THE PECULIARITY OF SELF-EXCITED OSCILLATIONS IN FRACTIONAL SYSTEMS

Aleksander A. Stanislavsky

Institute of Radio Astronomy
4 Chervonopraporna St., Kharkov 61002, Ukraine
alexstan@ira.kharkov.ua

(Received September 22, 2005; revised version received October 24, 2005)

In this paper we show that the compensation of loss in the linear fractional oscillator by an active device can result in auto-oscillations. Due to the main feature of linear fractional oscillations, namely a finite number of zeros, the limit cycle in such a generator has a short life time depending on the order of fractional derivative. The electronic circuit, leading to such auto-oscillations, is studied as an example, and its differential equation is derived. The active device characteristic is represented in the piecewise-linear approximation. The features of electric elements are discussed.

PACS numbers: 05.45.–a, 05.40.Fb, 07.50.–e

1. Introduction

The fractional generalization of harmonic oscillatory motion has attracted more attention in recent years [1–5]. After a formal study of some aspects of the fractional oscillator, basically from the mathematical point of view to the solution of a differential equation with fractional derivative [6,7], the main interest to this problem shifted onto physical aspects of the dynamical system such as the total energy and the dynamical response [9, 10]. The fractional oscillator behaves dynamically like a damped harmonic oscillator, but the similarity is only in appearance. If the damping in a damped harmonic oscillator rides on an external frictional force, the fractional oscillations damp because of entirely internal causes. It turns out [3] that the understanding of the intrinsic absorption arises from the interpretation of the fractional oscillator as an ensemble average of ordinary harmonic oscillators randomly interacting with environment. Since the harmonic oscillators differ slightly in frequency, then each response is compensated by an antiphase response of other oscillators. In this connection it should be pointed out that the interaction with environment is expressed in terms of the stochastic arrow of time [11]. This permits one also to reach a progress in one more issue. Formerly the equation of fractional oscillator was not obtained from any physical principles. Instead of the formal change of the second-order derivative in the harmonic oscillatory equation on a fractional derivative, the new approach derives the equation of fractional oscillator from the Hamilton formalism [4]. However, mainly the success concerns linear fractional systems. In fact, there are many cases in which linear treatments are not sufficient. The more general systems described by nonlinear fractional differential equations have been studied not enough [12–16]. The ordinary calculus brings out clearly that essentially new phenomena occur in nonlinear systems which cannot in principle occur in linear systems.

The purpose of this paper is to present the simplest case, when the fractional motion tends to self-excited or self-sustained fractional oscillations. Systems of this kind, when their differential equation are expressed in terms of integer-order derivatives, are very common in nature. They occur always when a periodic motion is maintained through absorption of energy from a constant flow of energy. But the fractional calculus introduces a novelty to this matter. The fractional oscillations have a finite number of zero. The mathematical model of the fractional generator is studied in Sec. 2. The generator may be reproduced in the form of the electronic circuit. Just in Sec. 3 we consider this device. The brief discussion of the obtained results together with the future perspectives sums up our analysis in Sec. 4.

2. Mathematical model

There are many physical models composed of linear system solutions which result in nonlinear phenomena. One of the best examples is the Adronov–Vitt–Khaikin model [17], where the limit cycle arises from solutions of two linear differential equations describing damped oscillations:

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x \;=\; \omega_0^2 g\,, \quad \dot{x} > 0\,, \tag{1}$$

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x \;=\; 0\,, \qquad \dot{x} \leq 0\,, \tag{2}$$

where g, k, ω_0 are constant. The presence of the right-hand term in Eq. (1) is interpreted as a driving force. The solution of Eq. (2) takes the form

$$x(t) = Re^{-kt}\cos(\gamma t + \theta)\,, \qquad \gamma = \sqrt{\omega_0^2 - k^2}$$

with an arbitrary phase θ. By the substitution of variables $y = x - g$ the Eq. (1) reduces to Eq. (2) in the variable y. The phase trajectory outgoing from the point $(x_1(t), 0)$ for $\dot{x} \leq 0$ is rotated around the point $(0, 0)$ whereas for $\dot{x} > 0$ it is winded around the point $(g, 0)$. As a result, there exists a close loop crossing the positive semi-axis x in $x_c = g/(1-\rho)$, where $\rho = \exp(-k\pi/\gamma) < 1$. In fact, this model describes a triode generator [17,18].

