THE SEARCH FOR INDEPENDENCE IN CHAOTIC SYSTEMS

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This paper is dedicated to the memory of two great Polish scientists: M. Kac and S. Ulam

We show that, given the generalized chaotic sequences $x_n = \cos[2\pi\theta z^n]$, where z is a typical real number, any string $x_s, x_{s+1}, x_{s+2}, \ldots, x_{s+r}$ (for any r) constitutes a set of statistically independent random variables. We will discuss the relevance of this result to dynamical systems, real physical experiments, and new technological devices used in secure communications.

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1. Introduction

The leitmotif of much of Mark Kac's work was the search for the meaning of independence [1]. Although independence is the central concept of probability theory, the work of the pioneers devoted to its foundations appeared to Kac awesomely abstract. He was interested in finding concrete mathematical objects that satisfy the normal law. The stochastically independent functions studied by Kac and Steinhaus [2–5] were quite concrete mathematical objects, unlike the "mysterious" objects used in the early books of probability theory [1].

In this context, two functions $f_1(t)$, $f_2(t)$ are independent in the sense that the proportion of time during which, simultaneously, $f_1(t) < \alpha_1$ and $f_2(t) < \alpha_2$ is equal to the product of the proportions of times during which separately $f_1(t) < \alpha_1$, $f_2(t) < \alpha_2$.

However, there is nothing random here. The functions studied by Kac and Steinhaus can be even periodic. For instance, $f_1(t) = \cos t$,

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$f_2(t) = \cos(\sqrt{2}t)$. We should say that Mark Kac was always fascinated by random phenomena and his dream was to answer the questions: "What is chance?", "What is random?" [6]. This quest has inspired us.

Mark Kac's search for independence in number theory has led to a very beautiful theorem [7], that marked the entry of normal law into number theory and gave birth to a new branch of this ancient discipline.

Ulam and von Neumann [8–10] were the first to prove that function $x_n = \sin^2(\theta\pi 2^n)$ is the general exact solution to the logistic map $x_{n+1} = 4x_n(1 - x_n)$.

Generalizing the exact solution $x_n = \sin^2[\theta\pi 2^n]$ to the logistic map $x_{n+1} = 4x_n(1 - x_n)$, several authors have obtained the solutions to new chaotic maps [11–16]. In general, the exact solutions can be written as $x_n = P(\delta k^n)$, where $P(t)$ is a periodic function, δ is a real parameter and k is an integer number.

Some examples are the following: $x_n = \cos[2\pi\theta 2^n]$ is the solution to map $x_{n+1} = 2x_n^2 - 1$, $x_n = \cos[2\pi\theta 3^n]$ is the solution to map $x_{n+1} = x_n(4x_n^2 - 3)$, ($\theta$ is a real parameter). These are special cases of the Chebyshev maps [11, 13].

In the present paper we will prove the statistical independence of the sequences values generated by function

$$x_n = \cos[2\pi\theta z^n]\,, \tag{1}$$

where z is a generic real number. We will discuss the relevance of this result to dynamical systems, real physical experiments, and new technological constructions.

2. Unpredictable dynamics

We have presented before [17–21] evidence that the functions $x_n = \sin^2[\theta\pi z^n]$ can produce very complex sequences. We will present here a brief explanation of why the dynamics of function (1) is unpredictable.

From the observation of a string of values $x_0, x_1, x_2, x_3, \ldots, x_m$ generated by function (1), when $z > 1$ is a non-integer number, it is impossible to determine the generation law. The fact is that it is impossible to determine which value of θ was used and the time-series produced for different values of θ satisfying the same string of values is different in most of the cases.

Let z be a rational number expressed as $z = p/q$, $p > q > 2$, where p and q are relative prime numbers. We are going to show that given $m + 1$ numbers generated by function (1): $x_0, x_1, \ldots, x_m$ (m can be as large as we wish), there is an infinite set of values of θ for which the generated string is the same, and still the next value x_{m+1} can take q different values. So we never can be sure of which the next value can be.

Let us define the following family of sequences:

$$x_n^{(k,m)} := \cos\left[2\pi\left(\theta_0 + q^m k\right)\left(\frac{p}{q}\right)^n\right], \tag{2}$$

where k is an integer. We have just re-defined $\theta = (\theta_0 + q^m k)$. The parameter k distinguishes the different sequences. For all sequences parametrized by k, the first $m+1$ values are the same.

