EVALUATION OF THE COINCIDENCE PROBABILITIES IN A GENERALIZED GAUSSIAN MODEL OF MULTIPLE PARTICLE PRODUCTION

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Coincidence probabilities, which yield Renyi entropies, are investigated in a generalized Gaussian model, which includes interparticle correlations.

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1. Introduction

Let us consider an M-particle state and assume that its density matrix is

$$\rho(\boldsymbol{K},\boldsymbol{q}) = e^{-v(\boldsymbol{K})-\frac{1}{2}\boldsymbol{q}\boldsymbol{L}^2\boldsymbol{q}+i\boldsymbol{q}\overline{\boldsymbol{X}}(\boldsymbol{K})}\,, \tag{1}$$

where $\boldsymbol{K}=(1/2)(\boldsymbol{p}+\boldsymbol{p}')$, $\boldsymbol{q}=\boldsymbol{p}-\boldsymbol{p}'$ and $\overline{\boldsymbol{X}}(\boldsymbol{K})$ are $3M$-dimensional vectors, and $\boldsymbol{L}^2$ is a $3M\times 3M$ dimensional matrix. Matrix $\boldsymbol{L}^2$ is in general a function of $\boldsymbol{K}$. The physical interpretation of formula (1) is naturally obtained when the density matrix is converted into the corresponding Wigner function:

$$W(\boldsymbol{K},\boldsymbol{X}) = \frac{\sqrt{2\pi}^{3M}}{\mathrm{Det}\boldsymbol{L}}\; e^{-v(\boldsymbol{K})-\frac{1}{2}\left(\boldsymbol{X}-\overline{\boldsymbol{X}}(\boldsymbol{K})\right)\boldsymbol{L}^{-2}\left(\boldsymbol{X}-\overline{\boldsymbol{X}}(\boldsymbol{K})\right)}\,. \tag{2}$$

Note that $\mathrm{Det}\,\boldsymbol{L}$ is an effective volume of the system.

In this paper we derive, assuming (1), an asymptotic formula, valid when the eigenvalues of matrix $\boldsymbol{L}$ are large, for the coincidence probabilities

$$C(l) = \mathrm{Tr}\,\rho^l = \int d^l p\, \prod_{j=1}^{l}\rho(\boldsymbol{K}_j,\boldsymbol{q}_j)\,, \tag{3}$$

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which are needed to obtain the Renyi entropies

$$H_l = \frac{1}{1-l} \log C(l) \,. \tag{4}$$

Here $\boldsymbol{K}_j = (1/2)(\boldsymbol{p}_j + \boldsymbol{p}_{j+1})$, $\boldsymbol{q}_j = \boldsymbol{p}_j - \boldsymbol{p}_{j+1}$ and $\boldsymbol{p}_{l+1} \equiv \boldsymbol{p}_1$.

As seen from formula (1), when the eigenvalues of matrix $\boldsymbol{L}$ are large, the integral is dominated by the region $q \approx 0$. Therefore, we propose to expand the exponent in the integrand in powers of the components of $\boldsymbol{q}$ and to keep only the terms up to second order. Thus, the problem reduces to the evaluation of a Gaussian integral.

Our final result is an elegant formula generalizing the formulae derived for the one-dimensional Gaussian model by a number of authors [2–9]. For the convenience of the reader, a short derivation of the formula for the one-dimensional case is given in Section 3. In Section 2 the integral (3) is worked out. In Section 4 this integral is significantly simplified using the result derived in Section 3. Section 5 contains our conclusions. Two Appendices contain discussions of the matrices $\boldsymbol{R}^2$ and S introduced in the text.

