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THE FOKKER–PLANCK EQUATION FOR CHAOTIC MAPS

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The Fokker–Planck equation for deterministic diffusion out of periodic, nonlinear maps is derived and compared with the random walk and Langevin approaches.

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1. Introduction

The mass transport that consists of two parts, drift and diffusion, can be described in quite a few ways. On the molecular level *i.e.* the Brownian particle behavior, the velocity and displacement are described by the Langevin equation [1–3]. The probability density field (*i.e.* normalized concentration of the matter in question) description is delivered, in turn, by the Fokker–Planck (Smoluchowski) type equations [1, 4–9].

This set, *i.e.* Langevin and Fokker–Planck equations provides a complete description of a given transport system, both on a molecular and phenomenological levels. Of course, the complexity of a particular equation and especially, the structure of their components, depend very much of a given case and the method of derivation.

If one is interested in enlighting the discrete properties of the system, one has to deal with either a random walk (RW) problem [10–12] or nonlinear iterated maps (NIM) of some special class [13–19]. In this paper we would like to contribute to the latter in the following way

- characterize the eligible class of maps;
- show the aspects of mass transport suitable for description by NIM;
- derive the Fokker–Planck (Smoluchowski) equation from NIM.

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Poincare has first recognized the importance of studying the dynamical behavior of mappings, as defined by a difference equation

$$x_{n+1} = T(x_n)\,, \tag{1}$$

where x is an d-dimensional vector and T is a nonlinear transformation. The index n refers to either discrete steps in time or successive returns to a surface of section.

It is clear that the nature of T generates the mapping with a particular properties. As for the "diffusive" mappings it is rather difficult to specify the detailed conditions on T (our understanding of this complex behavior is still incomplete). A sort of "experimental" reasoning leads to the following conditions on T: it should be a quasi periodic, non-trapping and symmetric mapping (see the next paragraph for details).

It is not difficult to realize an analogy between a random walk (RW) and NIM mechanisms of generating mass transport. While in the RW, jumps along with their prescribed probabilities build up a microscopic view of transport, in case of NIM — it is a "local dynamics", *i.e.* in a sense, a mesoscopic scale of a bath influence (coded by T) on the molecule behavior.

To provide the final step of deterministic analysis of diffusional transport system, the Smoluchowski equation for probability density (concentration) field dynamics will be derived. From the "technical" point of view the derivation is similar to that from RW to PDE [10, 12], however, the assumptions made in both cases are quite different, and lead to a different structure of the transport coefficients.

2. Deterministic generator of diffusion

Let us consider a periodic chaotic map F, defined on a single interval as [13]

$$x_{n+1} = \mathrm{int}(x_n) + F(x_n \mathrm{mod} 1)\,, \tag{2}$$

where x stands for a particle position and n denotes the iteration number (discrete time), $\mathrm{int}(x)$ denotes the integer value of x. F is symmetric ($F(x) = -F(1-x)$) and periodic with period 1 (this is handled by the mod1 operator). The escape of particles from the $(0, 1)$ single cell period subject to absorbing boundaries is described by an escape rate formalism [16, 19].

In such case, due to the symmetry assumption, no direction can be distinguished and the particles escape from the single intervals according to the escape rate with equal fluxes to either side of the cell.

The requirement for the description in the escape rate formalism is quite strong, and for many maps the single cell repellers cannot meet it. An example can be the Geisel–Thomae map [18], the maps with traps, ballistic maps, *etc.*

Using the above assumptions, we can write an equation for the time evolution of the number of particles in a single cell in a similar way as we do in the random walk problem

$$\begin{aligned} W(x,t+\Delta t) \; &= \; W(x,t) + \frac{1}{2}W(x-\Delta x,t)\left(1-e^{-\gamma\Delta t}\right) \\ &\quad + \frac{1}{2}W(x+\Delta x,t)\left(1-e^{-\gamma\Delta t}\right) - W(x,t)\left(1-e^{-\gamma\Delta t}\right), \end{aligned} \tag{3}$$

where W is the number of particles in a single cell, γ is the escape rate constant of a repeller, $1-e^{-\gamma\Delta t}$ is the fraction of particles that leave the single cell in the time Δt, and the coefficient $1/2$ comes from the fact that particles escape either to the left, or to the right from the single cell.