This can be observed in the following calculation

$$\begin{aligned} x_n^{(k,m)} &= \cos\left[2\pi\theta_0\left(\frac{p}{q}\right)^n + 2\pi k p^n q^{(m-n)}\right] \\ &= \cos\left[2\pi\theta_0\left(\frac{p}{q}\right)^n\right], \end{aligned} \tag{3}$$

for all $n \leq m$. Note that the number $kp^n q^{(m-n)}$ is an integer for $n \leq m$. Nevertheless, the next value

$$x_{m+1}^{(k,m)} = \cos\left[2\pi\theta_0\left(\frac{p}{q}\right)^{m+1} + \frac{2\pi k p^{(m+1)}}{q}\right] \tag{4}$$

can take q different values.

Figures 1, 2 and 3 show different examples of the dynamics produced by function (1) for different values of z.

Recent developments have shown that predictability can be used to characterize the complexity of dynamical systems (see *e.g.*, [22]).

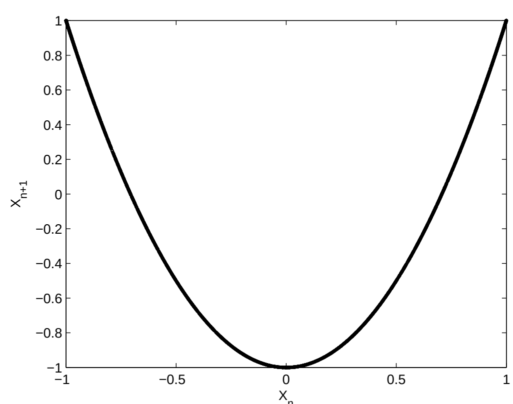

Fig. 1. First-return map constructed with the dynamics generated by function (1) $z = 2$.

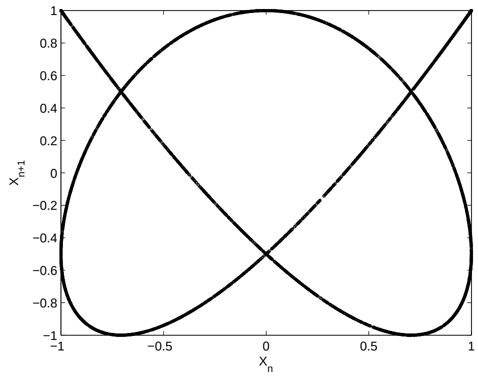

Fig. 2. First-return map constructed with the dynamics generated by function (1) with $z = 4/3$.

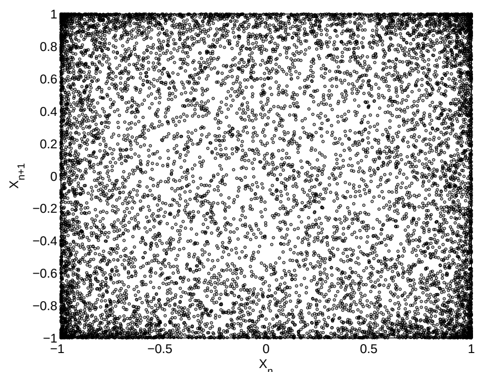

Fig. 3. First-return map constructed with the dynamics generated by function (1) with $z = \pi$.

3. Statistical independence

Considering that $E(x)$ is the expected value of quantity x, let us define the r-order correlations [23, 24]:

$$E(x_{n_1} x_{n_2} \cdots x_{n_r}) = \int_{-1}^{1} dx_0 \left[\rho(x_0) x_{n_1} x_{n_2} \cdots x_{n_r}\right]. \tag{5}$$

Note that the functions (1) possess zero mean, their values are bound in the interval $-1 \leq x_n \leq 1$, the invariant density is given by $\rho(x) = 1/\pi\sqrt{1-x^2}$ and $x_0 = \cos(2\pi\theta)$.

