GAUGE PRINCIPLE AND QED*

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One of the major developments of twentieth century physics has been the gradual recognition that a common feature of the known fundamental interactions is their gauge structure. In this talk the early history of gauge theory is reviewed, emphasising especially Weyl's seminal contributions of 1918 and 1929.

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1. Introduction

The organisers of this conference asked me to review the early history of gauge theories. Because of space and time limitations I shall concentrate on Weyl's seminal papers of 1918 and 1929. Important contributions by Fock, Klein and others, based on Kaluza's five-dimensional unification attempt, will not be discussed. (For this I refer to [31] and [32].)

The history of gauge theories begins with GR, which can be regarded as a non-Abelian gauge theory of a special type. To a large extent the other gauge theories emerged in a slow and complicated process gradually from GR. Their common geometrical structure — best expressed in terms of connections of fibre bundles — is now widely recognised.

It all began with Weyl [2] who made in 1918 the first attempt to extend GR in order to describe gravitation and electromagnetism within a unifying geometrical framework. This brilliant proposal contains the germs of all mathematical aspects of non-Abelian gauge theory. The word 'gauge' (German: 'Eich-') transformation appeared for the first time in this paper, but in the everyday meaning of change of length or change of calibration.

Einstein admired Weyl's theory as "a coup of genius of the first rate", but immediately realized that it was physically untenable. After a long discussion Weyl finally admitted that his attempt was a failure as a physical

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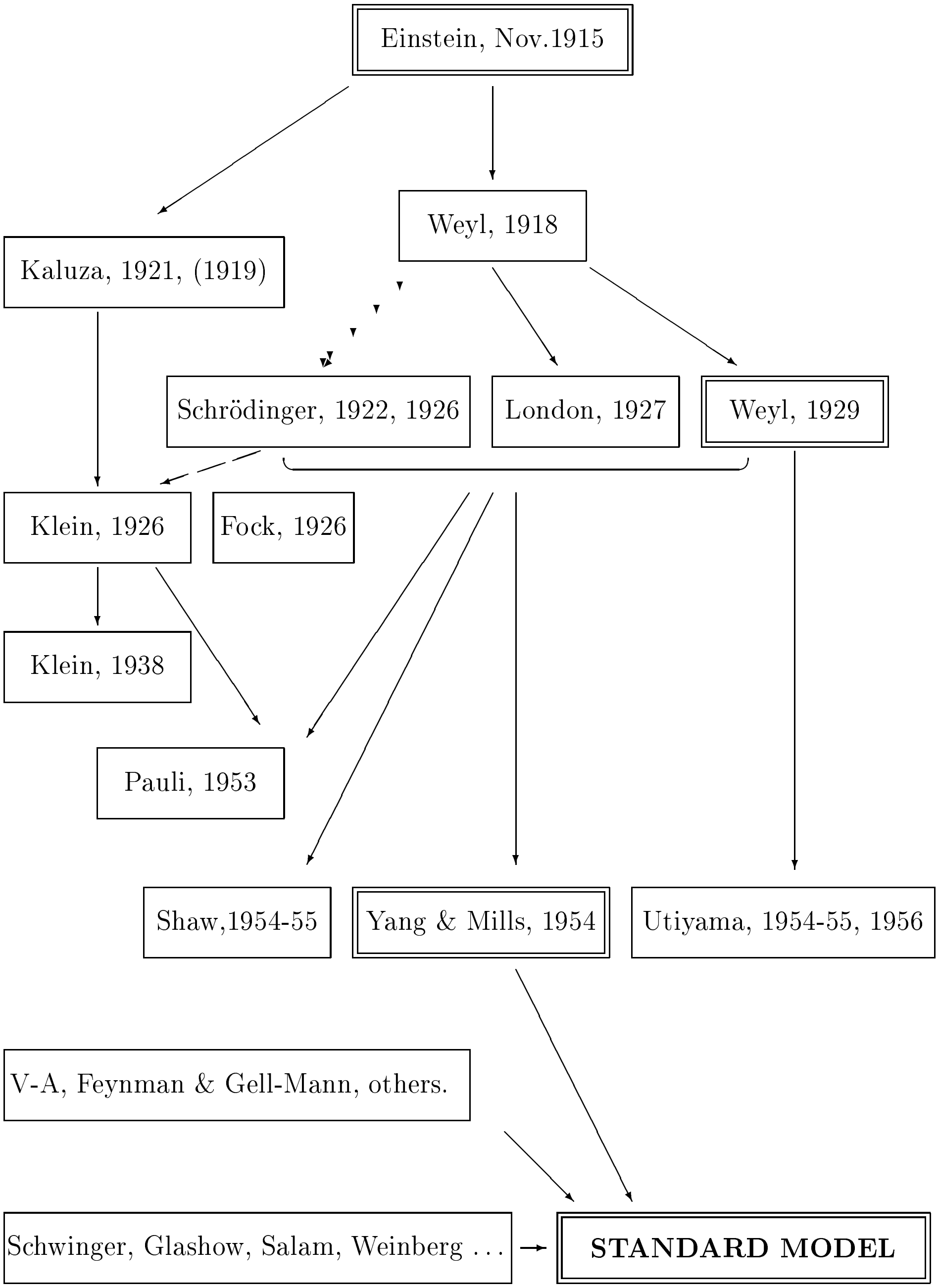


