GRAVITATIONAL FIELD ENERGY CONTRIBUTION TO THE NEUTRON STAR MASS*

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Neutron stars are discussed as laboratories of physics of strong gravitational fields. The mass of a neutron star is split into matter energy and gravitational field energy contributions. The energy of the gravitational field of neutron stars is calculated with three different approaches which give the same result. It is found that up to one half of the gravitational mass of maximum mass neutron stars is comprised by the gravitational field energy. Results are shown for a number of realistic equations of state of neutron star matter.

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1. Introduction

Gravity makes all self-gravitating fluid bodies spherical. Thus the spherical symmetry of non-rotating neutron stars, which are real objects, is their intrinsic property, not an idealization. Neutron stars are relativistic objects, with mass of order $1M_\odot$ enclosed within 10–20 km. Gravity of such a compact mass distribution is strong enough to require relativistic description. In the following we focus on gravitational field contribution to the neutron star masses in General Relativity.

The problem of the energy distribution of gravitational field remains an unsettled issue in General Relativity [1] mainly because of the lack of proper energy-momentum tensor for gravitational field. However, for static,

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spherically symmetric space-time no controversy exists and relevant matter and gravity energies can be unambiguously defined [2,3]. Our main concern here is to study whether neutron stars could shed some new light on the problem of gravitational field energy and its localization.

To calculate both matter and gravitational field contributions to the neutron star mass we solve the Einstein's field equations for the static, spherically symmetric metric and a given equation of state of dense matter. The metric is

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right) , \tag{1}$$

where ν, λ are functions of the radial coordinate r.

Dense matter in neutron stars is thought to be in a liquid state, hence it is described by the perfect fluid energy-momentum tensor,

$$T^{\mu\nu} = (\rho c^2 + P) u^{\mu} u^{\nu} - P g^{\mu\nu} ,$$

where ρc^2 is the energy density, P is the pressure and u^{μ} is the four velocity. For the neutron star matter both energy density and pressure are uniquely determined by the baryon number density, $\rho = \rho(n_B), P = P(n_B)$, hence the equation of state is of barotropic form, $P = P(\rho)$. In this paper we employ a number of realistic equations of state, described in some details in Sec. 4.

Using the Einstein's equations one finds that functions λ and ν satisfy ordinary differential equations

$$\lambda'(r) = \frac{1}{r}\left(1 - e^{\lambda} + \frac{8\, e^{\lambda}\, G\, \pi\, r^2\, \rho}{c^2}\right) , \tag{2a}$$

$$\nu'(r) = \frac{1}{r}\left(-1 + e^{\lambda} + \frac{8\, e^{\lambda}\, G\, P\, \pi\, r^2}{c^4}\right) , \tag{2b}$$

$$P'(r) = -\frac{1}{2}\left(p + \rho c^2\right) \nu' . \tag{2c}$$

The metric (1) has to be continuous in the whole space. For this to happen, at the stellar boundary, $r = R$, the internal metric has to go over into the external Schwarzschild metric. There is a vacuum outside the star and the matter energy-momentum tensor vanishes. The metric functions at the boundary should satisfy the condition $e^{\nu(R)} = e^{-\lambda(R)} = 1 - 2GM/Rc^2$, where M is the mass of the star.

From Eqs. (2) one derives the Tolman–Oppenheimer–Volkoff (TOV) equation that describes nonrotating compact objects in the hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2} \times \left(1+\frac{P(r)}{c^2\rho(r)}\right)\left(1+\frac{4\pi r^3 P(r)}{c^2 M(r)}\right)\left(1-\frac{2GM(r)}{c^2 r}\right)^{-1}. \quad (3)$$

Here we denote by $M(r)$ the integral $M(r) = \int_0^r \rho(r')d^3r'$. Usually $M(r)$ is referred to as "the mass within the radius r". This name is rather misleading, as it suggests that only matter energy density is responsible for building the neutron star mass $M = M(R)$. It is our prime goal here to calculate the gravitational energy contribution to M. In the following we shall be using the differential form of the $M(r)$ definition,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)\,. \quad (4)$$

Choosing an appropriate equation of state of dense matter,

$$P = P(\rho)\,, \quad (5)$$

and the boundary conditions

$$\rho(r=0) = \rho_{\rm c}\,, \quad \text{and} \quad P(r=R) = 0\,, \quad (6)$$

we solve this set of equations for different values of central density, $\rho_{\rm c}$, obtaining the whole family of configurations of compact objects in hydrostatic equilibrium. We report here calculations for four equations of state, which we denote as A18, A18+δv+UIX*, FPR and MS (see Sec. 4). These four EOS have been chosen from about twenty EOS we studied solely to make the plots clear. The results of our calculations are shown in Fig. 1.

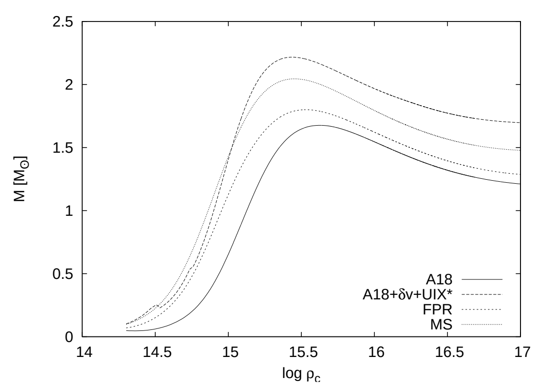

Fig. 1. Neutron star masses *versus* the central density for four equations of state.

Numerical values of the neutron stars masses for these equations of state are given in the Table I. We show parameters corresponding to the standard neutron star mass, $M = 1.44M_{\odot}$, and to the maximum mass neutron star, which is different for every equation of state.

TABLE I

Computed neutron star masses. For each equation of state the first line shows the value of the central density of 1.44 $M_{\odot}$ neutron star, and in the second line the central density of the maximum mass neutron star and the value of this mass are given.

Equation of state	Central density [10^{15}g/cm^3]	Mass [$M_{\odot}$]
A18	2.54	1.44
A18	4.23	1.68
A18+δv+UIX*	1.02	1.44
A18+δv+UIX*	2.74	2.22
FPR	1.36	1.44
FPR	3.38	1.80
MS	1.01	1.44
MS	2.84	2.04

2. Gravitational field energy of the compact objects

Boundary conditions at the surface of the neutron star, $r = R$, ensure that the external metric is just the Schwarzschild metric with the mass M. There is no more matter contributions to M at $r > R$ since the matter energy density is zero outside the star. Hence the same mass M survives in the metric functions in the asymptotic region, $r \to \infty$, and M is the total gravitational mass of the neutron star (sometimes called the ADM mass).