2. Evaluation of $C(l)$

In order to calculate the coincidence probabilities it is convenient to introduce the notation

$$\boldsymbol{K}_j = \overline{\boldsymbol{K}} + \boldsymbol{k}_j \,, \qquad \overline{\boldsymbol{K}} = \frac{1}{l}(\boldsymbol{p}_1 + \cdots + \boldsymbol{p}_l) \,. \tag{5}$$

Note that

$$\sum_{j=1}^{l} \boldsymbol{k}_j = 0 \,, \qquad \sum_{j=1}^{l} \boldsymbol{q}_j = 0 \,. \tag{6}$$

An immediate implication is that the terms with $\overline{\boldsymbol{X}}$ do not contribute. Indeed, let us consider the expansions of each term $q_j \overline{\boldsymbol{X}}(\boldsymbol{K}_j)$ in powers of the components of $\boldsymbol{q}$. The first terms are $q_j \overline{\boldsymbol{X}}(\overline{\boldsymbol{K}})$ and their total contribution to the exponent is

$$i\overline{\boldsymbol{X}}(\overline{\boldsymbol{K}}) \sum_{j=1}^{l} \boldsymbol{q}_j = 0 \,. \tag{7}$$

The total contribution from the second order terms of the expansion must be zero because of the hermiticity of the density matrix and all the further terms are negligible being cubic or higher order in the components of $\boldsymbol{q}$.

It is convenient to replace in (3) the variables $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_l$ by the variables $\overline{\boldsymbol{K}}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{l-1}$. The Jacobian of this change of variables equals one. Further, one introduces into the integrand the factor

$$\delta\left(\sum_{j=1}^{l}\boldsymbol{q}_j\right) = \int \frac{dt}{(2\pi)^{3M}}\; e^{-i\boldsymbol{t}\sum_{j=1}^{l}\boldsymbol{q}_j}\,, \tag{8}$$

where $\boldsymbol{t}$ is another $3M$-dimensional vector, and compensates it by the integration over $\boldsymbol{q}_j$. Thus, (3) takes the form

$$C(l) = \int dK \int \frac{dt}{(2\pi)^3 M} \int d^l\, q e^{\sum_{j=1}^{l}\left(-v(\boldsymbol{K}_j)-\frac{1}{2}\boldsymbol{q}_j L^2\boldsymbol{q}_j - i\boldsymbol{t}\boldsymbol{q}_j\right)}\,. \tag{9}$$

The sum of the v terms can be rewritten as follows

$$\sum_{j=1}^{l} v(\boldsymbol{K}_j) = lv(\overline{\boldsymbol{K}}) + \frac{1}{2}\sum_{j=1}^{l}(\boldsymbol{k}_j\nabla)(\boldsymbol{k}_j\nabla)v(\overline{\boldsymbol{K}})\,, \tag{10}$$

where the differential operators ∇_α act on the components of $\overline{\boldsymbol{K}}$. Expressing the vectors $\boldsymbol{k}_j$ as linear combinations of the vectors $\boldsymbol{q}_j$ and performing the summation over j one finds that

$$\sum_{j=1}^{l} v(\boldsymbol{K}_j) = lv(\overline{\boldsymbol{K}}) + \frac{1}{2}\sum_{j,k}^{l-1}\boldsymbol{q}_j S_{jk}\boldsymbol{R}^2\boldsymbol{q}_k\,, \tag{11}$$

where $\boldsymbol{R}^2$ is a $3M \times 3M$ matrix with elements

$$R^2_{\alpha\beta} = \nabla_\alpha\nabla_\beta v(\overline{\boldsymbol{K}}) \tag{12}$$

and S is an $l \times l$ matrix. Matrix $\boldsymbol{R}^2$ for a specific model is discussed in Appendix A. The matrix elements S_{jk} are not needed for our calculation. They are, however, useful for cross-checks. Therefore, they are calculated and discussed in Appendix B. Actually, as seen there, there are potentially useful ambiguities in the definitions of the matrices S.