Fig. 1. Key papers in the development of gauge theories.

theory. (For a discussion of the intense Einstein–Weyl correspondence, see Ref. [4].) It paved, however, the way for the correct understanding of gauge invariance. Weyl himself reinterpreted in 1929 his original theory after the advent of quantum theory in a grand paper [5]. Weyl's reinterpretation of his earlier speculative proposal had actually been suggested before by London [11]. Fock [15], Klein [16], and others arrived at the principle of gauge invariance in the framework of wave mechanics along a completely different line. It was, however, Weyl who emphasised the role of gauge invariance as a *constructive principle* from which electromagnetism can be derived. This point of view became very fruitful for our present understanding of fundamental interactions. We[1] have described this more extensively in [31].

These works underlie the diagram in Fig. 1.

2. Weyl's attempt to unify gravitation and electromagnetism

On the 1st of March 1918 Weyl writes in a letter to Einstein ([3], Vol. 8B, Doc. 472): "These days I succeeded, as I believe, to derive electricity and gravitation from a common source . . . ". Einstein's prompt reaction by postcard indicates already a physical objection which he explained in detail shortly afterwards. Before we come to this we have to describe Weyl's theory of 1918.

2.1. Weyl's generalisation of Riemannian geometry

Weyl's starting point was purely mathematical. He felt a certain uneasiness about Riemannian geometry, as is clearly expressed by the following sentences early in his paper:

> *But in Riemannian geometry described above there is contained a last element of geometry "at a distance" (ferngeometrisches Element) — with no good reason, as far as I can see; it is due only to the accidental development of Riemannian geometry from Euclidean geometry. The metric allows the two magnitudes of two vectors to be compared, not only at the same point, but at any arbitrarily separated points. A true infinitesimal geometry should, however, recognise only a principle for transferring the magnitude of a vector to an infinitesimally close point and then, on transfer to an arbitrary distant point, the integrability of the magnitude of a vector is no more to be expected that the integrability of its direction.*

[1] Soon after our joint paper appeared in print Lochlain O'Raifeartaigh died suddenly, to the great sorrow and surprise of his family and numerous friends. I would like to dedicate this contribution to the memory of Lochlain.

After these remarks Weyl turns to physical speculation and continues as follows:

> *On the removal of this inconsistency there appears a geometry that, surprisingly, when applied to the world, explains not only the gravitational phenomena but also the electrical. According to the resultant theory both spring from the same source, indeed in general one cannot separate gravitation and electromagnetism in a unique manner. In this theory all physical quantities have a world geometrical meaning; the action appears from the beginning as a pure number. It leads to an essentially unique universal law; it even allows us to understand in a certain sense why the world is four-dimensional.*

In brief, Weyl's geometry can be described as follows (see also Ref. [8]). First, the spacetime manifold M is equipped with a conformal structure, *i.e.*, with a class $[g]$ of conformally equivalent Lorentz metrics g (and not a definite metric as in GR). This corresponds to the requirement that it should only be possible to compare lengths at one and the same world point. Second, it is assumed, as in Riemannian geometry, that there is an affine (linear) torsion-free connection which defines a covariant derivative ∇, and respects the conformal structure. Differentially this means that for any $g \in [g]$ the covariant derivative ∇g should be proportional to g:

$$\nabla g = -2A \otimes g \qquad (\nabla_\lambda g_{\mu\nu} = -2A_\lambda g_{\mu\nu})\,, \tag{1}$$

where $A = A_\mu dx^\mu$ is a differential 1-form.