We have the following formula for the correlation functions:

$$E(x_{n_1}x_{n_2}\cdots x_{n_r}) = \int_0^1 d\theta\,[\cos(2\pi\theta z^{n_1})\cos(2\pi\theta z^{n_2})\cdots\cos(2\pi\theta z^{n_r})]\,. \tag{6}$$

Considering that $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, we obtain

$$E(x_{n_1}x_{n_2}\cdots x_{n_r}) = 2^{-r}\sum_{\sigma}\delta\left(\sigma_1 z^{n_1} + \sigma_2 z^{n_2} + \cdots + \sigma_r z^{n_r}, 0\right), \tag{7}$$

where $\sum_\sigma$ is the summation over all possible configurations $(\sigma_1, \sigma_2, \ldots, \sigma_r)$, with $\sigma = \pm 1$, and $\delta(n,m) = 1$, if $n = m$ or $\delta(n,m) = 0$, if $n \neq m$. We will have non-zero correlations only for the sets $n_1, n_2, \ldots, n_r$ that satisfy the equation

$$\sum_{i=1}^{r}\sigma_i z^{n_i} = 0, \tag{8}$$

where $\sigma_i = \pm 1$. These equation can always be written in the form

$$N_0 + N_1 z + \cdots + N_j z^j = 0, \tag{9}$$

where $N_0, N_1, \ldots, N_j$ are integers.

Let us recall here some very important definitions and results [2–5, 25–29] in probability theory. The bounded functions u and v are statistically independent if and only if

$$E(u^m v^n) = E(u^m)E(v^n), \tag{10}$$

for all integer numbers m, n.

Equivalently, the functions $f_1, f_2 \cdots, f_r$ constitute a set of statistically independent functions if and only if

$$E(f_1^{n_1} f_2^{n_2}\cdots f_r^{n_r}) = E(f_1^{n_1})E(f_2^{n_2})\cdots E(f_r^{n_r}), \tag{11}$$

for all integers $n_1, n_2, \ldots, n_r$.

Now we will prove that the functions $x_n = \cos[2\pi\theta z^n]$ (where z is a typical real number) constitute sequences of statistically independent values.

We must verify the equality:

$$E[x_s^{n_0} x_{s+1}^{n_1}\cdots x_{s+r}^{n_r}] = E(x_s^{n_0})E(x_{s+1}^{n_1})\cdots E(x_{s+r}^{n_r}) \tag{12}$$

for all positive integers $n_0, n_1, n_2, \ldots, n_r$.

Note that equation (12) means that any string of values produced by function $x_n = \cos[2\theta\pi z^n]$: $x_s, x_{s+1}, \ldots, x_{s+r}$ is a block of statistically independent values. And this string can be started at any point of the sequence. And the length of this block r can be any integer.

Let us make some calculations:

$$\begin{aligned} E(x_s^{n_0} x_{s+1}^{n_1} \cdots x_{s+r}^{n_r}) \quad &= 2^{-(n_0+n_1+\cdots+n_r)} \sum_{\sigma} \delta \left(\sigma_{s_1} z^s + \sigma_{s_2} z^s + \cdots + \sigma_{s_{n_0}} z^s\right. \\ &+\sigma_{(s+1)_1} z^{s+1} + \sigma_{(s+1)_2} z^{s+1} + \cdots + \sigma_{(s+1)_{n_1}} z^{s+1} + \cdots \\ &\left.+\sigma_{(s+r)_1} z^{s+r} + \sigma_{(s+r)_2} z^{s+r} + \cdots + \sigma_{(s+r)_{n_r}} z^{s+r}, 0\right) \\ &= 2^{-(n_0+n_1+\cdots+n_r)} \sum_{\sigma} \delta \left[\left(\sigma_{s_1} + \sigma_{s_2} + \cdots + \sigma_{s_{n_0}}\right) z^s\right. \\ &+ \left(\sigma_{(s+1)_1} + \sigma_{(s+1)_2} + \cdots + \sigma_{(s+1)_{n_1}}\right) z^{s+1} + \cdots \\ &\left.+ \left(\sigma_{(s+r)_1} + \sigma_{(s+r)_2} + \cdots + \sigma_{(s+r)_{n_r}}\right) z^{s+r}, 0\right] \end{aligned} \tag{13}$$