In General Relativity the mass of the star can be split into two parts: the matter energy and the gravitational field energy [4], $Mc^2 = E_{\rm Matter} + E_{\rm Field}$. Formally we can write

$$Mc^2 = 4\pi \int\limits_0^R \left(T_0^0 + t_0^0\right) e^{\nu/2} e^{\lambda/2} r^2 dr \,, \tag{7}$$

where T_0^0 is the energy density component of the matter energy-momentum tensor, and t_0^0 is the energy density of the gravitational field pseudotensor.

The matter energy is easily calculated for any given neutron star, as all three functions, $\rho(r), \lambda(r), \nu(r)$ entering the integral,

$$E_{\rm Matter} = 4\pi c^2 \int_0^R \rho e^{\nu/2} e^{\lambda/2}\, r^2 dr\,, \qquad (8)$$

are determined by solving Eqs. (2). One should notice that not only the integral (8) is uniquely determined, but also matter energy density is completely known, $\rho_{\rm Matter} = \rho e^{\nu/2}$. This energy density includes contributions from nuclear and electromagnetic interactions (through ρ), as well as the gravitational interaction (through $e^{\nu/2}$). It is important to stress that, in the Newtonian limit, the generally relativistic matter energy, $E_{\rm Matter}$, includes twice the (negative) potential energy contribution, as shown by Lynden-Bell and Katz [5].

Using this definition, Eq. (8), we can compute the matter energy for all equations of state we study. The results of our calculations are shown in the Fig. 2 as functions of the total gravitational mass.

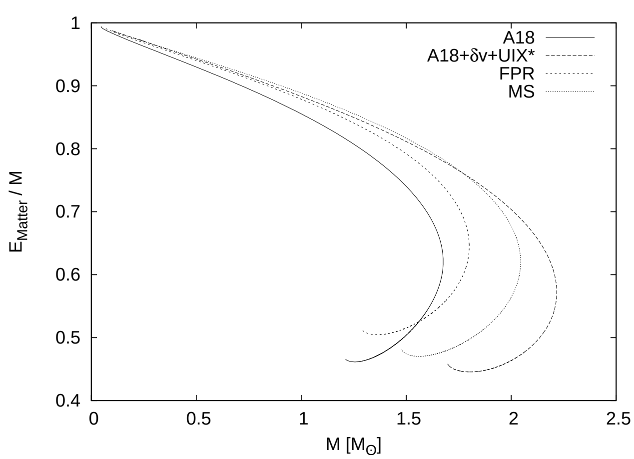

Fig. 2. Matter energy contribution to the neutron star mass as a function of the total gravitational mass.

One should perhaps stress once again, that for a given equation of state and a chosen central density, the matter part of the total mass integral (7) is fully determined. For spherically symmetric metric there does not exist any ambiguity as far as the integral (8) is concerned, despite the pseudotensorial nature of the gravitational field energy density t_0^0. The Eqs. (2) are fully immune to the particular choice of the gravitational field energy-momentum pseudotensor.

Having the matter energy $E_{\rm Matter}$ we can easily find the gravitational field energy contribution to the neutron star mass,

$$E_{\rm Field} = Mc^2 - E_{\rm Matter} = 4\pi c^2 \int\limits_0^R \left(e^{-\lambda/2} - e^{\nu/2}\right) \rho e^{\lambda/2} r^2 \, dr \,, \tag{9}$$

where, formally, we can refer to the function under the integral as the "gravitational field energy density",

$$\rho_{\rm Field} = \left(e^{-\frac{\lambda}{2}} - e^{\frac{\nu}{2}}\right) \rho \,, \tag{10}$$

in units of $\mathrm{g/cm^3}$. Unfortunately, in contrast to the gravitational field energy contribution to the neutron star mass, $E_{\rm Field}$, which is fully determined from Eq. (9), this function is not unique. The reason is that t_0^0 is not unique, only the integral in Eq. (9) has a well defined meaning.

In the following we calculate the gravitational field energy contribution to the neutron star mass in terms of often used pseudotensors (which are discussed below in some details). From Eq. (16) we have the field energy in the Einstein–Tolman prescription

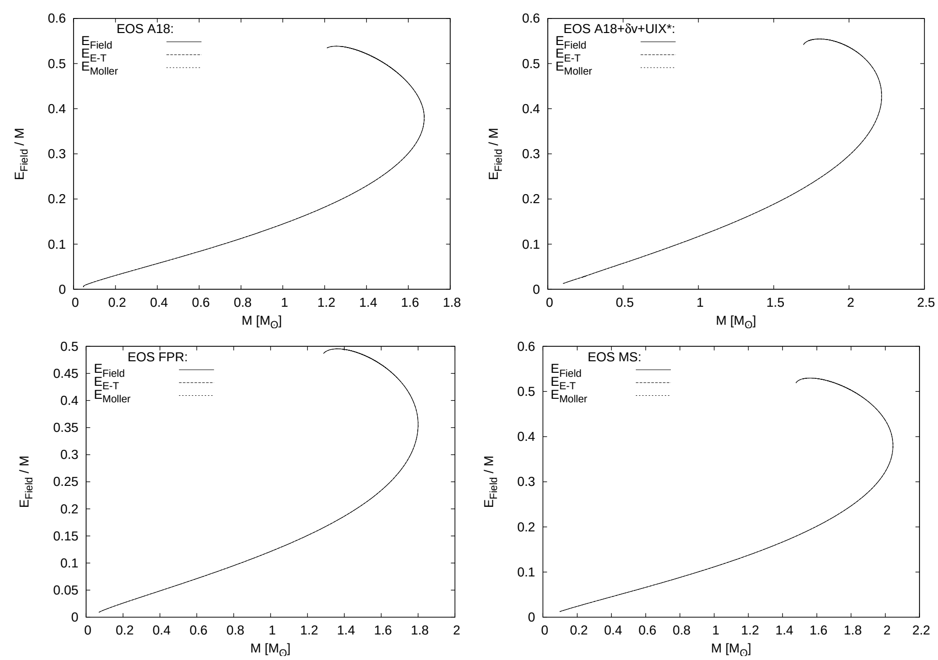

Fig. 3. Gravitational field energy of the neutron star. Curves, corresponding to all three definitions of the gravitational field energy (9), (11) and (12) coincide.

$$E_{\rm Field}^{\rm E-T} = \frac{4\pi G}{c^2} \int\limits_0^R e^{\nu/2}\, M(r) \left(\frac{1}{1 - 2\,G\,M(r)/c^2\,r}\right)^{3/2} \left(P + c^2\,\rho\right) r\, dr\,. \quad (11)$$

Using the Møller pseudotensor (19) we find the gravitational field energy in the form

$$E_{\rm Field}^{\rm Moller} = 12\pi \int\limits_0^R e^{\nu/2}\, P\, \sqrt{\frac{1}{1 - 2\,G\,M(r)/c^2\,r}}\; r^2\, dr\,. \quad (12)$$

The definitions of the gravitational field energy of the neutron star Eqs. (9), (11) and (12) give the same value of the field energy contribution. This holds for all equations of state we study as it is shown in Fig. 3.

