpha\beta}})\right|^{-\frac{1}{2}} \frac{\partial \mathcal{L}}{\partial \tilde{R}_{\mu\nu}} = \sqrt{\left|\frac{\det(\tilde{g}_{\alpha\beta})}{\det(g_{\alpha\beta})}\right|}\frac{\partial \mathcal{L}}{\partial \tilde{R}_{\mu\nu}}, \tag{4}$$

where $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$, $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}$. The Legendre transformation (4) is a map of the metric manifold $(M, \tilde{g}_{\mu\nu})$ to another one, $(M, g_{\mu\nu})$. One assumes that $\det(\partial\mathcal{L}/\partial\tilde{R}_{\mu\nu}) \neq 0$ to view $g^{\mu\nu}$ as a spacetime metric. In EF all indices are raised and lowered with the aid of g and, therefore, it is convenient to alter our notation and denote $\tilde{g}_{\mu\nu}$ by $\psi_{\mu\nu}$ and its inverse $\tilde{g}^{\mu\nu}$ by $\gamma^{\mu\nu}$. For the generic Lagrangian as in (3),

$$g^{\mu\nu} = \sqrt{\left|\frac{\det(\psi_{\mu\nu})}{\det(g_{\mu\nu})}\right|}\left[(1+2a\tilde{R})\gamma^{\mu\nu} + 2b\tilde{R}^{\mu\nu}\right] , \tag{5}$$

one introduces $\Phi_{\mu\nu}$

$$\tilde{g}_{\mu\nu} = \psi_{\mu\nu} = \Phi_{\mu\nu} + g_{\mu\nu} \,.$$

The doublet of the fields $\{g_{\mu\nu}, \Phi_{\mu\nu}\}$ defines the Einstein frame EF. In general $\Phi_{\mu\nu}$ is an admixture of spin-two and scalar fields. When $3a + b = 0$ this is a pure spin-2 field. Having constructed the Hamiltonian density and related Helmholtz Lagrangian density one may derive the equations of motion for the metric, which are Einstein ones, $G_{\mu\nu}(g) = T_{\mu\nu}(g, \psi)$ and nonlinear propagation equations for $\Phi_{\mu\nu}$, which reduce to the Klein–Gordon equations when linearized around the Minkowski space.

Although in both (HJF and EF) frames a higher derivative gravity provides a consistent description of a self-gravitating massive spin-two field, mathematical similarity to GR and physical arguments indicate that EF is physically more acceptable [4].

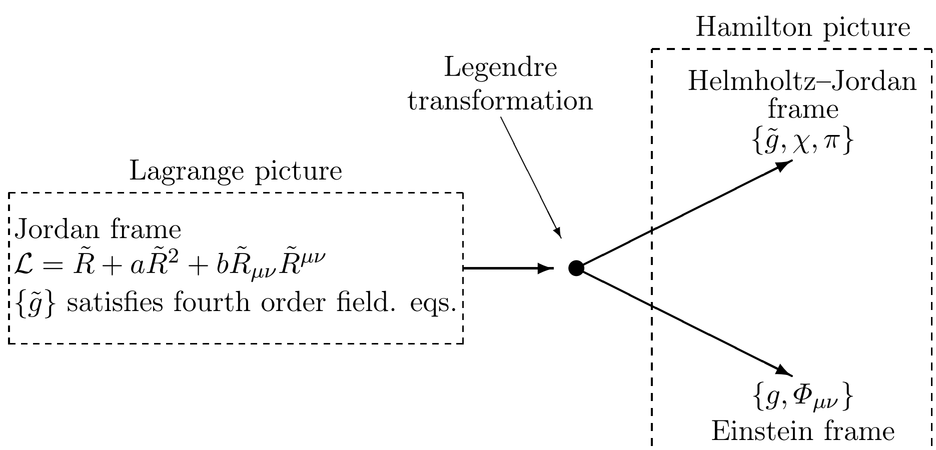

Fig. 1. The relations between possible frames.