Recently, the clear physical interpretation has been established for a linear fractional oscillator [3]. Briefly recall it. The fractional oscillator results from an ensemble average of identical harmonic oscillators noninteracting with each other, but interacting with environment. It should be pointed out that the dispersion properties of the fractional oscillator is enough similar to the case described by an ensemble of harmonic oscillators with damping.

Now we modify the system of Eqs. (1), (2) to the fractional form:

$$D^\alpha x + \omega_0^\alpha x \ = \ \omega_0^\alpha g, \quad \dot{x} > 0\,, \tag{3}$$

$$D^\alpha x + \omega_0^\alpha x \ = \ 0\,, \qquad \dot{x} \leq 0\,, \tag{4}$$

with $1 < \alpha < 2$. Here the fractional operator D^α is supposed from the definition [19]:

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau\,, \qquad n-1 < \alpha < n\,,$$

where $x^{(n)}(t)$ means the n-derivative of $x(t)$, and $\Gamma(z)$ is the gamma function. Then Eq. (4) describes fractional damped oscillations. The substitution $y = x - g$ reduces Eq. (3) to Eq. (4). In fact, the system of Eqs. (3), (4) describes a dissipative model. It is reasonable to ask what behavior is demonstrated by the dynamical system in comparison with the Adronov–Vitt–Khaikin's case. Here we are going to clear up this question. The model from Eqs. (3), (4) will be regarded as a peculiar generator. The sketch of its electronic scheme is shown in the next section.

For the sake of simplicity we choose the initial conditions with $\dot{x}(0) = 0$. Then the solution of Eq. (4) is written as

$$x(t) = x(0) E_\alpha(-\omega_0^\alpha t^\alpha)\,, \tag{5}$$

where $E_\alpha(z)$ is the one-parameter Mittag–Leffler function [20]. According to [7,8], for $1 < \alpha < 2$ the function $e_\alpha(t) = E_\alpha(-t^\alpha)$ can be decomposed into two parts. In other words, it may be expressed in terms of a simple sum [8]. The first contribution gives a completely monotonic function tending to zero as t tends to infinity. The second part has an oscillatory character with an exponential decay. Owing to two contributions, this fractional model exhibits a finite number of damped oscillations. It is easy to show that the same is in the case for the derivative $\dot{x}(t)$.

Really, the derivative of the Mittag–Leffler function $\dot{e}_\alpha(t)$ is also decomposed into two parts:

$$\dot{f}_\alpha(t) = -\frac{1}{\pi}\int_0^\infty e^{-rt}\,\frac{r^\alpha \sin\pi\alpha}{r^{2\alpha}+2r^\alpha\cos\pi\alpha+1}\,dr\,, \tag{6}$$

$$\dot{g}_\alpha(t) = \frac{2}{\alpha}\,e^{t\cos(\pi/\alpha)}\,\cos[t\sin(\pi/\alpha)+\pi/\alpha]\,. \tag{7}$$

The two terms compete with each other. Following this competition, the function $\dot{e}_\alpha(t)$ has zeros. Since $\dot{f}_\alpha(t)$ and $\dot{g}_\alpha(t)$ have a different character of decay, the number of the zeros is finite. For sufficiently large t the zeros of $\dot{e}_\alpha(t)$ may be found approximately from the equation

$$\frac{2}{\alpha}\,e^{t\cos(\pi/\alpha)} \approx \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}\,, \tag{8}$$

where the envelope of the expression (7) is compared with the first term from the asymptotic expansion of $\dot{f}_\alpha(t)$.

It is evident that only one zero is present for the derivative $\dot{e}_1(t)$. Now putting $\alpha = 1+\varepsilon$, in the limit $\varepsilon \to 0$ the first-order approximation gives

$$\Gamma(-\alpha) = -\frac{\Gamma(-\varepsilon)}{1+\varepsilon} \sim \frac{1}{\varepsilon}\,, \qquad \cos(\pi/\alpha) \sim -1\,.$$

The asymptotic position T of the zero having the largest argument is determined from Eq. (8) so that

$$e^{-T} \sim \frac{\varepsilon\,(1+\varepsilon)}{2}\,T^{-2-\varepsilon}\,.$$

This implies that

$$T \sim \ln\left(\frac{2}{\varepsilon}\right) + 2\ln(T)\,.$$

From that it is seen that for $\varepsilon \to 0$ the value T tends to infinity. Since the term T increases faster than $\ln(T)$, for the sake of simplicity one may neglect $\ln(T)$ with respect to T. Then we get

$$T \sim \ln\left(\frac{2}{\varepsilon}\right) \qquad \text{or} \qquad \varepsilon \sim 2\,e^{-T}\,.$$

The expression supports the tendency of T to infinity as $\varepsilon \to 0$.