Now we can compare contributions of the matter energy and the gravitational field energy to neutron star masses. In Fig. 4 we show both energies for all equations of state we use. For low masses of neutron stars the matter energy dominates. For "canonical" neutron star mass $M = 1.44 M_\odot$ the gravitational field energy contributes about 20% to the total gravitational mass.

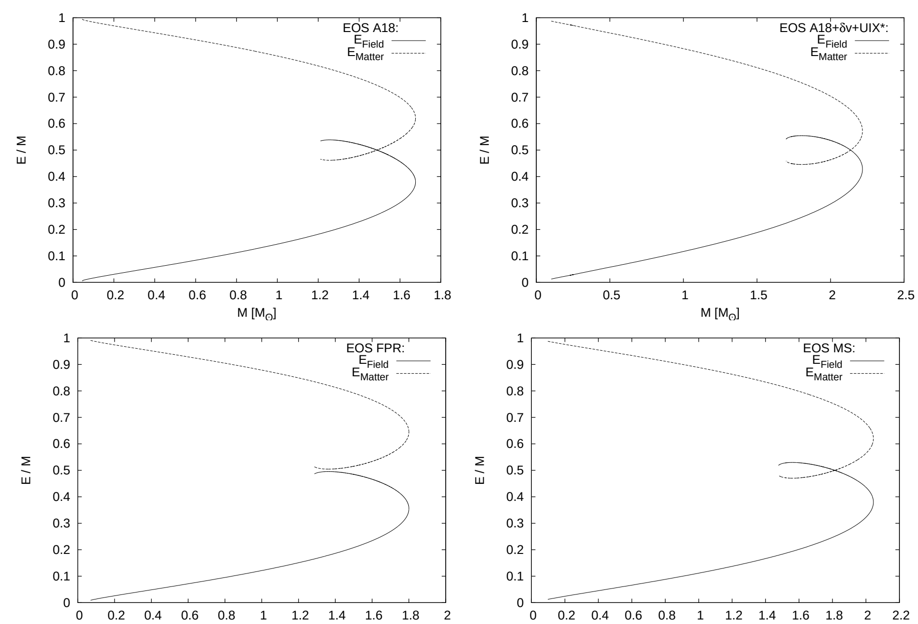

Fig. 4. Comparison of the gravitational field energy and matter energy of neutron star.

This contribution increases significantly for maximum mass neutron stars, exceeding 40% for the A18+δv+UIX* EOS. Crossing point of the curves in Fig. 4 indicates the star for which half of the gravitational mass is comprised by energy of its gravitational field. One can see that this occurs for three equations of state, albeit for unstable stars.

For completeness we provide here some formulae for two pseudotensors we used in our calculations. We tried a number of gravitational field pseudotensors, available in the literature. In particular we tried Einstein–Tolman, Møller, Landau–Lifshitz and Weinberg pseudotensors. However, not all of them were suitable for our calculations. To obtain the formulae given below we calculated all the components in Gallilean coordinates (see Appendix A) and employed the GRTensor program to perform the algebra. Eventually, we performed numerical integration indicated in Eqs. (8), (9), (11), (12) for Einstein–Tolman and Møller pseudotensors.

2.1. Einstein–Tolman pseudotensor

The energy-momentum pseudotensor of Einstein and Tolman [4] is given by the formulae:

$$\theta^{\nu}_{\mu} = \frac{c^4}{16\pi G} H^{\nu\gamma}_{\mu}{}_{,\gamma} \,, \tag{13}$$

where $\theta^{\nu}_{\mu} = \sqrt{-g}(t^{\nu}_{\mu} + T^{\nu}_{\mu})$, and

$$H^{\nu\gamma}_{\mu} = \frac{g_{\mu\rho}}{\sqrt{-g}} \left[-g\left(g^{\nu\rho}g^{\gamma\sigma} - g^{\gamma\rho}g^{\nu\sigma}\right)\right]_{,\sigma} \,. \tag{14}$$

Here g is the determinant of the metric tensor. This pseudotensor satisfies the local conservation laws:

$$\frac{\partial \theta^{\nu}_{\mu}}{\partial x^{\nu}} = 0\,. \tag{15}$$

For the metric (1) we find the energy density for the Einstein–Tolman pseudotensor to be

$$t^{0}_{0} = \frac{G\,M(r)\,\left(P + c^2\,\rho\right)}{-2\,G\,M(r) + c^2\,r}\,. \tag{16}$$

Some technical details of the calculations are shown in Appendix B. The same result was obtained in Ref. [6].

2.2. Møller pseudotensor

The energy-momentum complex of Møller [7] is given by the expression

$$M^{\nu}_{\mu} = \frac{c^4}{8\pi G}\, \chi^{\nu\gamma}_{\mu}{}_{,\gamma} \,, \tag{17}$$

where $M_\mu^\nu = \sqrt{-g}(t_\mu^\nu + T_\mu^\nu)$ and the antisymmetric tensor $\chi_\mu^{\nu\gamma}$ is

$$\chi_\mu^{\nu\gamma} = \sqrt{-g}\,(g_{\mu\sigma},_\rho - g_{\mu\rho},_\sigma)\, g^{\nu\rho} g^{\gamma\sigma}\,. \tag{18}$$

For metric (1) we have:

$$t_0^0 = 3\,P\,. \tag{19}$$

Some useful formulae are shown in the Appendix C.