We assume that the particles can escape only to nearby cells, but further generalization to more cells is also possible. However, it would require some knowledge on distribution of the escaping particles that cannot appear *a priori* like the knowledge of the equal fluxes that came from the symmetry of the mapping.

Note that the key point for writing above equation is an observation that we do not care what happens inside the single cell, as long as it is small, and the escape rate provides the interface for the interchange of the particles between cells. We will not always be allowed to make it, especially when going to the potential force influence. In such case we encounter a major difference of this, and a Langevin or RW approach.

Expanding the term $\exp(-\gamma\Delta t)$ on the RHS and LHS of the equation into a Taylor series, we can rewrite Eq. (3)

$$\begin{aligned} W(x,t) + \frac{\partial W(x,t)}{\partial t}\Delta t \; &+ \; \frac{\partial^2 W(x,t)}{2\partial t^2}\Delta t^2 \\ &= \; W(x,t) + \frac{1}{2}W(x-\Delta x,t)\left(1-1+\gamma\Delta t - \frac{\gamma^2}{2}\Delta t^2\right) \\ &\quad + \frac{1}{2}W(x+\Delta x,t)\left(1-1+\gamma\Delta t - \frac{\gamma^2}{2}\Delta t^2\right) \\ &\quad - W(x,t)\left(1-1+\gamma\Delta t - \frac{\gamma^2}{2}\Delta t^2\right), \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial W(x,t)}{\partial t}\Delta t + \frac{\partial^2 W(x,t)}{2\partial t^2}\Delta t^2 \; &= \; \frac{\Delta t}{2}W(x-\Delta x,t)\left[\gamma - \frac{\Delta t\gamma^2}{2}\right] \\ &\quad + \frac{\Delta t}{2}W(x+\Delta x,t)\left[\gamma - \frac{\Delta t\gamma^2}{2}\right] \\ &\quad - \Delta t W(x,t)\left[\gamma - \frac{\Delta t\gamma^2}{2}\right]. \end{aligned} \tag{5}$$

Now, we can use a Taylor expansion to the RHS of Eq. (5) with respect to position. We will expand it up to quadratic terms in space to obtain nonzero dynamics for the probability evolution. Of course, this is just an approximation of the real probability density that will be different for each particular mapping considered (each mapping has its own invariant measure, corresponding to probability distribution within cells [19]). Please notice, however, that Eq. (3) depends only on the coefficient γ and thus is independent of the particular form of the mapping (any mapping characterized by the same γ will behave identically in (3)). This is not surprising, since this equation does not describe what happens between unit cells but gives some macroscopic description at points separated by Δx. What happens between, can be modeled in various ways, depending on the mapping. Nevertheless, the results at Δx spaced points should be the same no matter the modeling of probability shape between cells (as long as we consider a mapping with the same γ coefficient). Therefore, we use the simplest possible model *i.e.* a quadratic expansion for the probability changes between cells. This is the expected probability distribution between cells for some maps with quenched disorder [13] which mimic the random walk (but have nonzero γ coefficient). After elementary calculations, we get

$$\begin{aligned}\frac{\partial W(x,t)}{\partial t}\Delta t + \frac{\partial^2 W(x,t)}{2\partial t^2}\Delta t^2 \;=\; & \left\{ \frac{\gamma}{2}W(x,t) + \frac{\gamma}{2}W(x,t) + \frac{\gamma}{2}\frac{\partial W(x,t)}{\partial x}\Delta x \right. \\ & - \frac{\Delta t\gamma}{2}\frac{\partial W(x,t)}{\partial x}\Delta x + \frac{\gamma}{4}\,\frac{\partial^2 W(x,t)}{\partial x^2}\,(\Delta x)^2 \\ & \left. + \frac{\gamma}{4}\,\frac{\partial^2 W(x,t)}{\partial x^2}(\Delta x)^2 - \gamma W(x,t) \right\} \\ & \times \left(\Delta t - \frac{\Delta t^2 \gamma}{2}\right) . \end{aligned} \tag{6}$$

Dividing Eq. (6) by Δt and letting this parameter to go to zero, reduces the quadratic terms in time, and consequently gives

$$\frac{\partial W(x,t)}{\partial t} = \frac{\gamma(\Delta x)^2}{2}\,\frac{\partial^2 W(x,t)}{\partial x^2} . \tag{7}$$

The final step is to introduce the probability density

$$P(x,t) = \frac{W(x,t)}{N} , \tag{8}$$

where N is the total number of particles in the system, and end up with

$$\frac{\partial P(x,t)}{\partial t} = \frac{D}{2}\,\frac{\partial^2 P(x,t)}{\partial x^2} , \tag{9}$$

where $D = \gamma(\Delta x)^2$, *i.e.* the Fokker–Planck or Smoluchowski equation without drift. (Putting $D = (1/2)\gamma(\Delta x)^2$ we get an analytical representation of, so called, second Fick's law.)

