THE "SQUARE ROOT" OF THE DIRAC EQUATION AND SOLUTIONS ON SUPERSPACE*

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The "square root" of the Dirac operator derived on the superspace is used to construct supersymmetric field equations. In addition to the recently found solution — a vector supermultiplet — it is demonstrated how another supermultiplet follows as solution: a set of spin 3/2 and spin 1 component fields obeying the appropriate equations of motion together with an auxiliary, spin 2 tensor field.

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1. Introduction

The idea to take the "square root" of the Dirac operator follows directly from the analogous procedure performed by Dirac on the Klein–Gordon operator [1]. Whereas in the first case the motivation was to linearize the operator in space-time derivatives, the form of the supersymmetry algebra [2] suggested that repeating this procedure would lead to the operator linear in supersymmetry generators or equivalently, spinorial derivatives. Such construction was presented some time ago [3] together with a set of supersymmetric field equations which result when acting with the "square root" operator on superfields.

To recall this construction in short let me write the Dirac equation in two-component notation and chiral representation:

$$-\begin{pmatrix} i\bar{\sigma}^{\mu\,\dot{\alpha}\alpha}\partial_\mu & m \\ m & i\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu \end{pmatrix}\begin{pmatrix} \varphi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \equiv \mathcal{D}\begin{pmatrix} \varphi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = 0\,. \qquad (1)$$

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The Lorenz indices are denoted here by μ, ν, λ and ρ, the spinor indices by α and β.

We are looking for the operator $\mathcal{S}$ satisfying:

$$\mathcal{S}^\dagger \mathcal{S} = \mathcal{D} \,. \tag{2}$$

The solution proposed in Ref. [3] is

$$\mathcal{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} D^\alpha & -\bar{D}_{\dot{\alpha}} \\ \bar{D}^{\dot{\alpha}} & D_\alpha \end{pmatrix} , \tag{3}$$

where the spinorial derivatives are defined on the superspace as

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i \sigma^\mu{}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \,, \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^\alpha \sigma^\mu{}_{\alpha\dot{\alpha}} \partial_\mu \,. \end{aligned} \tag{4}$$

Indeed, using the anticomutation relations

$$\begin{aligned} \{D_\alpha, D_\beta\} &= \left\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right\} = 0 \,, \\ \left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\} &= -2i\sigma^\mu{}_{\alpha\dot{\beta}} \partial_\mu \,, \end{aligned} \tag{5}$$

we get

$$\mathcal{S}^\dagger \mathcal{S} = - \begin{pmatrix} i\bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu & M \\ M & i\sigma^\mu{}_{\alpha\dot{\alpha}} \partial_\mu \end{pmatrix} , \tag{6}$$

where a scalar, hermitian operator

$$M = -\frac{1}{4} \left(DD + \bar{D}\bar{D} \right) \tag{7}$$

appears instead of the mass m. The operator $\mathcal{S}$ is thus the solution to our problem on the space of superfields Λ which satisfy

$$M \, \Lambda = m \, \Lambda \,. \tag{8}$$

Acting with the operator $\mathcal{S}$ on a superfield we are able to construct a free field equation — the "square root" of the Dirac equation. The simplest two choices of superfields are [3]

$$F = \begin{pmatrix} W_\alpha \\ \bar{\mathcal{H}}^{\dot{\alpha}} \end{pmatrix} \tag{9}$$

and

$$B = \begin{pmatrix} \Phi \\ V^{\alpha\dot{\alpha}} \end{pmatrix} \tag{10}$$

leading to the equations

$$\mathcal{S}F = 0\,, \qquad \mathcal{S}B = 0\,. \tag{11}$$

It is also obvious that due to Eq. (2) both superfields F and B satisfy the Dirac equation