The next step is to replace the integration variables $\boldsymbol{q}_j$ by their linear combinations $\boldsymbol{Q}_j$, which are the eigenvectors of matrix S: $\boldsymbol{q}_j = \sum_{k=1}^{l} U_{jk}\boldsymbol{Q}_k$. This transformation can be chosen orthogonal and then its Jacobian equals one. Using the notation

$$U^{\mathrm{T}}SU = \Lambda; \qquad \Lambda_{jk} = \Lambda_j\delta_{jk}\,, \tag{13}$$

one finds

$$\begin{aligned} C(l) \;=\; & \int d\overline{K}\; e^{-lv(\overline{\boldsymbol{K}})} \int \frac{dt}{(2\pi)^{3M}} \\ & \times \prod_{j=1}^{l}\int dQ \exp\left[-\frac{1}{2}\boldsymbol{Q}(\boldsymbol{L}^2 + \Lambda_j\boldsymbol{R}^2)\boldsymbol{Q} + i\boldsymbol{t}\sum_{k=1}^{l}U_{kj}\boldsymbol{Q}\right]. \end{aligned} \tag{14}$$

Further, for each j we introduce an orthogonal transformation $\boldsymbol{Q} = V^{(j)}\boldsymbol{z}$ which diagonalizes the quadratic form in the exponent

$$V^{(j)\mathrm{T}}(\boldsymbol{L}^2 + \Lambda_j \boldsymbol{R}^2)V^{(j)} = \lambda^{(j)}\,, \qquad \lambda^{(j)}_{\alpha\beta} = \lambda^{(j)}_{\alpha}\delta_{\alpha\beta}\,. \tag{15}$$

This reduces each integral over Q into a product of $3M$ single Gaussian integrals. Performing them one gets

$$\begin{aligned} C(l) &= \sqrt{2\pi}^{3M(l-2)} \int d\overline{\boldsymbol{K}}\, \frac{e^{-lv(\overline{\boldsymbol{K}})}}{\sqrt{\prod_{j=1}^{l} \mathrm{Det}(\boldsymbol{L}^2 + \Lambda_j \boldsymbol{R}^2)}} \\ &\quad \times \int dt \exp\left[-\frac{l}{2}\sum_{j=1}^{l} C_j \boldsymbol{t} V^{(j)} \frac{1}{\lambda^{(j)}} V^{(j)\mathrm{T}} \boldsymbol{t}\right], \end{aligned} \tag{16}$$

where

$$C_j = \frac{1}{l}\sum_{ik} U_{ij}U_{kj}\,, \tag{17}$$

and the identities

$$\prod_{\alpha=1}^{3M} \lambda^{(j)}_{\alpha} = \mathrm{Det}(\boldsymbol{L}^2 + \Lambda_j \boldsymbol{R}^2) \tag{18}$$

have been used.

Performing in (16) the integration over t and extracting $\mathrm{Det}\boldsymbol{L}^2 = (\mathrm{Det}\boldsymbol{L})^2$ one gets

$$C(l) = \left(\frac{\sqrt{2\pi}}{\mathrm{Det}\boldsymbol{L}}\right)^{3M(l-1)} l^{-\frac{3M}{2}} \int d\overline{\boldsymbol{K}}\, \frac{e^{-lv(\overline{\boldsymbol{K}})}}{\sqrt{\mathrm{Det}\boldsymbol{A} \prod_{j=1}^{l} \mathrm{Det}(1 + \Lambda_j \boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1})}}, \tag{19}$$

where

$$\boldsymbol{A} = l\sum_{j=1}^{l} C_j \boldsymbol{L}^{-1} V^{(j)} \frac{1}{\lambda^{(j)T}} V^{(j)\mathrm{T}} \boldsymbol{L}^{-1}. \tag{20}$$