On the other hand

$$\begin{aligned} E(x_s^{n_0}) \; &= \; 2^{-(n_0)} \sum_{\sigma} \delta \left[\left(\sigma_{s_1} + \sigma_{s_2} + \cdots + \sigma_{s_{n_0}}\right) z^s, 0\right], \\ E(x_{s+1}^{n_1}) \; &= \; 2^{-(n_1)} \sum_{\sigma} \delta \left[\left(\sigma_{(s+1)_1} + \sigma_{(s+1)_2} + \cdots + \sigma_{(s+1)_{n_1}}\right) z^{s+1}, 0\right], \\ &\vdots \\ E(x_{s+r}^{n_r}) \; &= \; 2^{-(n_r)} \sum_{\sigma} \delta \left[\left(\sigma_{(s+r)_1} + \sigma_{(s+r)_2} + \cdots + \sigma_{(s+r)_{n_r}}\right) z^{s+r}, 0\right]. \end{aligned} \tag{14}$$

We should notice that

$$E(x_n^m) = 0 \tag{15}$$

if m is odd. So any of the correlations E in Eq. (14) is zero if n_i is odd. However,

$$E(x_s^{n_0} x_{s+1}^{n_1} \cdots x_{s+r}^{n_r}) = 0 \tag{16}$$

if any n_i is odd, because the equations

$$\begin{aligned} (\sigma_{s_1} \; + \; \sigma_{s_2} + \cdots + \sigma_{s_{n_0}}) z^s + \left(\sigma_{(s+1)_1} + \sigma_{(s+1)_2} + \cdots + \sigma_{(s+1)_{n_1}}\right) z^{s+1} + \cdots \\ + \; \left(\sigma_{(s+r)_1} + \sigma_{(s+r)_2} + \cdots + \sigma_{(s+r)_{n_r}}\right) z^{s+r} = 0 \end{aligned} \tag{17}$$

are never satisfied for a transcendental z.

Suppose now that all n_i are even. After trivial combinatorial analysis, we get

$$E(x_s^{n_0} x_{s+1}^{n_1} \cdots x_{s+r}^{n_r}) = 2^{-(n_0+n_1+\cdots+n_r)} \begin{pmatrix} n_0 \\ \frac{n_0}{2} \end{pmatrix} \begin{pmatrix} n_1 \\ \frac{n_1}{2} \end{pmatrix} \cdots \begin{pmatrix} n_r \\ \frac{n_r}{2} \end{pmatrix}, \quad (18)$$

$$E(x_s^{n_0}) = 2^{-n_0} \begin{pmatrix} n_0 \\ \frac{n_0}{2} \end{pmatrix}, \quad (19)$$

$$E(x_{s+1}^{n_1}) = 2^{-n_1} \begin{pmatrix} n_1 \\ \frac{n_1}{2} \end{pmatrix}, \ldots, \quad (20)$$

$$E(x_{s+r}^{n_r}) = 2^{-n_r} \begin{pmatrix} n_r \\ \frac{n_r}{2} \end{pmatrix}. \quad (21)$$

Equations (15)–(21) imply that

$$E\left(x_s^{n_0} x_{s+1}^{n_1} \cdots x_{s+r}^{n_r}\right) = E\left(x_s^{n_0}\right) E\left(x_{s+1}^{n_1}\right) \cdots E\left(x_{s+r}^{n_r}\right), \quad (22)$$

for all integer $n_0, n_1, \ldots n_r$.

Thus the functions (1) with a transcendental z constitute sequences of statistically independent values. This result is so important that we wish to present an alternative proof.

Let us define $t = 2\pi\theta z^n$ in Eq.(1). Now we can consider the following functions $f_1 = \cos(\lambda_1 t), f_2 = \cos(\lambda_2 t), \ldots, f_r = \cos(\lambda_r t)$, where $\lambda_1 = z, \lambda_2 = z^2, \ldots, \lambda_r = z^r$.

Using a re-working of Theorem 3 [5] and other theorems and definitions of Refs. [2–5], we obtain the result that the functions $f_i = \cos(\lambda_i t)$ constitute a set of independent functions when the numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$ are linearly independent over the rationals (equivalently: integers). This means that if $\alpha_1, \alpha_2, \ldots, \alpha_r$ are rational, $\alpha_1\lambda_1 + \alpha_2\lambda_2 + \cdots + \alpha_r\lambda_r = 0$ only if $\alpha_1, \alpha_2, \ldots, \alpha_r$ are all zero. On the other hand, the powers of a transcendental number are linearly independent over the rationals.