When the index α changes from 1 to 2, the number of zeros of the function $\dot{e}_\alpha(t)$ increases up to infinity since $\dot{e}_2(t) = -\sin(t)$ has infinitely many zeros. Here we also estimate the value T (and the total number N of zeros) when $\alpha = 2-\delta$ in the limit $\delta \to 0$. In the framework of the first-order approximation we find

$$\Gamma(-\alpha) = -\frac{\Gamma(-1+\delta)}{2-\delta} \sim \frac{1}{2\delta}\,, \qquad \cos(\pi/\alpha) \sim -\frac{\pi\delta}{4}\,.$$

From Eq. (8) it follows that

$$e^{-\pi\delta T/4} \sim \delta\,(2-\delta)\,T^{-3+\delta}$$

from which

$$\frac{\pi\delta T}{4} \sim \ln\left(\frac{1}{2\delta}\right) + 3\ln(T)\,. \tag{9}$$

Since both terms in the right-hand side of Eq. (9) diverge, the production δT tends to infinity too. However, T and $1/\delta$ are of the same order. Therefore, we put either $T \sim -(A/\delta)\ln(2\delta)$ or $\delta \sim B\ln(T)/T$. Next, it is easy to define $A = B = 16/\pi$. The asymptotic expression (9) takes the equivalent form

$$T \sim \frac{16}{\pi\delta}\ln\left(\frac{1}{2\delta}\right)\,, \qquad \text{or} \qquad \delta \sim \frac{16}{\pi}\,\frac{\ln(T)}{T}\,.$$

For $\delta \to 0$ the length of the positive intervals of $\dot{g}_\alpha(t)$ becomes equal to π. Thus, the total number of zeros tends to $N \sim T/\pi$ in the limit $\delta \to 0$.

Next we use the recurrent formula [21]

$$\frac{dE_{\alpha,\beta}(z)}{dz} = \frac{E_{\alpha,\beta-1}(z) - (\beta-1)E_{\alpha,\beta}(z)}{\alpha z}\,,$$

where $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}(z^n/\Gamma(\alpha n+\beta))$ is the two-parameter Mittag–Leffler function so that $E_{\alpha,1}(z) = E_\alpha(z)$. Then the calculation of the first-order derivative of the Mittag–Leffler function is reduced to the calculation of the Mittag–Leffler functions with various indices. The accurate numerical algorithm for calculating the two-parameter Mittag–Leffler function is described in [22]. This approach helps to simulate the function $\dot{e}_\alpha(t)$ numerically. An example of this numerical treatment is presented in Fig. 1. The results support our analytical estimations.

The main conclusion of our analysis is that the derivative $\dot{x}(t)$ contains a finite number of damped oscillations with an algebraic decay. In this connection it should be pointed out that each solution of two linear differential equations of the Adronov–Vitt–Khaikin model has an infinite number of damped oscillations. This permits the nonlinear term to excite a self-oscillation, and the phase portrait shows a limit cycle. The system described

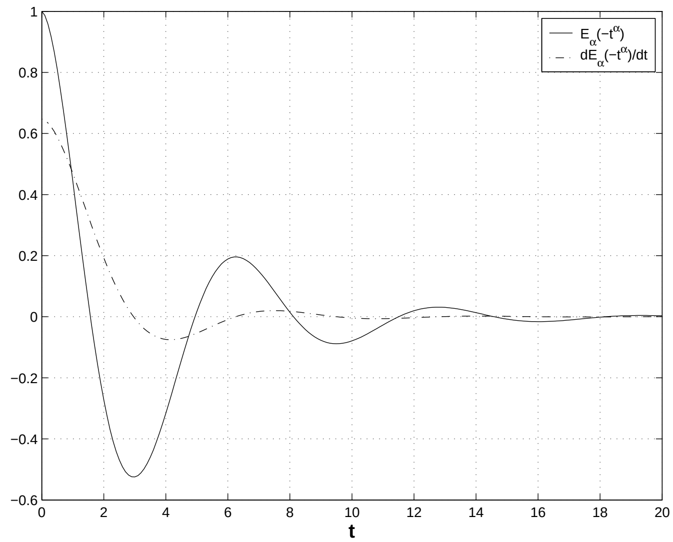

Fig. 1. Fractional oscillation $e_\alpha(t)$ and its derivative for $\alpha = 1.7$.

by Eqs. (3), (4) behaves in another way. The cause just consists in the number of zeros. A finite number of zeros cannot lead to an ordinary limit cycle. The threshold level (equal to zero in Eqs. (3), (4)) of $\dot{x}(t)$, switching the linear equations, is not of a vital importance. This dynamical system may generate only a short-living limit cycle. The numerical simulation of the self-oscillations is presented in Fig. 2. The system is qualitatively stable to small changes of its parameters like a threshold level, an index α and initial conditions.