Using (15) this can be also rewritten as

$$\boldsymbol{A} = l\sum_{j=1}^{l} \frac{C_j}{1 + \Lambda_j \boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1}}\,, \tag{21}$$

which yields

$$C(l) = \left(\frac{\sqrt{2\pi}}{\mathrm{Det}\boldsymbol{L}}\right)^{3M(l-1)} l^{-\frac{3M}{2}} \int d\overline{\boldsymbol{K}} \frac{e^{-lv(\overline{\boldsymbol{K}})}}{\sqrt{\mathrm{Det}\left[\sum_{j=1}^{l} C_j \prod_{k}^{l(j)} (1 + \Lambda_k \boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1})\right]}}. \tag{22}$$

The symbol $\prod_{k=1}^{l(j)}$ denotes the product $\prod_{k=1}^{l}$ with the j-th factor omitted. This is a usable expression for the coincidence probabilities $C(l)$ [1]. The formula, however, can be significantly simplified. In order to derive the simplification we will need a result concerning the one-dimensional Gaussian model. This is derived in the following section.

3. The one-dimensional Gaussian model

The model considered in the present paper is a generalization of the more common Gaussian model, where the M-particle density matrix is a product of $3M$ one dimensional Gaussian density matrices of the form:

$$\rho(K,q) = \frac{R}{\sqrt{2\pi}}\, e^{-\frac{1}{2}R^2K^2-\frac{1}{2}L^2q^2}\,, \tag{23}$$

where K, q, R, L are just real numbers. Let us calculate the coincidence probabilities $C_1(l)$ for the one-dimensional model (23). To this end we observe that $\rho(K,q)$ is diagonal in the representation of the wave functions of the harmonic oscillator [6–9] and can be written in the form

$$\rho(K,q) = \sum_{n=0}^{\infty} \psi_n(p)\lambda_n\psi_n^*(p')\,, \tag{24}$$

where

$$\lambda_n = (1-z)z^n\,, \tag{25}$$

$$\psi_n(p) = \sqrt{\frac{\alpha}{\sqrt{\pi}2^n n!}}\,\exp\left(-\frac{1}{2}\alpha^2p^2\right)H_n(\alpha p)\,, \tag{26}$$

$$\alpha = \sqrt{RL}\,, \qquad z = \frac{1-R/(2L)}{1+R/(2L)}\,. \tag{27}$$

Therefore,

$$\mathrm{Tr}\,\rho^l = \sum_{n=0}^{\infty}\lambda_n^l = (1-z)^l\frac{1}{1-z^l}\,, \tag{28}$$

where a geometric progression has been summed. It is seen from (27) that $|z|<1$, as it should.

[1] It requires, however, the determination of the eigenvalues Λ_k and of the coefficients C_j. This is a nontrivial task for large l.

To show that formulae (23) and (24) are equivalent, one may use the identity[2]

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x)H_n(y)=\frac{1}{\sqrt{1-4z^2}}\,\exp\left(-4z\frac{z(x^2+y^2)-xy}{1-4z^2}\right). \qquad (29)$$

Substituting the formula for z into (28) one easily finds

$$C_1(l)=\left(\frac{R}{L}\right)^l\frac{1}{\left(1+\frac{R}{2L}\right)^l-\left(1-\frac{R}{2L}\right)^l}. \qquad (30)$$

This is the formula we need.

4. Final expression for $C(l)$

The one-dimensional Gaussian model can be also studied using the methods from Section 2. There is a number of simplifications, however. Since the dimension of the matrices $3M$ is replaced by one

$$\lambda^{(j)}=L^2+\Lambda_jR^2; \qquad (d=1). \qquad (31)$$

Since there is no need to diagonalize $L^2+\Lambda_iR^2$: $V^{(j)}=V^{(j)T}=1$. Moreover,

$$e^{-lv(\overline{K})}=\left(\frac{R}{\sqrt{2\pi}}\right)^l e^{-\frac{l}{2}R^2\overline{K}^2}, \qquad (d=1). \qquad (32)$$