$$\mathcal{D}F = 0\,, \qquad \mathcal{D}B = 0\,. \tag{12}$$

Recently the equations

$$\mathcal{S}F = 0 \tag{13}$$

(together with the condition $MF = mF$) were studied and solved in Ref. [4]. In the simplest case when $W_\alpha = \mathcal{H}_\alpha$, the solution was found to be the Maxwell supermultiplet

$$\begin{aligned} W_\alpha = &- i\lambda_\alpha(y) + \left[\delta_\alpha{}^\beta d(y) - \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha{}^\beta(\partial_\mu w_\nu(y) - \partial_\nu w_\mu(y))\right]\theta_\beta \\ &+ \theta\theta\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y)\,, \end{aligned} \tag{14}$$

with $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$. The massless component fields $w_\mu(x)$ and $\lambda_\alpha(x)$ satisfy the Maxwell and Dirac equations, respectively, and $d = \text{const}$.

2. The equations and their solutions

In this paper we study the other set of Eqs. (11)

$$\mathcal{S}B = 0\,, \tag{15}$$

the superfield B satisfying in addition the condition (8). In terms of (in general complex) component superfields the equations read:

$$\begin{aligned} D^\alpha\Phi - \bar{D}_{\dot{\alpha}}V^{\alpha\dot{\alpha}} &= 0\,, \\ \bar{D}^{\dot{\alpha}}\Phi + D_\alpha V^{\alpha\dot{\alpha}} &= 0\,, \end{aligned} \tag{16}$$

and

$$\begin{aligned} M\Phi &= m\Phi\,, \\ MV^{\alpha\dot{\alpha}} &= mV^{\alpha\dot{\alpha}}\,. \end{aligned} \tag{17}$$

Multiplying the first Eq. (16) by D_α and making use of the second Eq. (16) we obtain

$$M\Phi = m\Phi = 0\,. \tag{18}$$

Out of two possibilities suppose first $\Phi = 0$ and the mass m arbitrary. The Eqs. (16) simplify and it is easy to show in this case that

$$D^2 V^{\alpha\dot{\alpha}} = \bar{D}^2 V^{\alpha\dot{\alpha}} = 0\,, \tag{19}$$

which implies

$$M V^{\alpha\dot{\alpha}} = m V^{\alpha\dot{\alpha}} = 0\,. \tag{20}$$

If we are interested in non-zero superfields, the mass m has to vanish, $m = 0$. The other possibility, $m = 0$ and Φ arbitrary, leads to similar conclusion. Indeed, a new constraint on Φ can be obtained by acting with the anticomutator $\left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\}$ on the superfield Φ and using Eqs. (16),

$$\left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\}\Phi = -2i\sigma^{\mu}{}_{\alpha\dot{\beta}}\partial_\mu \Phi = 2MV_{\alpha\dot{\beta}} = 0\,. \tag{21}$$

The above equality means that Φ is constant in space-time and depends only on θ and $\bar{\theta}$. Taking into account the condition (18), the most general form of Φ is

$$\Phi_{\mathrm{c}} = c_1 + c_2^{\alpha}\theta_\alpha + \bar{c}_{3\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + c_{4\mu}\theta\sigma^\mu\bar{\theta} + c_5(\theta\theta - \bar{\theta}\bar{\theta})\,, \tag{22}$$

with constant $c_1, c_2, \ldots, c_5$. One further notices that the Eqs. (16) are invariant under the simultaneous shift

$$\begin{aligned} \Phi \;&\rightarrow\; \Phi' = \Phi + \Phi_c\,, \\ V^{\alpha\dot{\alpha}} \;&\rightarrow\; V'^{\alpha\dot{\alpha}} = V^{\alpha\dot{\alpha}} + V_c^{\alpha\dot{\alpha}}\,, \end{aligned} \tag{23}$$

where

$$V_{\mathrm{c}}^{\alpha\dot{\alpha}} = c_2^{\alpha}\bar{\theta}^{\dot{\alpha}} - \theta^{\alpha}\bar{c}_3^{\dot{\alpha}} + c_{4\mu}\bar{\sigma}^{\mu\dot{\alpha}\alpha}(\theta\theta - \bar{\theta}\bar{\theta}) + c_5\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}\,. \tag{24}$$