The analysis of Eqs. (3), (4) will not be complete, unless one considers the condition (relating the sign of the derivative $\dot{x}(t)$ to a prescribed form of the Eqs. (3), (4)). It is easy to see that the condition is similar to the one in Eqs. (1), (2). In this particular case, the use of the dependence is fully legitimate [17, 18]. The derivative $\dot{x}(t)$ has a clear physical sense, namely velocity or momentum. It would be natural to use also a similar argument of the velocity (momentum) dependence of the dynamic equation in cases (3) and (4). The generalized momentum of the fractional oscillator is proposed in [9]. In this case such a momentum p is defined so that the expression $p^2/(2m)$ has the dimension of energy. Thus, the condition for the sign of $\dot{x}(t)$ may be replaced by the sign dependence of $D^{\alpha/2}x(t)$. Substituting the expression of $x(t)$ under the fractional derivative, we obtain

$$D^{\alpha/2}e_\alpha(t) = -t^{\alpha/2}E_{\alpha,\,1+\alpha/2}(-t^\alpha)\,.$$

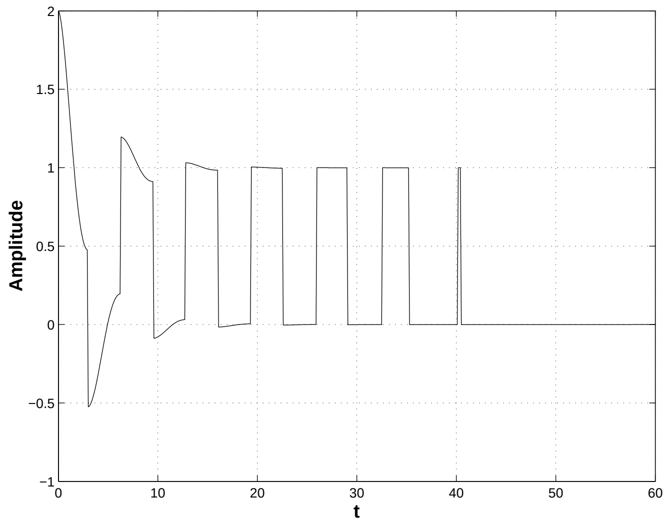

Fig. 2. Finite number of self-oscillations in the dynamical system following Eqs. (3), (4) with $\alpha = 1.7$.

Following the arguments of [4], the number of the zeros in $\dot{x}(t)$ turns out to be finite. Thus, the main conclusion of our paper remains true. The fractional oscillator under the piecewise-linear approximation generates a short-living limit cycle.

3. Electronic scheme of fractional generator

Now we consider an electric circuit represented in Fig. 3. It contains an active component (for example, either triod vacuum tube or field electronic transistor) and an oscillatory circuit. The oscillatory circuit consists of a coil of inductance L, a resistance R, and a condenser of capacitance C all in parallel. By means of the coupling coil L' it governs by the active component. The control potential is provided by a mutual inductance M, as indicated. Sometimes such a circuit is called feedback.

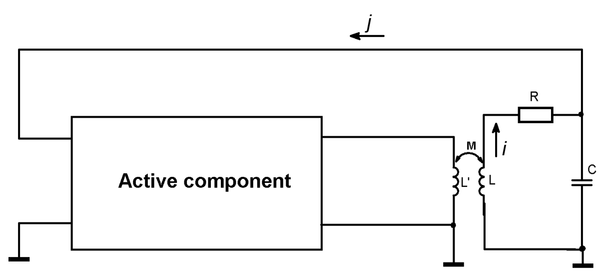

Fig. 3. Oscillator circuit with a piecewise-linear characteristic of the active component.