Thus

$$\begin{aligned}C_1(l) &= \frac{R^l}{2\pi\sqrt{\prod_{j=1}^{l}(L^2+\Lambda_jR^2)}}\int dK e^{-\frac{l}{2}K^2R^2}\\ &\quad\times\int dt\exp\left[-\frac{l}{2}\sum_{j=1}^{l}C_j\frac{t^2}{L^2+\Lambda_jR^2}\right]. \end{aligned} \qquad (33)$$

Note that in spite of all these simplifications the eigenvalues Λ_j and the coefficients C_j are the same as in the general case. Performing the integrations over t and K:

$$C_1(l)=\frac{R^{l-1}}{\sqrt{l}\sqrt{A\prod_{j=1}^{l}(L^2+\Lambda_jR^2)}}, \qquad (34)$$

[2] To prove (29) it is enough to substitute on the left-hand side twice the definition $H_n(u)=\frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}dt\,(u+it)^ne^{-t^2}$, perform the summation and a Gaussian integration.

where

$$A = l \sum_{j=1}^{l} \frac{C_j}{L^2 + \Lambda_j R^2}\,, \qquad (d = 1)\,. \tag{35}$$

The formula for $C_1(l)$ can be rewritten as

$$C_1(l) = \left(\frac{R}{L}\right)^{l-1} \frac{1}{l} \frac{1}{\sqrt{\sum_{j=1}^{l} C_j \prod_{k=1}^{l(j)} (1 + \Lambda_j \left(\frac{R}{L}\right)^2)}}\,. \tag{36}$$

Let us compare now this result with formula (30). The equivalence of the two formulae implies that

$$l \sqrt{\sum_{j=1}^{l} C_j \prod_{k=1}^{l(j)} \left(1 + \Lambda_j \left(\frac{R}{L}\right)^2\right)} = \frac{L}{R} \left[\left(1 + \frac{R}{2L}\right)^l - \left(1 - \frac{R}{2L}\right)^l\right]\,, \tag{37}$$

or equivalently

$$\sqrt{\sum_{j=1}^{l} C_j \prod_{k=1}^{l(j)} \left(1 + \Lambda_j \left(\frac{R^2}{L^2}\right)\right)} = 1 + \sum_{n=1}^{N(l)} 2^{-2n} \frac{(l-1)!}{(2n+1)!(l-1-2n)!} \left(\frac{R^2}{L^2}\right)^n\,, \tag{38}$$

where $N(l) = E\left[\frac{1}{2}(l-1)\right]$. Note that this identity is valid whatever is substituted for R^2/L^2.

Let us now go back to the general case. The argument of the determinant in formula (38) differs from the argument of the square root in formula (36) only by the substitution of the matrix $\boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1}$ for the number R^2/L^2. Therefore, the same substitution can be made in the identity (38) and one obtains

$$C(l) = \left(\frac{\sqrt{2\pi}}{\mathrm{Det}\boldsymbol{L}}\right)^{3M(l-1)} l^{-\frac{3M}{2}} \int dK \frac{e^{-lv(\boldsymbol{K})}}{\mathrm{Det}\left[1 + \sum_{n=1}^{N(l)} a_n \left(\boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1}\right)^n\right]}\,, \tag{39}$$

where

$$a_n = 2^{-2n} \frac{(l-1)!}{(2n+1)!(l-1-2n)!}\,. \tag{40}$$

This is our final formula and the main result of this paper. Let us note that when the eigenvalues $t_1, \ldots, t_{3M}$ of matrix

$$\boldsymbol{V} = \boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1} \tag{41}$$

are known, the determinant in the integrand of formula (39) can be evaluated and then we get

$$C(l)=\left(\frac{\sqrt{2\pi}}{\mathrm{Det}\boldsymbol{L}}\right)^{3M(l-1)} l^{\frac{3M}{2}} \int dK\, \frac{e^{-lv(K)}\sqrt{\prod_{\alpha=1}^{3M} t_\alpha}}{\prod_{\alpha=1}^{3M}\left[(1+\frac{1}{2}\sqrt{t_\alpha})^l-(1-\frac{1}{2}\sqrt{t_\alpha})^l\right]}\,. \quad (42)$$