We can perform this shift so that

$$\Phi = 0\,. \tag{25}$$

Eqs. (16) reduce then to

$$\bar{D}_{\dot{\alpha}} V^{\alpha\dot{\alpha}} = D_\alpha V^{\alpha\dot{\alpha}} = 0\,. \tag{26}$$

We first express the bi-spinor superfield $V^{\alpha\dot{\alpha}}$ through the vector superfield V_μ

$$V^{\alpha\dot{\alpha}} = \sigma^{\mu\dot{\alpha}\alpha}V_\mu\,. \tag{27}$$

Acting with the anticomutator $\left\{D_\alpha, \bar{D}_{\dot{\beta}}\right\}$ on the superfield $V^{\alpha\dot{\alpha}}$ one notices that the superfield V_μ is divergenceless

$$\partial^\mu V_\mu = 0 \tag{28}$$

and due to Eq. (19)

$$D^2 V_\mu = \bar{D}^2 V_\mu = 0\,. \tag{29}$$

It is interesting to note that the Eqs. (29), (19) are not independent and follow from Eqs. (26). To find the general solution to Eqs. (26) let me expand V_μ in terms of component fields:

$$\begin{aligned} V_\mu \;=\; & a_\mu(x) + \sqrt{2}\theta\psi_\mu(x) + \sqrt{2}\bar{\theta}\bar{\chi}_\mu(x) + \theta\sigma^\nu\bar{\theta}v_{\nu\mu}(x) + \theta\theta f_\mu(x) + \bar{\theta}\bar{\theta}\bar{h}_\mu(x) \\ & + \bar{\theta}\bar{\theta}\theta^\beta\Big(\eta_{\mu\beta}(x) - \frac{1}{\sqrt{2}}\,\varepsilon_{\mu\nu\lambda\rho}\sigma^\nu_{\beta\dot{\beta}}\partial^\lambda\bar{\chi}^{\rho\dot{\beta}}(x)\Big) \\ & + \theta\theta\bar{\theta}_{\dot{\beta}}\Big(\bar{\rho}^{\dot{\beta}}_\mu(x) + \frac{1}{\sqrt{2}}\,\varepsilon_{\mu\nu\lambda\rho}\bar{\sigma}^{\nu\dot{\beta}\beta}\partial^\lambda\psi^\rho_\beta(x)\Big) \\ & + \theta\theta\bar{\theta}\bar{\theta}\Big(c_\mu(x) + \frac{1}{4}\Box\, a_\mu(x)\Big)\,. \end{aligned} \tag{30}$$

The conditions (26), (28), (29) lead to the following relations among the component fields (x dependence suppressed):

— bosons

$$\partial^\mu a_\mu \;=\; 0\,, \tag{31}$$

$$f_\mu = h_\mu = c_\mu \;=\; 0\,, \tag{32}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\sigma^\lambda_{\alpha\dot{\beta}}(i\partial_\lambda a_\mu + v_{\lambda\mu}) \;=\; 0\,, \tag{33}$$

$$\Box\, a_\mu - \varepsilon^{\mu\nu\lambda\rho}\partial_\nu v_{\lambda\rho} \;=\; 0\,, \tag{34}$$

$$\partial^\nu v_{\nu\mu} = \partial^\mu v_{\nu\mu} \;=\; 0\,, \tag{35}$$

— fermions

$$\partial^\mu\psi_{\mu\alpha} = \partial^\mu\bar{\chi}_{\mu\dot{\alpha}} \;=\; 0\,, \tag{36}$$