It is interesting to compare the structure of the diffusion coefficients of the Smoluchowski equation obtained in different ways, *i.e.* RW, Langevin and chaotic dynamics,

$$D = \frac{\delta^2}{\tau} = \gamma(\Delta x)^2 = \frac{\xi^2}{2\eta^2}\,, \tag{10}$$

where δ is the jump distance in RW, τ is the time of jump, ξ is the amplitude of noise in stochastic Langevin equation, and η is the friction coefficient.

It shows, among other things, the formal connections between these three mechanisms of diffusional transport. A drift term can be introduced when the mapping is not symmetrical and we cannot assure the equality of fluxes for particles escaping from a unit cell. By a suitable modification of the previous procedure (Eq. (4)), we can write

$$\begin{aligned}\frac{W(x,t+\Delta t) - W(x,t)}{\Delta t} &= p\gamma\, W(x,t) + q\gamma\, W(x,t) \\ &\quad + p\gamma\, \frac{\partial W(x,t)}{\partial x}\Delta x - q\gamma\, \frac{\partial W(x,t)}{\partial x}\Delta x \\ &\quad + \frac{p}{2}\gamma \frac{\partial^2 W(x,t)}{\partial x^2}\,(\Delta x)^2 + \frac{q}{2}\gamma \frac{\partial^2 W(x,t)}{\partial x^2}\,(\Delta x)^2 \\ &\quad - \gamma\, W(x,t)\,,\end{aligned} \tag{11}$$

where p, q are the fractions of the particles, escaping to the left and right respectively (they were both equal 1/2 in the Eq. (4)). The first two terms on RHS cancel with the last term of RHS. However, the third and fourth terms do not. This leads to

$$\begin{aligned}\frac{W(x,t+\Delta t) - W(x,t)}{\Delta t} &= (p-q)\gamma \frac{\partial W(x,t)}{\partial x}\Delta x \\ &\quad + \frac{1}{2}\gamma \frac{\partial^2 W(x,t)}{\partial x^2}(\Delta x)^2\,,\end{aligned} \tag{12}$$

after expanding the LHS of Eq. (11) we get finally

$$\frac{\partial W(x,t)}{\partial t} = (p-q)\gamma\Delta x \frac{\partial W(x,t)}{\partial x} + \frac{1}{2}\gamma(\Delta x)^2 \frac{\partial^2 W(x,t)}{\partial x^2}\,, \tag{13}$$

or

$$\frac{\partial P(x,t)}{\partial t} = c\frac{\partial P(x,t)}{\partial x} + \frac{1}{2}D\frac{\partial^2 P(x,t)}{\partial x^2}\,, \tag{14}$$

where $P(x,t)$ is the probability density (*c.f.* Eq. (8)), $c = (p-q)\gamma\Delta x$ is a drift coefficient, and $D = \gamma(\Delta x)^2$ stands for the diffusion coefficient.

It is interesting to compare the drift and diffusion coefficients obtained here with those delivered by random walk, and Langevin approaches [1, 3, 10, 12] (see Table I).

TABLE I

Comparison of diffusion and drift coefficients for different generating mechanisms. Δx is the width of a cell in chaotic dynamics, δ is the size of jump in RW, τ is the waiting time for a jump in RW, F_{p} is the potential force in the Langevin equation, η is the friction coefficient and ξ is the amplitude of the noise.