3. Spherically symmetric solutions

3.1. The field equations

The Einstein–Bach equations which follow for the Lagrangian (2) are

$$G_{\mu\nu}+\frac{1}{m^2}\left(-\Box R_{\mu\nu}+2R_{\alpha\mu\nu\beta}R^{\alpha\beta}+\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\right)$$
$$+\frac{1}{m^2}\left(\frac{1}{6}g_{\mu\nu}(\Box R-R^2)+\frac{1}{3}\nabla_\mu\nabla_\nu R+\frac{2}{3}RR_{\mu\nu}\right)=0\,.$$

Since now, for the metric in JF we use g without a tilde. Taking the trace of this tensor equations we obtain the following constraint

$$R=0\,. \tag{6}$$

Therefore, the field equations reduce to

$$R_{\mu\nu}+\frac{1}{m^2}\left(-\Box R_{\mu\nu}+2R_{\alpha\mu\nu\beta}R^{\alpha\beta}+\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\right)=0\,. \tag{7}$$

Our field equations contain explicitly parameter m and, therefore, it is convenient to use variables rescaled by m. Under static spherically symmetric ansatz (1) we define $t\to\tau=mt$, $r\to x=mr$ and $A(r)\to A(x)$, $B(r)\to B(x)$. Then m drops out from the field equations. Though one can assume more general spherically symmetric ansatz including an explicit dependence on time coordinates of the metric, but our aim here is to study only static solutions. Taking $\mu=\nu=1$ in (7) and reducing it with help of (6) one obtains the first of the following equations. The second one is just (6) expressed in terms of $A(x)$ and $B(x)$.

$$\frac{A'B''}{2xAB^2}-\frac{B''}{x^2B^2}+\frac{1}{x}\left(\frac{A'^3}{4A^3B}+\frac{A'^2B'}{4A^2B^2}-\frac{A'B'^2}{AB^3}\right)$$
$$+\frac{1}{x^2}\left(-\frac{3A'^2}{4A^2B}+\frac{A'B'}{2AB^2}+\frac{5B'^2}{4B^3}\right)$$
$$+(B-1)\left(-\frac{B'}{x^3B^2}+\frac{2}{x^4B}-\frac{1}{x^2}\right)+\frac{A'}{xA}=0\,, \tag{8}$$

$$\frac{A''}{A}-\frac{A'^2}{2A^2}-\frac{A'B'}{2AB}+\frac{2}{x}\left(\frac{A'}{A}-\frac{B'}{B}\right)+\frac{2}{x^2}(1-B)=0\,. \tag{9}$$

3.2. Aspects of integrability

The first question concerning the integrability of our system is whether there exists an explicit solution. Up to now, the only one known for the full equations (not reduced) is the Schwarzschild solution. However, it corresponds to the vacuum one with vanishing spin-2 field (in both HJF and EF), namely to the Schwarzschild solution.

The second question is whether there exist a first integral or an invariant which could be used to give a local explicit solution via the implicit function theorem. One can ask also whether there exists a sufficient number of symmetries either to reduce the differential equations of the system to algebraic equations or to obtain an independent first integral. There is only one Lie point symmetry, that of re-scalings of $A(x)$ which is obvious since the system is time-independent. There are no Lie–Bäcklund symmetries, at least up to the second order derivative transformations, however one may expect there exist non-local symmetries. Eq. (9) is linear with respect to $\sqrt{A(x)}$ (homogenous second order) as well as with respect to $1/B(x)$ (inhomogenous first order), solving it explicitly one gets

$$\frac{1}{B(x)} = \frac{2\exp\left(3\mathit{\Gamma}(x)\right)}{xA(x)\left(2+\frac{xA'(x)}{2A(x)}\right)^2}\int dx\left(2+\frac{xA'(x)}{2A(x)}\right)\frac{A(x)}{\exp\left(3\mathit{\Gamma}(x)\right)}\,, \tag{10}$$

where $\mathit{\Gamma}(x) = \int dx \frac{A'(x)}{\left(4A(x)+xA'(x)\right)}$. Putting $B(x)$ from the above formula one decouples (8) and obtains a nonlinear integro-differential equation for $A(x)$.