Hence the values $x_n = \cos[2\pi\theta z^n] = \cos t$, $x_{n+1} = \cos[2\pi\theta z^{n+1}] = \cos(zt)$, $x_{n+2} = \cos[2\pi\theta z^{n+2}] = \cos(z^2 t)$,..., $x_{n+r} = \cos[2\pi\theta z^{n+r}] = \cos(z^r t)$ constitute a set of independent values.

Here it is important to recall several notions. The set of all algebraic numbers is countable. Real numbers that are not algebraic are called transcendental. The set of all transcendental numbers is not countable. Most real numbers are transcendental. The cardinality of the set of transcendental numbers is the same as that of the set of real numbers, *i.e.* the order of the continuum. So for most numbers z, function (1) is a sequence of statistically independent random variables.

It is easy to see that any set of values (not necessarily consecutive) produced by function (1) is also a system of independent variables, because

$$E\left[x_{n_0}^{k_0}x_{n_1}^{k_1}\cdots x_{n_r}^{k_r}\right]=E\left[x_{n_0}^{k_0}\right]E\left[x_{n_1}^{k_1}\right]\cdots E\left[x_{n_r}^{k_r}\right] \tag{23}$$

for all positive integers $n_0, n_1 \ldots, n_r, k_0, k_1, \ldots, k_r$.

It is also trivial to verify that for any integer z, the variables x_n and x_m $(n \neq m)$ are not statistically independent [23, 24]. For instance, for $z = 2$, $E[x_n^2 x_{n+1}] = 1/4$ and $E[x_n^2]E[x_{n+1}] = 0$. So $E[x_n^2 x_{n+1}] \neq E[x_n^2]E[x_{n+1}]$.

Other particular (but interesting) facts are the following. For a rational $z = p/q$, where p and q are relative prime numbers $(p > q > 2)$, there are non-zero higher-order correlations. The largest non-zero correlations that involve powers of x_n and x_{n+1} are the following $E(x_n^p x_{n+1}^q)$, which are of the order of $2^{-(p+q)}$. (So for large p and q, these correlations are exponentially small.) All the correlations involving x_{n+1} as a linear term are zero. That is $E\left[x_{n+1}x_n^{k_n}x_{n-1}^{k_{n-1}}\cdots x_{n-r}^{k_{n-r}}\right] = 0$, for all positive integers $r, k_n, k_{n-1}, \ldots, k_{n-r}$. This result reflects the fact that x_{n+1} cannot be expressed as a one-valued function of past values. In other words, this result shows that the dynamics is not predictable.

Another important fact is that x_n and x_{n+1} are independent variables even for an algebraic irrational z. For a general irrational z, the equation

$$E\left(x_n^{k_n}x_{n+1}^{k_{n+1}}\right)=E\left(x_n^{k_n}\right)E\left(x_{n+1}^{k_{n+1}}\right) \tag{24}$$

is always satisfied.

When k_n and k_{n+1} are both even, we again have

$$E\left(x_n^{k_n}x_{n+1}^{k_{n+1}}\right) = 2^{-(k_n+k_{n+1})}\begin{pmatrix} k_n \\ \frac{k_n}{2}\end{pmatrix}\begin{pmatrix} k_{n+1} \\ \frac{k_{n+1}}{2}\end{pmatrix}, \tag{25}$$

$$E\left(x_n^{k_n}\right) = 2^{-k_n}\begin{pmatrix} k_n \\ \frac{k_n}{2}\end{pmatrix}, \tag{26}$$

$$E\left(x_{n+1}^{k_{n+1}}\right) = 2^{-k_{n+1}}\begin{pmatrix} k_{n+1} \\ \frac{k_{n+1}}{2}\end{pmatrix}. \tag{27}$$

The main problem in Eq. (24) is when one of the numbers k_n or k_{n+1} is odd. In fact, this is the case when Eq. (24) is not satisfied for all maps of type $x_{n+1} = f(x_n)$. We will show that the correlation functions $E\left(x_n^{k_n}x_{n+1}^{k_{n+1}}\right)$ are zero when k_n (or k_{n+1}) is odd. Suppose $k_n = i$, $k_{n+1} = 2j + 1$. The calculation yields

$$E\left(x_n^i x_{n+1}^{2j+1}\right) = \frac{1}{2^{(i+2j+1)}} \sum_\sigma \delta\left[\sigma_1 z^n + \cdots + \sigma_i z^n + \sigma_{i+1} z^{n+1} + \cdots + \sigma_{i+2j+1} z^{n+1}, 0\right]. \tag{28}$$