Assume that the current through the mutual inductance may be neglected, and the values R, L, C are constant. If v_L, v_C, v_R are the drop potentials through L, C, R, respectively, the Kirchhoff law of voltages gives $v_L = v_C + v_R$. The resistance voltage v_R is iR, the current $j = C dv_C/dt$, and the drop potential v_L through the inductance coil equal to $L di/dt$. The currents i, j obey the equation $i + j = f(v)$, where the function $f(v)$ is sometimes called the characteristic (of tube, for example). This function determines the current which flows through the active component depending on the control potential value $v = M di/dt$. If f takes the form of a step function, then we obtain

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{d\,i}{dt} + \frac{i}{LC} = \begin{cases} i_0/(LC)\,, & \text{if } di/dt > 0\,, \\ 0\,, & \text{if } di/dt \le 0\,. \end{cases}$$

This differential equation of second order exactly corresponds to the Adronov–Vitt–Khaikin model [17, 18], briefly described in the previous section.

From the analysis of the electric circuit (Fig. 3) it follows that the characteristic $f(v)$ depends on features of the active component, whereas the differential form of the equation is determined merely by the oscillatory circuit. In particular, the resistance R characterizes dissipative processes in the model, and the frequency of oscillations is calculated by the relation $1/\sqrt{LC}$. This case expects idealized properties of electrical elements (capacitor and coil) so that their voltage (or current) evolution is expressed in terms of the temporal derivative of the first order. However, this is not necessarily so. As is shown, for example, in [23], no ideal capacitor exists in nature. The condenser C is discharged exponentially because of the exponential relaxation of charges (Debay law) in the dielectric between capacitor plates. If the relaxation adheres to another response, then the temporal evolution of the capacitor discharge takes a nonexponential form. In particular, for the Cole–Cole response the polarization, induced in such a dielectric medium, satisfies the fractional-order differential equation [24, 25]. It is relevant to remark here that the fractional capacitor theory started with Curie's empirical law (1889), describing the power relaxation of current in a capacitor [26]. Later the fractional-order capacitor models were developed in [27, 28]. Now the term *fractance* has been generally recognized. It denotes an electric element with a non-integer impedance [29].

Let an appropriate dielectric material be located between capacitor plates so that the discharge rate $j_\alpha(t)$ from the electric element and the potential difference $v_\alpha(t)$ between the plates are written as

$$\mathcal{C}\, D^\alpha v_\alpha(t) = j_\alpha(t)\,,$$

where $\mathcal{C}$ is the generalized capacity (constant). Assume that the ferromagnetic core of the inductance coil consists of magnetic domains having a similar response of magnetic susceptibility. This leads to the following relation for the current and the drop of potential through the inductance coil, namely

$$\mathcal{L}\, D^{\mu} i_{\mu}(t) = u_{\mu}(t)\,,$$

where $\mathcal{L}$ is the generalized inductance (constant). For the sake of mathematical simplicity, the resistance R will be ignored. Notice that the fractional derivative itself already accounts for dissipative effects. Next set $\alpha = \mu > 1/2$. Since $v_{\alpha}(t) = u_{\alpha}(t)$, the current j_{α} is expressed in terms of $\mathcal{C}D^{\alpha}v_{\alpha}(t) = \mathcal{C}D^{\alpha}u_{\alpha}(t) = \mathcal{L}\mathcal{C}D^{2\alpha}i_{\alpha}(t)$. The sum of currents $j_{\alpha}(t) + i_{\alpha}(t)$ equals to the characteristic of the active component $f(v)$. Using the interrelation, we arrive at the equation describing the electronic generator:

$$D^{2\alpha} i_{\alpha}(t) + \frac{i_{\alpha}(t)}{\mathcal{L}\mathcal{C}} = \begin{cases} g\,, & \text{if } D^{\alpha} i_{\alpha}(t) > 0\,, \\ 0\,, & \text{if } D^{\alpha} i_{\alpha}(t) \le 0\,, \end{cases}$$

where $g = i_{\alpha}(0)/(\mathcal{L}\mathcal{C})$ is the constant determined by the characteristic of the active component. The value $1/\sqrt{\mathcal{L}\mathcal{C}}$ describes the frequency of free fractional oscillations. Recall that the fractional oscillations have a finite number of zeros. Note that the sign condition in the latter equation may take another form, namely $di_{\alpha}/dt > 0$ and $di_{\alpha}/dt \le 0$. This depends on a coupling coil and its mutual inductance.

A different model of the fractional oscillator was studied by Heaviside (1922) and Bush (1929) [30–32] who analyzed a semi-infinite lossy transmition line terminated by an inductor. However, this circuit gives only one fixed index 3/2 for the fractional order of derivative.