2.3. Landau pseudotensor

The Landau pseudotensor reads [8]:

$$L^{\mu\nu} = \frac{c^4}{16\pi G} S^{\mu\nu\rho\sigma},_{\rho\sigma}\,, \tag{20}$$

where $S^{\mu\nu\rho\sigma} = -g(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})$ and $L^{\mu\nu} = -g(T^{\mu\nu} + t^{\mu\nu})$. The energy density is then given by

$$t_0^0 = \frac{c^2\,G\,M(r)\,\left[M(r) - 4\,\pi\,r^3\,\rho\right]}{2\,\pi\,r^3\,\left(2\,G\,M(r) - c^2\,r\right)}\,. \tag{21}$$

One can notice that this energy density becomes negative outside the stellar radius. This is unphysical and we discard the Landau pseudotensor in the following. The fact that the Landau pseudotensor gives the negative energy density of the gravitational field of the Schwarzschild metric was first recognized by Virbhadra [9]. One could probably attribute this behavior to a peculiar dependence on $-g$, as observed by Cooperstock and Rosen [10]. Some details of derivation of the t_0^0 component of the Landau–Lifshitz pseudotensor, are given in Appendix D .

As far as the Weinberg's pseudotensor is concerned, the relevant components vanish (Appendix E).

3. The problem of localization of the gravitational field energy

The fact that all three definitions of gravitational field energy of neutron stars give the same value is reassuring. One can be quite confident that what we find is really the prediction of the General Relativity theory as to the gravitational energy field contribution to the neutron star mass.

There is another important conclusion to be drawn: In all cases we consider the gravitational field energy is rather well localized. To show this we plot in Figs. 5 and 6 "gravitational field energy density" for all three definitions we use. Although these are not to be understood as physically meaningful, plots in Figs. 5 and 6 show that they share the same feature: namely all densities are localized within the star.

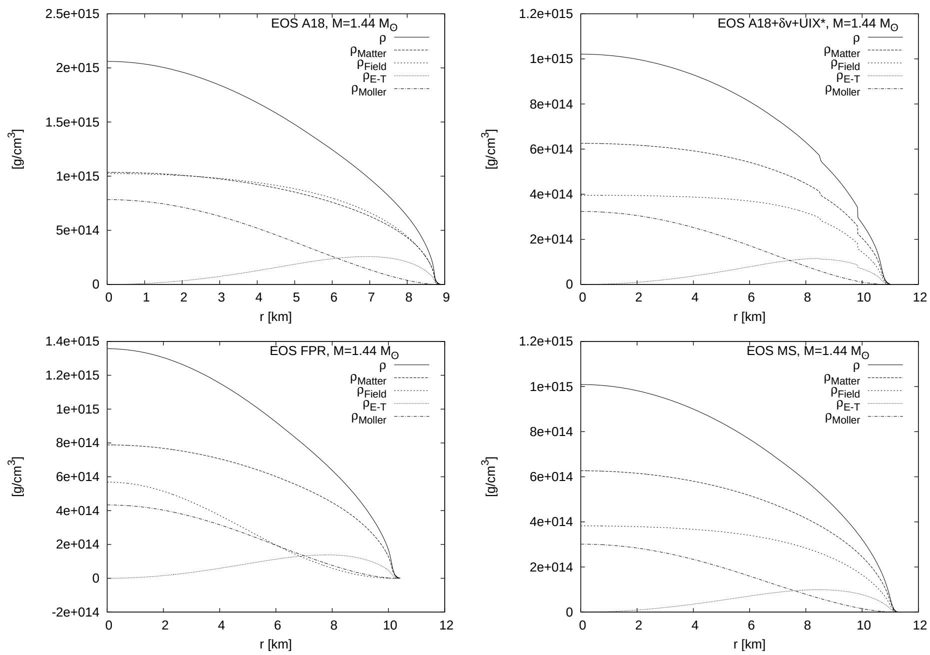

Fig. 5. The energy density of the neutron star gravitational field according to different definitions for the canonical mass neutron star $1.44\,M_\odot$.

The first density corresponds to our definition of the field energy (9) and is given by Eq. (10). The other two gravitational field energy densities correspond to Einstein–Tolman pseudotensor

$$\rho^{\mathrm{E-T}}_{\mathrm{Field}} = \frac{e^{\nu/2} G\,M\,\left(P + c^2\,\rho\right)}{-2\,G\,M + c^2\,r} \tag{22}$$

and to the Møller pseudotensor

$$\rho^{\mathrm{Moller}}_{\mathrm{Field}} = 3\,e^{\nu/2} P\,. \tag{23}$$

In Figs. 5 and 6 we show the results for “canonical” neutron star and for maximum mass neutron stars, respectively.

The fact that the whole gravitational field energy is accounted for within the star radius is fully consistent with the findings of Nahmad-Achar and Schutz [11] who found that the energy associated with the Schwarzschild space-time was independent of radius for radii exceeding the radius of spherically symmetric mass. We thus arrive at the conclusion that for spherically symmetric compact objects with Einstein–Tolman and Møller gravitational field pseudotensors, the whole gravitational mass is localized within the star,

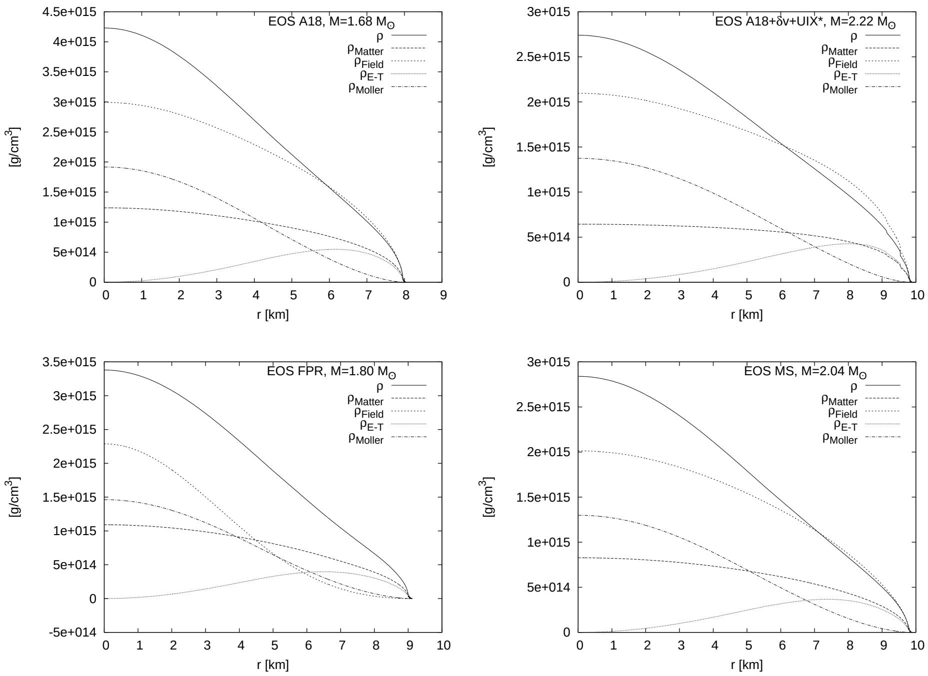

Fig. 6. The same as in Fig. 5 for maximum mass neutron stars.

including the gravitational field contribution which for neutron stars is substantial — reaching 40% of the total mass for maximum mass neutron stars.