Approach	Chaotic dynamics	RW	Langevin
D	$\gamma(\Delta x)^2$	$\frac{\delta^2}{\tau}$	$\frac{\xi^2}{2\eta^2}$
c	$(p-q)\gamma\Delta x$	$(p-q)\frac{\delta}{\tau}$	$\frac{F_{\mathrm{p}}(x)}{\eta}$

As can be seen from Table I, the mechanism behind the mass transport can express itself in quite a different formal form of its coefficients. On the other hand there are striking similarities on a physical meaning level. Comparing the three expressions for diffusion coefficient it is enough to realize that γ is proportional to $1/\tau$. Moreover, the jumps of the Brownian particles in the Langevin approach are proportional to the noise amplitude, and inversely proportional to the environmental friction (so again we have proportionality to δ^2). As for the term $F_{\mathrm{p}}(x)/\eta$ (potential force divided by a friction coefficient), in high friction limit it simply stands for the average velocity of particles in the potential field, *i.e.*

$$\bar{v} = \frac{F_{\mathrm{p}}(x)}{\eta} . \tag{15}$$

Similar interpretation can be given to the $(p-q)$ terms of chaotic and RW approaches.

Looking at this relation, we can feel temptation to implement the potential forces into chaotic maps, since it seems to make no problem to implement such systematic velocity correction to the considered particle dynamics.

However, we must be careful since such additional velocity term affects the internal dynamics of a single cell, not only the dynamics of transitions between the cells.

We must be aware of the fact that a particle can circulate within a unit cell for considerably long time, going back and forward. The regions of backward motion are localized in different places with respect to the most frequently occupied one, so it may happen that when we add potential force, particles will hit such places more often, and no drift will occur, or even opposite drift may appear.

It may also happen that due to potential force, particles will become attracted to some fixed trapping points that were of minor importance without this force.

It is worth to notice that the most frequently occupied place of the unit cell itself will also change under the potential force.

It is, therefore, different than in case of the RW or Langevin approaches, and all this happens because the particles within single cells in NIM have some density distribution which rarely corresponds to an uniform distribution. Therefore, no simple linearization of this effect is possible, and this is a reason for difficulties in precise mathematical description.

The mapping with a potential force can be defined as

$$x_{n+1} = \mathrm{int}(x_n) + \bar{v}\Delta t + F(x_n \mathrm{mod} 1)t = \mathrm{int}(x_n) + F_a(x_n)\,, \qquad (16)$$

where $\bar{v}\Delta t$ is an additional Δx_n, caused by the potential force in time Δt between the moments n and $n+1$.

For positive velocities in Eq. (16) the region that causes the escape to the second cell may grow (or remain unchanged), and the region that causes escape to the previous cell may decrease (or remain unchanged). We cannot tell anything about the frequency of occupation for such regions.

If we assume a smooth mapping, then the regions will grow and decrease (consider a simple lifting of the chart of F, and check the width of regions that go above single cell interval).

Nevertheless, we cannot talk about any proportionality, because this depends on the structure of mapping. Proportionality could appear only in case, when the slopes have constant angle in the mapping, and the number of escaping regions does not grow (example of such mapping could be a sort of the triangular mapping).

All these facts together support the claim that the influence of potential force in a chaotic mapping is far more complicated than it is for the random walk or Langevin model.

Example values of the velocity of the center of the mass of particles ensemble (caused by a potential force in the symmetrical mapping) *versus* the drift velocity (systematic velocity correction, coming from potential force), as for the sine mapping

$$F_a(x_n) = v_{\mathrm{drift}} + 2.0 \sin 2\pi x_n\,, \qquad (17)$$

(comparing to Eq. (16) we assume $\Delta t = 1$ for simplicity) are gathered in the Fig. 1. We see that the positive values of this velocity correspond exactly to the negative values (they are symmetrical about 0 value), but their behavior as a function of the external potential force is highly irregular.