Note that $E(x_n^i x_{n+1}^{2j+1})$ is not zero only if

$$(\sigma_1 + \sigma_2 + \cdots + \sigma_i) z^n + (\sigma_{i+1} + \sigma_{i+2} + \cdots + \sigma_{i+2j+1}) z^{n+1} = 0. \tag{29}$$

For irrational z, this equation has not solutions. The same result is obtained if we assume that k_n is odd.

In the language of functions $f_1 = \cos(\lambda_1 t)$, $f_2 = \cos(\lambda_2 t)$, where $\lambda_1 = z$, $\lambda_2 = z^2$, $t = 2\pi\theta z^n$, the necessary and sufficient condition for the statistical independence is the linear independence of λ_1 and λ_2. However, for the particular case of two numbers λ_1 and λ_2, the linear independence is equivalent to the irrationality of λ_2/λ_1 [1]. Of course, here $\lambda_2/\lambda_1 = z$.

An alternative checking is the following. Let us define the vector $\vec{r} = (x_n, y_n)$, where $y_n = x_{n+1}$. The probability densities $P(x,y)$, $P(x)$, $P(y)$ can be calculated as

$$\begin{aligned} P(x,y) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta(\vec{r} - \vec{r}_n), \\ P(x) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta(x - x_n), \\ P(y) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta(y - y_n). \end{aligned}$$

There are different methods for the calculation of these quantities [30–32]. Following these methods we can check that $P(x,y) = P(x)P(y)$. In fact,

$$P(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad P(y) = \frac{1}{\pi\sqrt{1-y^2}}, \quad P(x,y) = \frac{1}{\pi^2\sqrt{(1-x^2)(1-y^2)}}.$$

In general, the numerical calculations corroborate the statistical independence of variables x_{n+1}, x_n, *etc.*

Figure 3 is a manifestation of this property. It is evident that for any map of type $x_{n+1} = f(x_n)$ (including the logistic) the property $P(x,y) = P(x)P(y)$ cannot be satisfied (compare the figures 1 and 3).

Let us discuss a formulation of the Central Limit Theorem. Using the theorems proved in Refs. [2–5] and the results of the present paper, we can obtain the following formula. If z is a transcendental number and $x_n = \cos[2\pi\theta z^n]$, then

$$\lim_{r\to\infty} P\left\{\alpha < \frac{x_1 + x_2 + \cdots x_r}{\sqrt{r}} < \beta\right\} = \frac{1}{\sqrt{\pi}} \int_\alpha^\beta e^{-\xi^2} d\xi. \tag{30}$$

Fig. 4 shows an approximation to the normal law constructed with the sum of a finite number of values generated by function (1).

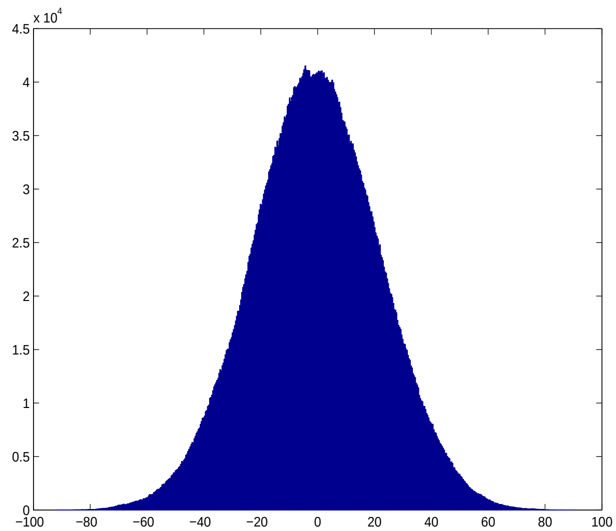

Fig. 4. Normal law constructed with the sum of a finite number of values generated by function (1).