In this case the total gravitational mass, M, which is the mass of the exterior Schwarzschild metric has all the contributions inside the stellar radius, and thus can be used for calculations of such properties as the last stable orbit around the neutron star, for sufficiently massive stars. However, there is some unusual property of the gravitational field in this approach: the field just below the surface of the star has the energy which contributes to the stellar mass, whereas above the stellar surface the field does not contribute to the neutron star mass: no gravitational field energy is localized outside the stellar surface.

4. Equations of state and nuclear models

In this section equations of state of dense matter, used for calculating curves in the above plots, are briefly discussed.

The density dependence of the pressure $P(\rho)$ plays the crucial role in calculations of neutron star models. The mass density of matter, ρ, includes kinetic and interaction energy density of the nucleons and rest masses of neutrons and protons, $\rho(n_B) = \varepsilon(n_N, n_P) + n_N m_N + n_P m_P$, while the pressure of the nuclear fluid is

$$P\left(n_B\right) = n_B^2\,\frac{d\left(\varepsilon/n_B\right)}{dn_B}\,. \tag{24}$$

The baryon number density, $n_B = n_N + n_P$, is the sum of the neutron and proton number densities.

We chose four realistic nuclear interaction models which give the energy density as a function of the baryon number density, $\epsilon(n_B)$. These are interactions derived by Myers and Swiatecki [12] (MS), the Friedman and Pandharipande interaction model [13] (as parametrized by Ravenhall in Ref. [14]) (FPR) and two modern potential models, A18 and A18+δv+UIX* [15], which provide a good fit to the nucleon–nucleon scattering data in the Nijmegen data base.

Myers and Swiatecki proposed a nuclear model based on the Thomas–Fermi approximation [16, 17]. They extended the model of Seyler and Blanchard [18] by improving four-parameter interaction force between nucleons. By fitting two additional parameters they obtained an excellent agreement with the ground state energies of about 1600 atomic nuclei. The energy density reads

$$\begin{aligned}\varepsilon_{\mathrm{MS}}\left(n_N,n_P\right) = {}& A\left(n_N^{\frac{5}{3}}+n_P^{\frac{5}{3}}\right)-B_1\left(n_N^2+n_P^2\right)-2B_2n_Nn_P+C_1\left(n_N^{\frac{8}{3}}+n_P^{\frac{8}{3}}\right)\\ &+C_2\left(n_N^{\frac{5}{3}}n_P+n_Nn_P^{\frac{5}{3}}\right)-D\left(5n_N^{\frac{2}{3}}n_P-n_P^{\frac{5}{3}}\right),\end{aligned} \tag{25}$$

where the parameters are: $A = 88.247\,\mathrm{MeV\,fm}^2$, $B_1 = 169.217\,\mathrm{MeV\,fm}^3$, $B_2 = 286.715\,\mathrm{MeV\,fm}^3$, $C_1 = 358.664\,\mathrm{MeV\,fm}^5$, $C_2 = 1167.630\,\mathrm{MeV\,fm}^5$ and $D = 50.148\,\mathrm{MeV\,fm}^2$.

Ravenhall has found a simple formula to reproduce nuclear matter calculations of Friedman and Pandharipande [13]. The interaction model fits the s-, p-, d- and f-wave phase shifts in the low density limit, and at high density the hypernetted chain calculations which makes the effective interactions similar to a contact two-body force. Also adjustable tensor force contribution to obtain good saturation properties is included. These model forces have been fitted to the nuclear matter calculations at low and high densities. The energy density is

$$\begin{aligned}\varepsilon_{\mathrm{FPR}}\left(n_N,n_P\right) = {}& \left(\frac{1}{2m_N}+B_N\right)\tau_N+\left(\frac{1}{2m_P}+B_P\right)\tau_P\\ &+n_B^2\left[a_1+a_2e^{-b_1n_B}+\left(\frac{1}{2}-x\right)^2\left(a_3+a_4e^{-b_1n_B}\right)\right]\\ &+n_Be^{-b_2n_B^2}\left[a_5+a_6n_B+\left(\frac{1}{2}-x\right)^2\left(a_7+a_8n_B\right)\right],\end{aligned} \tag{26}$$

where $n_B = n_N + n_P$, $x = n_P/n_B$, $B_i = (a_9 n_B + a_{10} n_i)\, e^{-b_3 n_B}$, $\tau_i = \frac{3}{5}\left(3\pi^2\right)^{\frac{2}{3}} n_i^{\frac{5}{3}}$ for $i = N, P$. The parameter values are:

$$\begin{aligned}
a_1 &= 1054.0\,\mathrm{MeV\,fm}^3\,, \qquad a_2 = -1393.0\,\mathrm{MeV\,fm}^3\,,\\
a_3 &= -2316.0\,\mathrm{MeV\,fm}^3\,, \qquad a_4 = 2859.0\,\mathrm{MeV\,fm}^3\,,\\
a_5 &= -1.78\,\mathrm{MeV}\,, \qquad a_6 = -52.0\,\mathrm{MeV\,fm}^3\,,\\
a_7 &= 5.5\,\mathrm{MeV}\,, \qquad a_8 = 197.0\,\mathrm{MeV\,fm}^3\,,\\
a_9 &= 89.8\,\mathrm{MeV\,fm}^5\,, \qquad a_{10} = -59.0\,\mathrm{MeV\,fm}^5\,,\\
b_1 &= 0.284\,\mathrm{fm}^3\,, \qquad b_2 = 42.25\,\mathrm{fm}^6 \quad \text{and} \quad b_3 = 0.457\,\mathrm{fm}^3\,.
\end{aligned}$$

Akmal, Pandharipande and Ravenhall used variational chain summation methods and created new Argonne v_{18} two-nucleon interaction which fits all the nucleon–nucleon scattering data in the Nijmegen data base.