At last we take a closer look at the Painlevé property. For linear ordinary differential equations (ODE's) all singularities are fixed (points where solutions are not analytic do not depend on constants of integration). However, for nonlinear ODE's there exist also movable singularities. There is strong evidence that all integrable equations have the Painlevé property, that is, all solutions are single valued (in the complex plane) around all movable singularities [6]. To find out whether the reduced system has the Painlevé property one can use the Ablowitz Ramani Segur (ARS) algorithm [5] or one of its implementation in Mathematica [7]. Passing the Painlevé test implies for the system (8), (9) that its solutions may be expressed in terms of elementary functions (as in the case of the Schwarzschild solution) or special functions: the Painlevé transcendents or the linear special functions.

3.3. A general picture of static spherically symmetric solutions around the center

Using standard notations and methods we assume a dominant behavior at the origin:

$$A = \alpha x^p , \qquad B = \beta x^q ,$$

where α, β, p, q are constants to be determined. From the system we obtain the following three possibilities:

$$\begin{array}{llll} (i) & p = 0 , \quad q = 0 , & \alpha \text{ arbitrary, } \beta = 1 , \\ (ii) & p = -1 , q = 1 , & \alpha, \beta \text{ arbitrary,} \\ (iii) & p = 2 , \quad q = 2 , & \alpha, \beta \text{ arbitrary.} \end{array}$$

Now we construct formal solutions with given singularity structures in terms of generalized power series expansions truncated at some power n by performing calculations with help of computer algebra. Proceeding step by step, we can determine every coefficient of the Laurent expansions in terms of a few first ones. A formal solution is an actual solution when the corresponding Laurent series has a non-zero radius of convergence.

$$A(x) = x^p \left(\sum_{i=0}^{n} a_i x^i + O(x)^{n+1} \right) , \quad B(x) = x^q \left(\sum_{i=0}^{n} b_i x^i + O(x)^{n+1} \right) .$$

The formal expansions may be used to analyze geometric invariants: squares of the Riemann and Ricci tensors are sufficient to distinguish regular and singular geometry of the origin.

- case (i) Up to the sixth order we find

$$\begin{aligned} A(x) &= a_0 + a_2 x^2 + \frac{a_2 \left(a_0 + 12 a_2\right)}{20 a_0} x^4 + \frac{a_2 \left(a_0^2 + 72 a_0 a_2 + 240 a_2^2\right)}{840 a_0^2} x^6 + O\left(x^7\right) , \\ B(x) &= 1 + \frac{a_2}{a_0} x^2 + \frac{a_2 (a_0 + 6 a_2)}{10 a_0^2} x^4 + \frac{a_2 \left(a_0^2 + 50 a_0 a_2 + 80 a_2^2\right)}{280 a_0^3} x^6 + O\left(x^7\right) . \end{aligned}$$

$\mathrm{Ric}^2 = R_{\mu\nu} R^{\mu\nu}$ and $\mathrm{Riem}^2 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$ are finite when $x \to 0$, therefore, around the center the spacetime geometry is regular.

- case (ii)

$$\begin{aligned} A(x) &= x^{-1} \left(a_0 + a_0 b_0 x + a_3 x^3 + \frac{a_3 b_0}{6} x^4 - \frac{a_3 b_0^2}{5} x^5 + O\left(x^6\right) \right) \\ B(x) &= x \left(b_0 - b_0^2 x + b_0^3 x^2 + \left(\frac{5 a_3 b_0}{3 a_0} - b_0^4 \right) x^3 + \left(-\frac{23 a_3 b_0^2}{6 a_0} + b_0^5 \right) x^4 + O\left(x^5\right) \right) . \end{aligned}$$

Riem^2 blows up at $x = 0$ as $1/x^6$, this implies a singularity at the center. Taking $a_3 = 0$ and $a_0 b_0 = 1$ we obtain the Schwarzschild solution.