4. Summary

We have established that the fractional oscillatory circuit may generate nonlinear oscillations in the form of pulses. Their number directly depends on the order of derivative. The closer this order is to two, the more number of pulses is observed. This fact patently indicates that by means of such a generator one can estimate the order index of the fractional system. As for the piecewise-linear approximation for the active component characteristic, much of the engineering literature on this subject is based on the assumption that the characteristic can be taken as such a function without too much error. The influence of the characteristic non-linearity on the fractional generator work will be examined in future.

The author thanks the referee for his useful remarks.

REFERENCES

[1] Ya.E. Ryabov, A. Puzenko, *Phys. Rev.* **B66**, 184201 (2002).

[2] B.N. Narahari Achar, J.W. Hanneken, T. Clarke, *Physica* **A339**, 311 (2004).

[3] A.A. Stanislavsky, *Phys. Rev.* **E70**, 051103 (2004).

[4] A.A. Stanislavsky, *Physica* **A354**, 101 (2005).

[5] T.M. Atanackovic, M. Budincevic, S. Pilipovic, *J. Phys.* **A38**, 6703 (2005).

[6] F. Mainardi, *Chaos, Solitons Fractals* **7**, 1461 (1996).

[7] R. Gorenflo, F. Mainardi, *Fractional Oscillations and Mittag–Leffler Functions*, Proceedings of RAAM 1996, Kuwait University, Kuwait 1996, p. 193–196.

[8] A. Wiman, *Acta Math.* **29**, 217 (1905).

[9] B.N. Narahari Achar, J.W. Hanneken, T. Enck, T. Clarke, *Physica* **A287**, 361 (2001).

[10] B.N. Narahari Achar, J.W. Hanneken, T. Clarke, *Physica* **A309**, 257 (2002).

[11] A.A. Stanislavsky, *Phys. Rev.* **E67**, 021111 (2003).

[12] T. Hartley, C. Lorenzo, H. Qammar, *IEEE Transactions Circuits and Systems* **42**, 485 (1995).

[13] P. Arena, R. Caponetto, L. Fortuna, D. Porto, *Chaos in a Fractional Order Duffing System*, Proceedings of ECCTD'97, Budapest University, Budapest 1997, p. 1259–1262.

[14] I. Grigorenko, E. Grigorenko, *Phys. Rev. Lett.* **91**, 034101 (2003).

[15] X. Gao, J. Yu, *Chaos, Solitons Fractals* **24**, 1097 (2005).

[16] G.M. Zaslavsky, A.A. Stanislavsky, M. Edelman, `nlin.CD/0508018`.

[17] A.A. Andronov, A.A. Vitt, S.E. Khaikin, *Theory of Oscillators*, Pergamon, Oxford 1966.

[18] D. Arrowsmith, C. Place, *Ordinary Differential Equations: A Qualitative Approach with Applications*, Chapman & Hall, London 1982.

[19] M. Caputo, *J. Roy. Astral. Soc.* **13**, 529 (1967).

[20] A. Erdélyi, *Higher Transcendental Functions*, Vol. III, McGraw-Hill, New York 1955.

[21] M. Dzherbashyan, *Integral Transforms and Representations of Functions in the Complex Plane*, Nauka, Moscow 1966.

[22] K. Diethelm, N.J. Ford, A.D. Freed, Yu. Luchko, *Comp. Methods Appl. Mech. Eng.* **194**, 743 (2005).

[23] A.K. Jonscher, *Dielectric Relaxation in Solids*, Chelsea Dielectric Press, London 1983.

[24] K. Weron, A. Klauser, *Ferroelectrics* **236**, 59 (2000).

[25] P.G. Petropoulos, *IEEE Trans. on Antennas and Propagation*, (2005), to appear.

[26] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego 1999.

[27] G.E. Carlson, C.A. Halijak, *IRE Trans. on Circuit Theory* **CT-11**, 210 (1964).

[28] S. Westerlund, L. Ekstam, *IEEE Trans. on Dielectrics and Electrical Insulation* **1**, 826 (1994).

[29] A. Le Méhauté, G. Crepy, *Solid State Ionics* **9–10**, 17 (1983).

[30] T. Hartley, C. Lorenzo, `NASA/TP-1998-208693`.

[31] T. Hartley, C. Lorenzo, `NASA/TP-1999-208919`.

[32] T. Clarke, B.N. Narahari Achar, J.W. Hanneken, *J. Mol. Liq.* **114**, 159 (2004).