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{\mu\alpha} = \sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}_\mu \;=\; 0\,, \tag{37}$$

$$\varepsilon^{\mu\nu\lambda\rho}\bar{\sigma}^{\dot{\alpha}\alpha}_\nu\partial_\lambda\psi_{\rho\alpha} + i\bar{\sigma}^{\nu\dot{\alpha}\alpha}\partial_\nu\psi^\mu_\alpha \;=\; 0\,, \tag{38}$$

$$\varepsilon^{\mu\nu\lambda\rho}\sigma_{\nu\alpha\dot{\alpha}}\partial_\lambda\bar{\chi}^{\dot{\alpha}}_\rho - i\sigma^\nu_{\alpha\dot{\alpha}}\partial_\nu\bar{\chi}^{\mu\dot{\alpha}} \;=\; 0\,, \tag{39}$$

$$\eta^\alpha_\mu = \bar{\rho}^{\dot{\alpha}}_\mu \;=\; 0\,. \tag{40}$$

As can be seen from the above equations we can eliminate the fields f_μ, h_μ, c_μ, $\eta_{\mu\alpha}$ and $\bar{\rho}_{\mu\dot{\alpha}}$ from the supermultiplet, so that we are left with the component fields a_μ, $\psi_{\mu\alpha}$, $\bar{\chi}_{\mu\dot{\alpha}}$ and $v_{\lambda\mu}$ only.

Let us look first at the bosonic content of the supermultiplet. Decomposing the tensor field $v_{\mu\nu}$ into symmetric and antisymmetric part

$$v_{\mu\nu} = v^{\rm A}_{\mu\nu} + v^{\rm S}_{\mu\nu}\,, \tag{41}$$

one notices that the symmetric part obeys the spin-2 field constraints

$$\begin{aligned} \partial^\mu v^{\rm S}_{\mu\nu} &= 0\,, \\ v^{{\rm S}\mu}_{\mu} v &= 0\,. \end{aligned} \tag{42}$$

It decouples as well from Eqs. (33)–(35), [7].

The antisymmetric part $v^{{\rm A}\mu\nu}$ can be expressed through a vector field v^μ

$$v^{{\rm A}\mu\nu} = \partial^\mu v^\nu - \partial^\nu v^\mu\,, \tag{43}$$

and related to the tensor field

$$a_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\,, \tag{44}$$

through (imaginary) duality relation

$$v^{\rm A}_{\mu\nu} = \frac{i}{4}\varepsilon_{\mu\nu\lambda\rho} a^{\lambda\rho}\,. \tag{45}$$

The relations (31)–(35) can be expressed by one of these fields (*e.g.* $a^{\mu\nu}$) and lead to the Maxwell field equations:

$$\begin{aligned} \partial^\nu a_{\nu\mu} &= 0\,, \\ \varepsilon_{\nu\mu\lambda\rho}\partial^\mu a^{\lambda\rho} &= 0\,. \end{aligned} \tag{46}$$

with the Lorenz condition $\partial^\mu a_\mu = 0$.

Let us look now at the fermionic sector. The component fields η^α_μ and $\bar\rho^{\dot\alpha}_\mu$ vanish. The vector-spin fields obey Eqs. (36)–(38) which means that only the spin 3/2 is present. To summarize, the solution to Eqs. (16) is built of two spin 3/2 vector-spinor component fields $\psi^\alpha_\mu(x)$ and $\bar\chi^{\dot\alpha}_\mu(x)$ obeying the (massless) Rarita–Schwinger equations (38)–(39), a spin 1 vector field $a_\mu(x)$ obeying the Maxwell equations (46) and a symmetric spin 2 tensor field $v^{\rm S}_{\mu\nu}$ with no eqution of motion.

3. Specific solution — chiral supermultiplet

The chiral and antichiral supermultiplets can be obtained as solutions to Eqs. (16) by assuming

$$V_\mu = \partial_\mu V\,, \tag{47}$$

with

$$\bar{D}_{\dot{\alpha}}V = 0\,, \qquad \text{or} \qquad D_\alpha V = 0\,. \tag{48}$$

Eqs. (31)–(40) simplify significantly and the solution to the Eqs. (16) consists of one scalar field a satisfying the (massless) Klein–Gordon equation one spinor field ψ satisfying the (massless) Dirac equation and one auxiliary (constant) scalar field F.