- case (iii) Up to the fourth order we find

$$\begin{aligned} A(x) &= x^2\Big(a_0 + a_1 x + a_2 x^2 \\ &\quad + \frac{-8a_1^4 + a_0 a_1^2 (19a_2 - a_0 b_0) + 6a_0^2 \left(a_2^2 - a_0^2 b_0 (9 + b_0)\right)}{18a_0^2 a_1} x^3 + O\left(x^4\right)\Big), \\ B(x) &= b_0 x^2\Big(1 + \frac{a_1}{a_0}\, x + \frac{a_1^2 + 2a_0 (-4a_2 + a_0 b_0)}{6a_0^2} x^2 \\ &\quad - \frac{12a_1^4 + a_0 a_1^2 (-25a_2 + 11a_0 b_0) + 10\left(-a_0^2 a_2^2 + a_0^4 b_0 (9 + b_0)\right)}{18a_0^3 a_1} x^3 \\ &\quad + O\left(x^4\right)\Big). \end{aligned}$$

Generically Ric^2 and Riem^2 blows up as $1/x^8$ and the center is singular. When the Ricci tensor is non-zero in JF, then the corresponding solutions have non-vanishing spin-2 fields in HJF and EF, this describes a generic situation. Nevertheless, the expansions around the center say nothing about the asymptotic behavior of the related solutions.

type	free parameters	p	q	Ric^2	Riem^2
(i)	a_0, a_2	0	0	regular	regular
(ii)	a_0, a_3, b_0	-1	1	regular	x^{-6}
(iii)	a_0, a_1, a_2, b_0	2	2	x^{-8}	x^{-8}

4. Conclusions

- Every vacuum solution in General Relativity corresponds to the same vacuum solution in JF, HJF and EF, especially the Schwarzschild solution in GR is also a solution of JF, HJF and EF.
- The static spherically symmetric case is exactly solvable; in general, the solutions may be expressed in terms of the nonlinear transcendental functions.
- There are non-trivial solutions of the Einstein–Bach equations, since we can construct static spherically symmetric solutions different from the Schwarzschild as well as the Minkowski space. The formal series solutions around the center exhibit three distinct behaviors, among which there are perfectly regular ones.

- Since the linearization procedure around the Minkowski space of the equations of motion for the spin-2 field results in the Klein–Gordon equations (in both HJF and EF), it clearly suggests that there are solutions of the Einstein–Bach equations with the asymptotic behavior $A \approx 1-(2M)/x+$ "Yukawa-like term", *i.e.* for which a deviation from the Schwarzschild solution vanishes exponentially. In consequence, *e.g.* $\pi_{00} \approx (\beta_{H00} \exp(-x))/x$, where β_{H00} is constant. The question of the behavior of such solutions in the neighborhood of the center is open.
- In EF we have a new metric g which determines a new canonical radial variable (let us call it s), nevertheless, geometric behavior of the solutions in the asymptotic region cannot be qualitatively different, $\Phi_{00} \approx (\beta_{E00} \exp(-x))/x \approx (\beta \exp(-s))/s$ and consequently $A_E \approx 1-(2M)/x+$ "Yukawa-like term", though their analytic expressions in both variables x and s may differ significantly. Here the Yukawa-like term means, that it vanishes asymptotically faster then any rational function.
- The above arguments clarify that our consistent description of a massive spin-2 field interacting gravitationally is reasonable at least in EF.
- Possible applications in cosmology and astrophysical systems seem to be very attractive. We point out *e.g.* the galactic rotation curves problem, dark matter, dark energy.

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