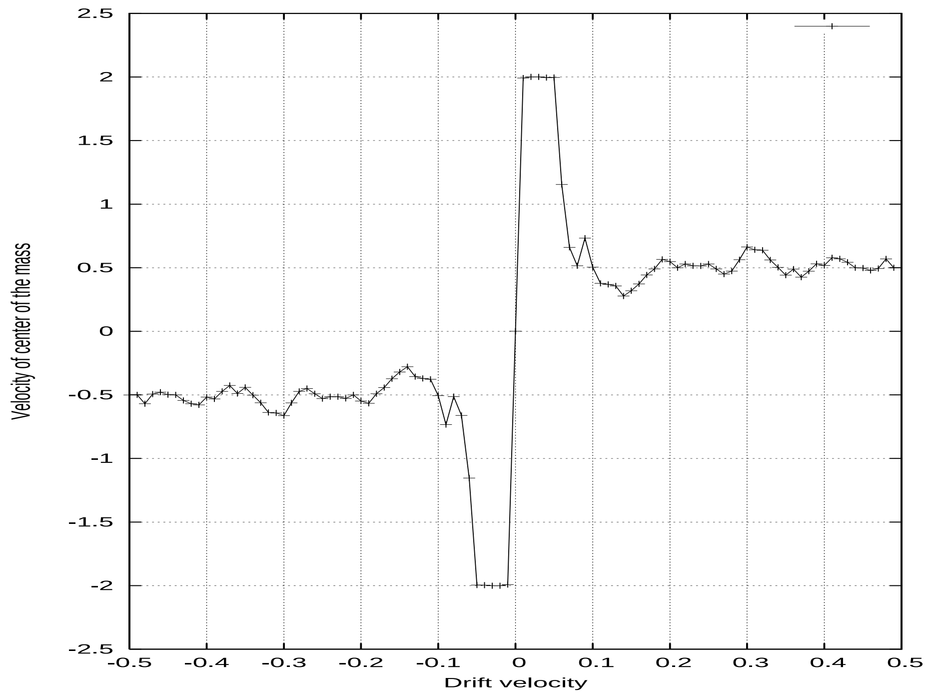

Fig. 1. The velocity of center of the mass of particles in the sine mapping *versus* the drift velocity caused by a potential force.

One can ask if it is wrong? The answer is no, it is not. Such system reactions may occur in the reality, since there are known transport processes that base on deterministic generators (like the Lorentz gas problems, the scattering transport theory [16]). Thus we need to be prepared for such, non-obvious, reactions of the systems.

3. Concluding remarks

A variety of phenomena can produce a probability propagation described by Fokker–Planck equations. These equations take into account different features of the underlying generators and have some limitations. However, the structure of coefficients for diffusion and drift remains similar in all of the examined approaches: RW, Langevin and chaotic.

Certainly the Langevin model of a Brownian particle dynamics breaks on the micro scale, where we definitely do not have a continuous spectrum of collisions with particles. The simple random walk model on the other hand implements a discontinuous position domain and needs tricky definitions for the jump probabilities in case, when we go with the limit of discretization to zero.

Also, the RW model stays apart of the microscopic dynamics. It makes some abstraction of dynamics in terms of probabilities, but does not keep a direct connection to it. The NIM approach is different in this respect, and can provide means to connect the micro dynamics with a macroscopic probability flow. This is, in a way, similar to Langevin approach that connects part of the dynamics (the friction) to probability propagation. The NIM approach goes further, and allows to inspect the effect of nonlinearities that are hidden in the noise term of Langevin approach.

The detailed derivation of suitable equations for deterministic generators is the most important part of this work. It shows that a Fokker–Planck equation can arise in this case coming purely from the chaotic properties of a map (usually such equations are derived basing on some assumptions of random walk similarities). It appears that the cells must be characterized by the escape rate formalism, and should have a finite and constant length.

This discretization, however, does not mean that we loose connection to what happens between, or inside single cells. This part of description is provided by invariant measure, and other tools of chaotic dynamics, not covered by Fokker–Planck equation [19]. Such sub-distributions can be viewed as second order corrections to the Fokker–Planck distribution that covers the macroscopic view of many single cells, interacting together.

As can be seen from this derivation the limit of $\Delta t \to 0$ can be taken without going into trouble of infinitely small differences in parameters, as it is in case of RW [12].

A drift term can arise from considerations of an external potential force, or from internal asymmetries of a mapping. In the first case, we see that reaction on this force depends on the mapping, and is not as straightforward as it was in the case of Langevin dynamics or random walk. This is because the mapping comes with some invariant measure that says about the distribution of particles within a single cell, and thus emphasizes the role of some parts of the mapping or diminishes it.

Introducing drift by a potential force can modify the invariant measure, and therefore, a simple proportionality cannot be expected. Additionally the regions that cause escape from the unit cell do not grow proportionally to the drift velocity which forms another problem. This sets a limit within the RW–NIM analogy in the description of transport phenomena.

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