The A18+δv+UIX* model comprises the boost correction δv to the two-nucleon interaction, which gives the leading relativistic effect of order $(v/c)^2$ and the three-nucleon interaction Urbana model IX* which predicts a transition in nucleon matter to a phase with neutral pion condensate at the density $0.195\,\mathrm{fm}^{-3}$ for pure neutron matter and $0.32\,\mathrm{fm}^{-3}$ for symmetric nuclear matter.

The energy density for both A18 and A18+δv+UIX* models can be given by a common formula

$$\begin{aligned}
\varepsilon_{\mathrm{APR}}\left(n_N, n_P\right) = {} & \left(\frac{1}{2m_N} + B_N\right)\tau_N + \left(\frac{1}{2m_P} + B_P\right)\tau_P \\
& - 4 n_N n_P \left(p_1 + p_2 n_B + p_6 n_B^2 + \left(p_{10} + p_{11} n_B\right) e^{-p_9^2 n_B^2}\right) \\
& - \left(n_N - n_P\right)^2 \left(p_{12}/n_B + p_7 + p_8 n_B + p_{13} e^{-p_9^2 n_B^2}\right)\,, \qquad (27)
\end{aligned}$$

for A18+δv+UIX* with $n_B > p_{19}$ or $n_B > p_{20}$ additionaly:

$$\begin{aligned}
& - 4 n_N n_P \left(n_B - p_{19}\right)\left(p_{17} + p_{21}\left(n_B - p_{19}\right)\right) e^{p_{18}\left(n_B - p_{19}\right)} \\
& - \left(n_N - n_P\right)^2 \left(n_B - p_{20}\right)\left(p_{15} + p_{14}\left(n_B - p_{20}\right)\right) e^{p_{16}\left(n_B - p_{20}\right)}\,,
\end{aligned}$$

where $B_i = (p_3 n_B + p_5 n_i)\, e^{-p_4 n_B}$, $\tau_i = \frac{3}{5}\left(3\pi^2\right)^{\frac{2}{3}} n_i^{\frac{5}{3}}$ for $i = N, P$. The parameters $p_3 = 89.8\,\mathrm{MeV\,fm}^5$, $p_4 = 0.457\,\mathrm{fm}^3$ and $p_5 = -59.0\,\mathrm{MeV\,fm}^5$ are common for these two models.

For A18 model the parameters are:

$$\begin{aligned} p_1 &= 297.6\,\mathrm{MeV\,fm^3}\,, & p_2 &= -134.6\,\mathrm{MeV\,fm^6}\,,\\ p_6 &= -15.9\,\mathrm{MeV\,fm^9}\,, & p_7 &= 215.0\,\mathrm{MeV\,fm^3}\,,\\ p_8 &= -116.5\,\mathrm{MeV\,fm^6}\,, & p_9 &= 6.42\,\mathrm{fm^3}\,,\\ p_{10} &= 51\,\mathrm{MeV\,fm^3}\,, & p_{11} &= -35\,\mathrm{MeV\,fm^6}\,,\\ p_{12} &= -0.2\,\mathrm{MeV} \text{ and} & p_{13} &= 0\,\mathrm{MeV\,fm^3}\,. \end{aligned}$$

For A18+δv+UIX* model the parameters are:

$$\begin{aligned} p_1 &= 337.2\,\mathrm{MeV\,fm^3}\,, & p_2 &= -382\,\mathrm{MeV\,fm^6}\,,\\ p_6 &= -19.1\,\mathrm{MeV\,fm^9}\,, & p_7 &= 214.6\,\mathrm{MeV\,fm^3}\,,\\ p_8 &= -384\,\mathrm{MeV\,fm^6}\,, & p_9 &= 6.4\,\mathrm{fm^3}\,,\\ p_{10} &= 69\,\mathrm{MeV\,fm^3}\,, & p_{11} &= -33\,\mathrm{MeV\,fm^6}\,,\\ p_{12} &= 0.35\,\mathrm{MeV}\,, & p_{13} &= 0\,\mathrm{MeV\,fm^3}\,,\\ p_{14} &= 0\,\mathrm{MeV\,fm^9}\,, & p_{15} &= 287\,\mathrm{MeV\,fm^6}\,,\\ p_{16} &= -1.54\,\mathrm{fm^3}\,, & p_{17} &= 175.0\,\mathrm{MeV\,fm^6}\,,\\ p_{18} &= -1.45\,\mathrm{fm^3}\,, & p_{19} &= 0.32\,\mathrm{fm^{-3}}\,,\\ p_{20} &= 0.195\,\mathrm{fm^{-3}} \quad \text{and} & p_{21} &= 0\,\mathrm{MeV\,fm^9}\,. \end{aligned}$$

Appendix A

Metric in Cartesian coordinates

First we change the spherical coordinates to the Cartesian ones. Using transformation $r=\sqrt{x^2+y^2+z^2}$, $\phi=\arctan(y/x)$, $\theta=\arccos(z/\sqrt{x^2+y^2+z^2})$ we obtain the metric in Cartesian coordinates,

$$\begin{aligned} ds^2 &= e^{\nu}c^2dt^2 - \frac{e^{\lambda}x^2+y^2+z^2}{x^+y^2+z^2}\,dx^2 - \frac{x^2+e^{\lambda}y^2+z^2}{x^+y^2+z^2}\,dy^2\\ &\quad - \frac{x^2+y^2+e^{\lambda}z^2}{x^+y^2+z^2}dz^2 - \frac{(e^{\lambda}-1)2yz}{x^+y^2+z^2}\,dy\,dz\\ &\quad + \left(-\frac{(e^{\lambda}-1)2xy}{x^+y^2+z^2}dy - \frac{(e^{\lambda}-1)2xz}{x^+y^2+z^2}dz\right)\,dx\,. \end{aligned} \tag{A.1}$$

The components of the covariant metric tensor are

$$g_{\mu\nu} = \begin{pmatrix} e^{\nu} & 0 & 0 & 0\\ 0 & -\frac{e^{\lambda}x^2+y^2+z^2}{x^2+y^2+z^2} & -\frac{\left(-1+e^{\lambda}\right)xy}{x^2+y^2+z^2} & -\frac{\left(-1+e^{\lambda}\right)xz}{x^2+y^2+z^2}\\ 0 & -\frac{\left(-1+e^{\lambda}\right)xy}{x^2+y^2+z^2} & -\frac{x^2+e^{\lambda}y^2+z^2}{x^2+y^2+z^2} & -\frac{-(yz)+e^{\lambda}yz}{x^2+y^2+z^2}\\ 0 & -\frac{\left(-1+e^{\lambda}\right)xz}{x^2+y^2+z^2} & -\frac{-(yz)+e^{\lambda}yz}{x^2+y^2+z^2} & \frac{x^2+y^2+e^{\lambda}z^2}{x^2+y^2+z^2} \end{pmatrix}. \tag{A.2}$$