The results about the independence of functions $x_n = \cos[2\pi\theta z^n]$ can be extended to more general functions as the following:

$$x_n = P[\theta T z^n] , \tag{31}$$

where $P(t)$ is a periodic functions, using the Fourier representation [23]. A very important example of function (31) is the following:

$$x_n = [\theta z^n] \quad \mathrm{mod}\ 1 . \tag{32}$$

A typical example of the dynamics produced by function (32) with $z = \pi$ can be seen in Fig. 5. Note that for function (1) the generated values are not uniformly distributed because the probability density is $\rho(x) = 1/\pi\sqrt{1-x^2}$. Nevertheless, for function (32), the produced values are uniformly distributed ($\rho(x) = 1$). Compare the distributions of points in the maps shown in figures 3 and 5.

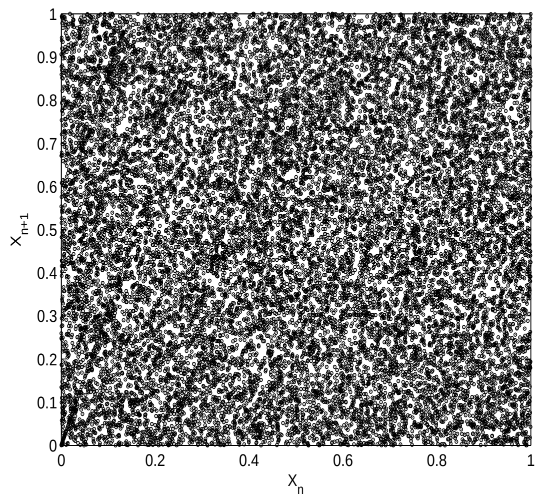

Fig. 5. First-return map produced by function (32) with $z = \pi$.

4. Dynamical systems and other nonlinear models

Let us consider the following dynamical system

$$x_{n+1} = \begin{cases} ax_n, & \text{if } x_n < Q, \\ by_n, & \text{if } x_n > Q, \end{cases} \tag{33}$$

$$y_{n+1} = cz_n, \tag{34}$$

$$z_{n+1} = \cos(2\pi x_n). \tag{35}$$

The parameters $a > 1$, $b > 1$, $c > 1$ are typical real numbers. Note that for $0 < x_n < Q$, the behavior of variable z_n is like that of function (1). For $x_n > Q$ the dynamics is re-injected to the region $0 < x_n < Q$ with a new initial condition which is obtained from Eq. (34) that depends on the unpredictable dynamics produced by Eq. (35). Due to the non-invertibility of function $y = \cos x$, it is impossible to determine the initial condition if the only observable is z_n.

A typical example of the dynamics generated by dynamical systems (33)–(35) is shown in Fig. 6.

When subjected to the same statistical analysis, the time-series generated by function (1) and variable z_n in dynamical system (33)–(35) are indistinguishable. Another application of these results can be illustrated by the input–output system shown in Fig. 7.

Function $x_n = P[\phi(n)]$, where $P(t)$ is a periodic function and $\phi(n)$ is a non-periodic oscillating function with intermittent intervals of truncated

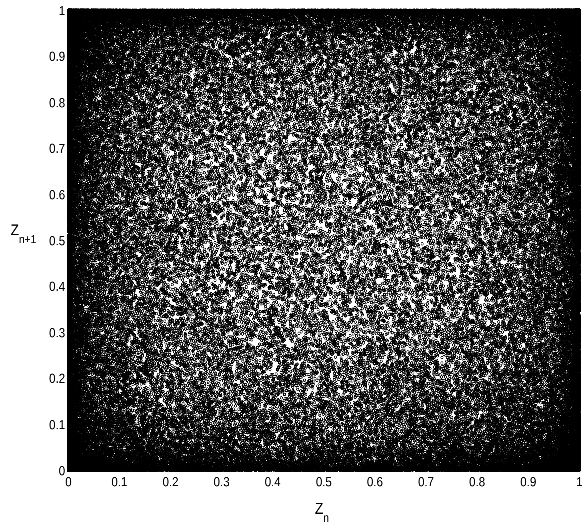

Fig. 6. First-return map constructed using the variable z_n in dynamical system (33)–(35). $a = \pi$, $b = 200$, $c = 70$, $Q = 500$.