4. Remarks

It is easy to show that the above equations of motion (38), (39), (46) follow also from the supersymmetric Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} V^{\mu\dagger} V_\mu = \int d^2\theta d^2\bar{\theta} V^\dagger_{\alpha\dot{\alpha}} V^{\dot{\alpha}\alpha}\,. \tag{49}$$

In addition, this Lagrangian forces the tensor field $v^{\mathrm{S}}_{\nu\mu}$, otherwise constrained only by Eq. (42), to vanish. The Lagrangian (49) has the "supergauge" symmetry

$$V_\mu \to V_\mu + \partial_\mu (V + V^\dagger)\,, \tag{50}$$

which transforms the component fields in the following way:

$$\begin{aligned}
a_\mu &\to a_\mu + \partial_\mu (a + a^\star)\,, \\
\psi_\mu &\to \psi_\mu + \partial_\mu \psi\,, \\
\bar{\chi}_\mu &\to \bar{\chi}_\mu + \partial_\mu \bar{\psi}\,, \\
v^{\mathrm{A}}_{\nu\mu} &\to v^{\mathrm{A}}_{\nu\mu}\,, \\
v^{\mathrm{S}}_{\nu\mu} &\to v^{\mathrm{S}}_{\nu\mu} + \partial_\nu \partial_\mu (a - a^\star)\,.
\end{aligned} \tag{51}$$

5. Summary

Together with earlier results [3, 4, 6] the "square root" of the Dirac operator supplies us with two different gauge supermultiplets. The first one coincides with the known vector gauge supermultiplet used to construct gauge theories together with chiral matter superfields. The supermultiplet V_μ studied in the present paper is less known. It was constructed and used in connection with the supergravity vector current [10], but in that case the "active" component fields were rather spin 2 (graviton) and spin 3/2 (gravitino). In our case the dynamical fields are rather the spin 1 and 3/2 ones. Their coupling to other supermultiplets is now under study.

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REFERENCES

[1] P.A.M. Dirac, *Proc. R. Soc. (London)* **A117**, 610 (1928); **A118**, 351 (1928).

[2] Y.A. Gol'fand, E.P. Likhman, *JETP Lett.* **B13**, 323 (1971); D. Volkov, N.P. Akulov, *Phys. Lett.* **B46**, 109 (1973); J. Wes, B. Zumino, *Nucl. Phys.* **B70**, 39 (1974).

[3] J. Szwed, *Phys. Lett.* **B181**, 305 (1986).

[4] A. Bzdak, L. Hadasz, *Phys. Lett.* **B582**, 113 (2004).

[5] V.I. Ogievetsky, E. Sokatchev, *JETP Lett.* **23**, 58 (1976); *J. Phys.* **A10**, 2021 (1977); S.J. Gates Jr., *Phys. Rev.* **D16**, 1727 (1977); S.J. Gates Jr., V.A. Kostelecky, *Nucl. Phys.* **B248**, 570 (1984); S.J. Gates Jr., R. Grimm, *Z. Phys.* **C26**, 621 (1985).

[6] A. Bzdak, J. Szwed, *Europhys. Lett.* **69**, 189 (2005).

[7] A. Bzdak, R. Grimm, private communication.

[8] J. Wess, J. Bagger, *Supersymmetry and Supergravity,* Princeton Univeristy Press, Princeton 1983.

[9] S. Weinberg, *The Quantum Theory of Fields. Vol. III Supersymmetry,* Cambridge University Press, Cambridge 2000.

[10] S. Ferrara, B. Zumino, *Nucl. Phys.* **B87**, 207 (1975); V. Ogievetsky, E. Sokatchev, *Nucl. Phys.* **B124**, 309 (1977).