5. Discussion

We have derived an explicit formula for the coincidence probabilities in a general, multidimensional Gaussian model. This model can be considered as an approximation, at large volume of the system, of a model with an arbitrary momentum distribution. Therefore, our result may be useful for a rather wide class of physical situations, particularly for analyses of the systems created in heavy ion collisions. The formula obtained in the present paper is elegant, transparent and easy to use.

Some comments are in order.

- Since we do not assume factorization of the multiparticle density matrix, our calculation takes into account possible correlations between particles. Correlations show up as the non-diagonal terms in the matrix $\boldsymbol{R}^2$. There is no restriction on their character and magnitude.

- It should be emphasized that, although the model is considered as an asymptotic expansion for large $\boldsymbol{L}$, the terms in Eq. (39) of higher order in $\boldsymbol{L}^{-2}$ should, in general, be included in spite of the fact that terms of these orders have been already neglected in the exponent of (1). The point is that in the presence of correlations some eigenvalues of matrix $\boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1}$ may be of the order of the number of particles M. Then the expansion in formula (39) contains terms of the order $(M/L^2)^n$, which for high multiplicities are not necessarily small even for a large system.

In order to illustrate the last point consider the matrices[3]

$$R^2_{ij}=\alpha(\delta_{ij}+\beta)\,,\qquad L_{ij}=L\delta_{ij}\,,\qquad i,j=1,\ldots,3M\,. \quad (43)$$

For these matrices the eigenvalues of matrix $\boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1}$ are

$$\lambda_1=\frac{\alpha(1+3M\beta)}{L^2}\,,\qquad \lambda_2=\cdots=\lambda_{3M}=\frac{\alpha}{L^2}\,. \quad (44)$$

[3] The model could be made more realistic by distinguishing between the interparticle correlations and the correlations between the directions x,y,z. This, however, would not change the qualitative results we need.

Therefore,

$$\mathrm{Det}\left[1+\sum_{n=1}^{N(l)} a_n \left(\boldsymbol{L}^{-1}\boldsymbol{R}^2\boldsymbol{L}^{-1}\right)^n\right] = \left[1+\sum_{n=1}^{N(l)} a_n \left(\frac{\alpha(1+3M\beta)}{L^2}\right)^n\right] \times \left[1+\sum_{n=1}^{N(l)} a_n \frac{\alpha^n}{L^{2n}}\right]^{3M-1}, \qquad (45)$$

which shows that terms of the kind $(3M\alpha\beta/L^2)^n$ do occur in the expansion.

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Appendix A

Matrix $V = L^{-1}R^2L^{-1}$ in a model

Let us consider a model where

$$\begin{aligned} v(\boldsymbol{K}) &= \sum_{n=1}^{M} v(K_n)\,, \\ v(K) &= \frac{1}{T}\sqrt{K_{\mathrm{T}}^2+m^2}+\frac{Y^2}{2A}+\log\left[\tilde{A}E\right], \\ \tilde{A} &= \pi T(m+T)\sqrt{8\pi A}\, e^{-\frac{m}{T}}\,. \end{aligned} \qquad (\mathrm{A}.1)$$