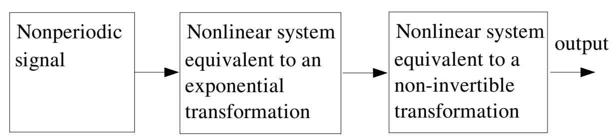

Fig. 7. Input–output system that can produce very complex dynamics.

exponential behavior, would produce also very complex dynamics. Similar results can be obtained with functions of type $x_n = h[\phi(n)]$, where $\phi(n)$ possesses the same properties as before, and $h(t)$ is a non-invertible function.

Function $\phi(n)$ can be produced by a chaotic system. Another example of this kind of function is the following

$$x_n = \cos[\phi(n)], \tag{36}$$

where $\phi(n) = A\exp[Q(n)]$, $Q(n) = a_1 \sin(\omega_1 n) + a_2 \sin(\omega_2 n) + a_3 \sin(\omega_3 n)$, $\frac{\omega_2}{\omega_1}$, $\frac{\omega_3}{\omega_2}$, $\frac{\omega_3}{\omega_1}$ are irrational numbers. An example of this dynamics can be observed in Fig. 8.

Systems like these can be realized in real experiments with electronic circuits. For instance, a quasiperiodic signal can be used as the input to an electronic circuit that simulates an exponential function. The output of the exponential circuit is used as the input to a circuit that simulates a sine function or cosine function [34].

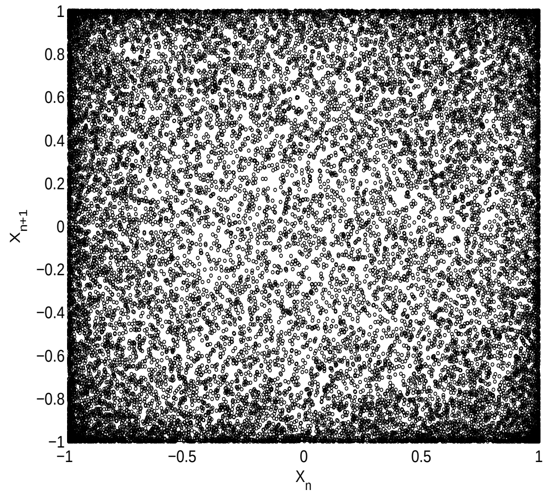

Fig. 8. First-return map constructed using the function (36).

Recently, an experimental setup that permits the direct realization of function

$$y_n = \sin^2[2\theta\pi z^n] \equiv 1 - \cos^2[2\theta\pi z^n] = 1 - x_n^2 \tag{37}$$

has been constructed [35].

Umeno *et al.*, [35] have created an optical device composed of several Mach–Zehnder interferometers for which Eq. (37) is the mathematical model. This development is related to secure communication technologies based on chaos.

5. Conclusions

The independent functions $f_1(t), \ldots, f_m(t)$ studied by Kac and Steinhaus could be periodic. They were independent in the sense that the proportions of times during which $f_1(t) < \alpha_1, \ldots, f_m(t) < \alpha_m$ behave as if they were probabilities of independent events. If we take *e.g.* two of them, one is independent of the other. But if we take one of them alone, it is a very simple dynamics. On the other hand, the dynamics generated by function (1) is very complex. It behaves as noise. It is difficult to distinguish it from a random phenomenon. Moreover, to speak about independence, we do not need to take unrelated sequences (*e.g.*, produced using different values of θ), which, in fact, can be independent.

We can speak about independence "inside" the same dynamics. That is, different subsequences of the same sequence $x_n = \cos[2\pi\theta z^n]$, for a given θ, are independent. For instance, x_n and x_{n+1} are independent. Their probabilities behave as the probabilities of independent events. This is what we would expect from a time-dependent random process.

In conclusion, we have shown that given the function $x_n = \cos[2\pi\theta z^n]$, where z is a typical real number, any string $x_s, x_{s+1}, x_{s+2}, \ldots, x_{s+r}$ (for any r) constitutes a set of statistically independent random variables.

In a forthcoming work we will present further developments that include the construction of different continuous functions that possess the properties of chaotic or stochastic processes. With the help of this theory, we hope to find analytical solutions to many nonlinear chaotic and stochastic systems. There are many open problems in this area. Some of them are related to the following subjects: spatiotemporal chaos [36–45], turbulence [46–51], Brownian motion (and related stochastic processes) [52], stochastic resonance [41, 53–61], just to mention a few.

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