The determinant of this metric tensor is $g=-e^{\lambda+\nu}$. Then we find the components of the contravariant metric tensor in Cartesian coordinates to be

$$g^{\mu\nu}=\begin{pmatrix} e^{-\nu} & 0 & 0 & 0 \\ 0 & -\frac{x^2+e^{\lambda}\left(y^2+z^2\right)}{e^{\lambda}\left(x^2+y^2+z^2\right)} & \frac{\left(-1+e^{\lambda}\right)x\,y}{e^{\lambda}\left(x^2+y^2+z^2\right)} & \frac{\left(-1+e^{\lambda}\right)x\,z}{e^{\lambda}\left(x^2+y^2+z^2\right)} \\ 0 & \frac{\left(-1+e^{\lambda}\right)x\,y}{e^{\lambda}\left(x^2+y^2+z^2\right)} & -\frac{y^2+e^{\lambda}\left(x^2+z^2\right)}{e^{\lambda}\left(x^2+y^2+z^2\right)} & \frac{\left(-1+e^{\lambda}\right)y\,z}{e^{\lambda}\left(x^2+y^2+z^2\right)} \\ 0 & \frac{\left(-1+e^{\lambda}\right)x\,z}{e^{\lambda}\left(x^2+y^2+z^2\right)} & \frac{\left(-1+e^{\lambda}\right)y\,z}{e^{\lambda}\left(x^2+y^2+z^2\right)} & -\frac{e^{\lambda}\left(x^2+y^2\right)+z^2}{e^{\lambda}\left(x^2+y^2+z^2\right)} \end{pmatrix}. \quad \text{(A.3)}$$

Note, that now λ and ν are functions of Cartesian coordinates x, y and z.

Appendix B

Calculation of the Einstein–Tolman pseudotensor

We are interested in the energy component of the Einstein–Tolman pseudotensor [4]. Using Eqs. (13) and (14) one can calculate the nonvanishing componets of $H_0^{0\alpha}$

$$\begin{aligned} H_0^{01} &= e^{(\nu-\lambda)/2}\left(\partial_x(e^{\lambda}g^{11})+\partial_y(e^{\lambda}g^{12})+\partial_z(e^{\lambda}g^{13})\right)\,, \\ H_0^{02} &= e^{(\nu-\lambda)/2}\left(\partial_x(e^{\lambda}g^{21})+\partial_y(e^{\lambda}g^{22})+\partial_z(e^{\lambda}g^{23})\right)\,, \\ H_0^{03} &= e^{(\nu-\lambda)/2}\left(\partial_x(e^{\lambda}g^{31})+\partial_y(e^{\lambda}g^{32})+\partial_z(e^{\lambda}g^{33})\right)\,. \end{aligned} \quad \text{(B.1)}$$

Inserting the components of the contravariant metric tensor we have

$$\begin{aligned} H_0^{01} &= \frac{2\,\left(e^{\lambda}-1\right)\,x}{x^2+y^2+z^2}\,, \\ H_0^{02} &= \frac{2\,\left(e^{\lambda}-1\right)\,y}{x^2+y^2+z^2}\,, \\ H_0^{03} &= \frac{2\,\left(e^{\lambda}-1\right)\,z}{x^2+y^2+z^2}\,. \end{aligned} \quad \text{(B.2)}$$

The 00 componet of θ pseudotensor is given by

$$\theta_0^0=\frac{c^4}{16\pi G}\left(\partial_x H_0^{01}+\partial_y H_0^{02}+\partial_z H_0^{03}\right)\,. \quad \text{(B.3)}$$

Taking the partial derivatives of $H_0^{0\alpha}$ we obtain

$$\theta_0^0=\frac{e^{(\lambda+\nu)/2}\,\left[\left(-1+e^{\lambda}\right)\,P+c^2\,\left(1+e^{\lambda}\right)\,\rho\right]}{2}\,. \quad \text{(B.4)}$$

Notice that $e^{-\lambda} = 1 - 2M/r$, so the 00 component of the Einstein–Tolman pseudotensor is

$$\theta_0^0 = \frac{e^{\frac{\nu}{2}} \left(\frac{c^2\, r}{-2\, G\, M + c^2\, r}\right)^{\frac{3}{2}} \left(c^4\, r\, \rho + G\, M\, \left(P - c^2\, \rho\right)\right)}{c^2\, r}\,. \tag{B.5}$$

Using definition of θ (13) we obtain the equation (16) which is the gravitational field energy density in the Einstein–Tolman prescription. The same result was obtained in Ref. [6].

$$t_0^0 = \frac{G\, M(r)\, \left(P + c^2\, \rho\right)}{-2\, G\, M(r) + c^2\, r}\,. \tag{B.6}$$

Appendix C

Møller prescription

For metric in Cartesian coordinates the χ potential is

$$\chi_0^{0\gamma} = -\sqrt{-g}\left(\frac{\partial g_{00}}{\partial x^\rho} g^{00} g^{\gamma\rho}\right)\,. \tag{C.1}$$

The nonzero componets of χ are

$$\begin{aligned}
\chi_0^{01} &= -e^{(\lambda+\nu)/2} \frac{\nu'}{\sqrt{x^2+y^2+z^2}} \left(x\, g^{11} + y\, g^{12} + z\, g^{13}\right)\,,\\
\chi_0^{02} &= -e^{(\lambda+\nu)/2} \frac{\nu'}{\sqrt{x^2+y^2+z^2}} \left(x\, g^{21} + y\, g^{22} + z\, g^{23}\right)\,,\\
\chi_0^{03} &= -e^{(\lambda+\nu)/2} \frac{\nu'}{\sqrt{x^2+y^2+z^2}} \left(x\, g^{31} + y\, g^{32} + z\, g^{33}\right)\,,
\end{aligned} \tag{C.2}$$

where "prime" denotes derivative of the function ν with respect to $r = \sqrt{x^2+y^2+z^2}$. Inserting the components of the contravariant metric tensor and using the Eq. (2b) we have

$$\begin{aligned}
\chi_0^{01} &= \frac{e^{\frac{-\lambda+\nu}{2}}\, x\, \left(c^4\, \left(-1+e^\lambda\right) + 8\, \pi\, G\, e^\lambda\, P\, \left(x^2+y^2+z^2\right)\right)}{c^3\, \left(x^2+y^2+z^2\right)}\,,\\
\chi_0^{02} &= \frac{e^{\frac{-\lambda+\nu}{2}}\, y\, \left(c^4\, \left(-1+e^\lambda\right) + 8\pi\, G\, e^\lambda\, P\, \left(x^2+y^2+z^2\right)\right)}{c^3\, \left(x^2+y^2+z^2\right)}\,,\\
\chi_0^{03} &= \frac{e^{\frac{-\lambda+\nu}{2}}\, z\, \left(c^4\, \left(-1+e^\lambda\right) + 8\pi\, G\, e^\lambda\, P\, \left(x^2+y^2+z^2\right)\right)}{c^3\, \left(x^2+y^2+z^2\right)}\,.
\end{aligned} \tag{C.3}$$