In this formula T and A are parameters, m is the mass of the particle; K are the three-dimensional momentum vectors composing the $3M$ dimensional vector $\overline{\boldsymbol{K}}$, K_{T} is the transverse component of K; the energy $E = \sqrt{m^2+K^2}$, the rapidity $Y = \frac{1}{2}\log\frac{E+K_z}{E-K_z}$. This is a commonly used model, where the particles are uncorrelated, but there are some correlations between the x, y, z components of the momentum vectors. Thus, matrix $\boldsymbol{R}^2$ consists of M diagonal 3×3 blocks which will be denoted R^2. The transverse momenta have a Boltzmann distribution, while in the longitudinal direction there is a mild cut-off by a Gaussian in rapidity. The last term in the expression for v ensures the correct normalization. Matrix $\boldsymbol{L}$ is assumed to be diagonal with diagonal elements $L_x = L_y \equiv L_{\mathrm{T}}, L_z$ in every diagonal block corresponding to the diagonal blocks of $\boldsymbol{R}^2$. Further we use the labels α, β for the components x, y, z and the labels a, b for the transverse components x, y, the transverse mass $m_{\mathrm{T}} = \sqrt{m^2+K_{\mathrm{T}}^2}$.

Let us define

$$V_{\alpha\beta} = \frac{R^2_{\alpha\beta}}{L_\alpha L_\beta}\,. \tag{A.2}$$

This gives

$$V_{ab} = \Theta K_a K_b + \Phi\,\delta_{ab}\,, \tag{A.3}$$

$$V_{az} = \omega K_a\,, \tag{A.4}$$

$$V_{zz} = \Psi\,, \tag{A.5}$$

where

$$L_{\rm T}^2\Theta = -\frac{1}{Tm_{\rm T}^3} - \frac{2}{E^4} + \frac{K_z}{AEm_{\rm T}^2}\left(\frac{K_z}{Em_{\rm T}^2} + \frac{2Y}{m_{\rm T}^2} + \frac{Y}{E^2}\right), \tag{A.6}$$

$$L_{\rm T}^2\Phi = \frac{1}{Tm_{\rm T}} + \frac{1}{E^2} - \frac{YK_z}{AEm_{\rm T}^2}\,, \tag{A.7}$$

$$L_{\rm T}L_z\omega = -\frac{1}{AE^2}\left(\frac{K_z}{m_{\rm T}^2} + \frac{Y}{E}\right) - \frac{2K_z}{E^4}, \tag{A.8}$$

$$L_z^2\Psi = \frac{1}{E^2}\left[\frac{A+1}{A} - \frac{K_z}{E}\frac{2AK_z + EY}{EA}\right]. \tag{A.9}$$

As seen from (A.3)–(A.5) matrix T can be written in the form

$$V_{\alpha\beta} = \Phi\delta_{\alpha\beta} + V'_{\alpha\beta}\,. \tag{A.10}$$

The second row of matrix V' is proportional to the first. Therefore, $\mathrm{Det}V' = 0$. Consequently, one of the eigenvalues of matrix V equals Φ and the other two can be found by solving a quadratic equation. The result is

$$t_1 = \Phi, \tag{A.11}$$

$$t_2 = \frac{1}{2}\left(\Theta K_{\rm T}^2 + \Phi + \Psi + \sqrt{(\Theta K_{\rm T}^2 + \Phi - \Psi)^2 + 4\omega^2 K_{\rm T}^2}\right), \tag{A.12}$$

$$t_3 = \frac{1}{2}\left(\Theta K_{\rm T}^2 + \Phi + \Psi - \sqrt{(\Theta K_{\rm T}^2 + \Phi - \Psi)^2 + 4\omega^2 K_{\rm T}^2}\right). \tag{A.13}$$

Since the eigenvalues are known formula (42) can be used to calculate $C(l)$. In particular for $l = 3$ the result is

$$C(3) = \left(\frac{\sqrt{2\pi}}{L_{\rm T}^2 L_z}\right)^{6M} 3^{-\frac{3M}{2}} \int dK\, \frac{e^{-3v(K)}}{\left[\left(1+\frac{\Phi}{12}\right)\left(1+\frac{\Theta K_{\rm T}^2+\Phi+\Psi}{12} + \frac{(\Theta K_{\rm T}^2+\Phi)\Psi - \omega^2 K_{\rm T}^2}{144}\right)\right]^M}. \tag{A.14}$$

Appendix B

Evaluation of the matrix S

Let us represent each vector $\boldsymbol{k}_j,\ j = 1,\ldots,l$ as a linear combination of the vectors $\boldsymbol{q}_j,\ j = 1,\ldots,l-1$

$$\boldsymbol{k}_j = \frac{1}{2}(\boldsymbol{p}_j + \boldsymbol{p}_{j+1}) - \frac{1}{l}(\boldsymbol{p}_1 + \cdots + \boldsymbol{p}_l) = \sum_{k=1}^{l-1} c_{jk}\boldsymbol{q}_k\,. \tag{B.1}$$

Because of the second identity (6) it is not mandatory to include q_l in the expansion. Equating the coefficients of the vectors $\boldsymbol{p}_1,\ldots \boldsymbol{p}_l$ on both sides of the second equality we get the equation system

$$\frac{1}{2}(\delta_{jn} + \delta_{j+1,n}) - \frac{1}{l} = c_{jn} - c_{j,n-1}; \qquad n = 1,\ldots,l\,, \tag{B.2}$$

where $c_{j0} = c_{j,l} = 0$ and $\delta_{l+1,n}$ stands for $\delta_{1,n}$. Summing each side of the equations over n from $n = k+1$ to $n = l$ one gets from the resulting equalities

$$c_{jk} = -\frac{k}{l} + 1 + \frac{1}{2}\delta_{jl} - \frac{1}{2}\left[\Theta(j-k) + \Theta(j-k+1)\right]\,, \tag{B.3}$$

where $\Theta(n)$ equals one for $n > 0$ and zero otherwise.

The coefficients S_{mn} are defined by

$$k_{j\alpha}k_{j\beta} = \sum_{m,n}^{l} S_{mn}q_{m\alpha}q_{n\beta}\,. \tag{B.4}$$

Substituting on the left-hand side the expansions of $\boldsymbol{k}_j$ in terms of the $\boldsymbol{q}_j$ one finds

$$S_{mn} = \sum_{j=1}^{l} c_{jm}c_{jn}\,. \tag{B.5}$$

Note that this matrix is symmetric and that by construction $S_{ml} = S_{lm} = 0$. Using the explicit formulae to perform the summation

$$S_{mn} = \frac{m(l-n)}{l} - \frac{1}{4}(1 + \delta_{mn}), \quad \text{for} \quad m \le n < l\,. \tag{B.6}$$

For $m > n$ one finds the matrix elements from the symmetry $S_{mn} = S_{nm}$.

For $l = 2$ one gets $S = 0$, which greatly simplifies the analysis [10]. Note also that because of the second identity (6) one can add on the left-hand side of relation (B.4) a term

$$\sum_{j=0}^{l} \boldsymbol{q}_j \sum_{k=1}^{l} c_k\boldsymbol{q}_k\,, \tag{B.7}$$

where c_k are arbitrary constants. This modifies S without affecting the products $\boldsymbol{k}_{j\alpha}\boldsymbol{k}_{j\beta}$ and sometimes can be used to simplify the calculations.

Choosing for instance: $c_1 = c_2 = -c_3 = -\frac{1}{12}$ for $l = 3$ and $c_1 = c_3 = -c_4 = \frac{1}{3}c_2 = \frac{1}{8}$ for $l = 4$ one gets the simple formulae:

$$S(3) = \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{1}{12} & 0 \\ 0 & 0 & \frac{1}{12} \end{pmatrix},$$

$$S(4) = \begin{pmatrix} \frac{1}{8} & 0 & -\frac{1}{8} & 0 \\ 0 & \frac{1}{8} & 0 & -\frac{1}{8} \\ -\frac{1}{8} & 0 & \frac{1}{8} & 0 \\ 0 & -\frac{1}{8} & 0 & \frac{1}{8} \end{pmatrix}, \tag{B.8}$$

from which the eigenvalues are immediately visible.

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