According to the definition (17) one finds the 00 component to be

$$M_0^0 = \frac{c^4}{8\pi G}\left(\partial_x \chi_0^{01} + \partial_y \chi_0^{02} + \partial_z \chi_0^{03}\right) . \tag{C.4}$$

After taking partial derivatives of potential $\chi_0^{0\gamma}$ and using Eqs. (2a), (2b), (2c) the Møller complex reads

$$\begin{aligned} M_0^0 \;=\; & e^{(\lambda+\nu)/2}\left(\frac{-8\,c^2\,G\,P^2\,\pi\,r^3 + c^8\,r\,\rho}{-4\,c^3\,G\,M + 2\,c^5\,r}\right) \\ & + e^{(\lambda+\nu)/2}\left(\frac{e^{\lambda}\,\left(-2\,G\,M + c^2\,r\right)\,\left(c^4 + 8\,G\,P\,\pi\,r^2\right)\,\left(P + c^2\,\rho\right)}{-4\,c^3\,G\,M + 2\,c^5\,r}\right) \\ & + e^{(\lambda+\nu)/2}\left(\frac{c^6\,(5Pr - 4GM\rho) - 4c^4GP\,\left(3\,M + 2\,\pi\,r^3\,\rho\right)}{-4c^3GM + 2c^5r}\right) . \end{aligned} \tag{C.5}$$

Using the definition of the function λ and rearranging terms we obtain

$$M_0^0 = c\,e^{\nu/2}\sqrt{\frac{c^2\,r}{-2\,G\,M + c^2\,r}}\,\left(3\,P + c^2\,\rho\right) . \tag{C.6}$$

The Møller pseudotensor is $M_0^0 = \sqrt{-g}(t_0^0 - T_0^0)$ and one can easily find that in this prescription the gravitational energy density is given by very simple formula $t_0^0 = 3\,P$.

Appendix D

Landau–Lifshitz prescription

For the metric in Cartesian coordinates we have

$$L^{00} = \frac{c^4}{16\pi G}\,\partial_\rho \partial_\sigma \left(-g g^{00} g^{\sigma\rho}\right) . \tag{D.1}$$

The determinant of the metric tensor is $-e^{\nu+\lambda}$ and the contravariant component g^{00} is $e^{-\nu}$, so Landau–Lifshitz complex reads

$$\begin{aligned} L^{00} \;=\; & \frac{c^4}{16\pi G}\left(\partial_x^2 e^{\lambda} g^{11} + \partial_y^2 e^{\lambda} g^{22} + \partial_z^2 e^{\lambda} g^{33}\right) \\ & + \frac{c^4}{8\pi G}\left(\partial_x \partial_y e^{\lambda} g^{12} + \partial_x \partial_z e^{\lambda} g^{13} + \partial_y \partial_z e^{\lambda} g^{23}\right) . \end{aligned} \tag{D.2}$$

Inserting the components of contravariant metric tensor, taking partial derivatives and using Eqs. (2a), (2b), (2c) we obtain

$$L^{00} = \frac{c^4\,\left(-G\,M^2 + 2\,c^2\,\pi\,r^4\,\rho\right)}{2\,\pi\,r^2\,(-2\,G\,M + c^2\,r)^2}. \tag{D.3}$$

For $r > R$ we have the Schwarzschild solution. So taking $\rho = 0$ we find how does the Landau–Lifszyc complex look like in the vacuum outside the star: the vacuum energy density is negative. The same result was obtained by Virbhadra (see Eq. (16) in [9] with $Q = 0$ and wiht $G = c = 1$). Because $L_0^0 = L^{00} g_{00}$ we finally find the gravitational field energy density in Landau–Lifshitz form to be

$$t_0^0 = \frac{c^2\,G\,M\,\left(M - 4\,\pi\,r^3\,\rho\right)}{2\,\pi\,r^3\,\left(2\,G\,M - c^2\,r\right)}\,. \tag{D.4}$$

Appendix E

Weinberg pseudotensor

In the Weinberg prescription [3] the complex of matter and field energy density is

$$W^{\mu\nu} = \frac{c^4}{16\pi G}\Delta^{\mu\nu\alpha}{}_{,\alpha}\,. \tag{E.1}$$

The tensor $\Delta^{\mu\nu\alpha}$ is given by

$$\Delta^{\mu\nu\alpha} = \frac{\partial h^\beta_\beta}{\partial x_\mu}\eta^{\nu\alpha} - \frac{\partial h^\beta_\beta}{\partial x_\nu}\eta^{\mu\alpha} - \frac{\partial h^{\beta\mu}}{\partial x^\beta}\eta^{\nu\alpha} + \frac{\partial h^{\beta\nu}}{\partial x^\beta}\eta^{\mu\alpha} + \frac{h^{\mu\alpha}}{\partial x_\nu} - \frac{h^{\nu\alpha}}{\partial x_\mu}\,, \tag{E.2}$$

where $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and $\eta_{\mu\nu}$ is Minkowski metric. We look for W^{00}. Using $h^\beta_\beta = \eta^\mu_\beta h_{\beta\nu}$ we find that

$$\Delta^{00\alpha} = \frac{\partial h^\beta_\beta}{\partial x_0}\eta^{0\alpha} - \frac{\partial h^\beta_\beta}{\partial x_0}\eta^{0\alpha} - \frac{\partial h^{\beta 0}}{\partial x^\beta}\eta^{0\alpha} + \frac{\partial h^{\beta 0}}{\partial x^\beta}\eta^{0\alpha} + \frac{\partial h^{0\alpha}}{\partial x_0} - \frac{\partial h^{0\alpha}}{\partial x_0} = 0\,. \tag{E.3}$$

So the Weinberg complex 00 component equals to 0 for any α and β as it was observed by Xulu [19].

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