Consider now a curve $\gamma : [0,1] \to M$ and a parallel-transported vector field X along γ. If l is the length of X, measured with a representative $g \in [g]$, we obtain from (1) the following relation between $l(p)$ for the initial point $p = \gamma(0)$ and $l(q)$ for the end point $q = \gamma(1)$:

$$l(q) = \exp\left(-\int_\gamma A\right)\, l(p)\,. \tag{2}$$

Thus, the ratio of lengths in q and p (measured with $g \in [g]$) *depends in general on the connecting path* γ (see Fig. 2). The length is only independent of γ if the curl of A,

$$F = dA \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)\,, \tag{3}$$

vanishes.

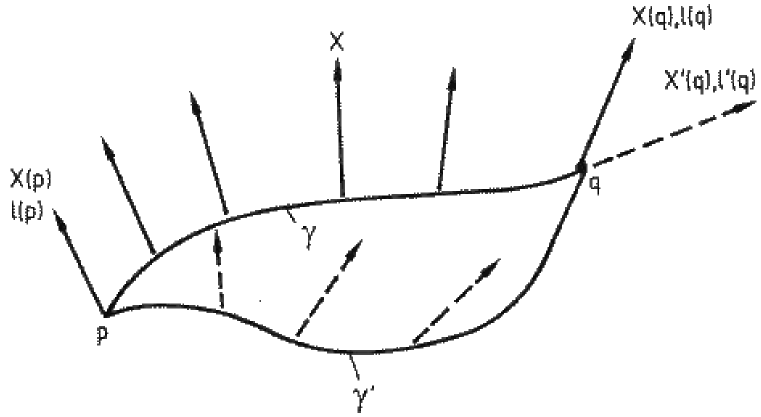


Fig. 2. Path dependence of parallel displacement and transport of length in Weyl space.

The compatibility requirement (1) leads to the following expression for the Christoffel symbols in Weyl's geometry:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(g_{\lambda\sigma,\nu} + g_{\sigma\nu,\lambda} - g_{\nu\lambda,\sigma}) + g^{\mu\sigma}(g_{\lambda\sigma}A_\nu + g_{\sigma\nu}A_\lambda - g_{\nu\lambda}A_\sigma) \,. \tag{4}$$

The second A-dependent term is a characteristic new piece in Weyl's geometry which has to be added to the Christoffel symbols of Riemannian geometry.

Until now we have chosen a fixed, but arbitrary metric in the conformal class $[g]$. This corresponds to a choice of calibration (or gauge). Passing to another calibration with metric $\bar{g}$, related to g by

$$\bar{g} = e^{2\lambda} g \,, \tag{5}$$

the potential A in (1) will also change to $\bar{A}$, say. Since the covariant derivative has an absolute meaning, $\bar{A}$ can easily be worked out: On the one hand we have by definition

$$\nabla \bar{g} = -2\bar{A} \otimes \bar{g}, \tag{6}$$

and on the other hand we find for the left side with (1)

$$\nabla \bar{g} = \nabla(e^{2\lambda} g) = 2d\lambda \otimes \bar{g} + e^{2\lambda}\nabla g = 2d\lambda \otimes \bar{g} - 2A \otimes \bar{g} \,. \tag{7}$$

Thus

$$\bar{A} = A - d\lambda \qquad (\bar{A}_\mu = A_\mu - \partial_\mu \lambda). \tag{8}$$

This shows that a change of calibration of the metric induces a *"gauge transformation"* for A:

$$g \to e^{2\lambda} g \,, \qquad A \to A - d\lambda \,. \tag{9}$$

Only gauge classes have an absolute meaning. (The Weyl connection is, however, gauge-invariant. This is conceptually clear, but can also be verified by direct calculation from expression Eq. (4).)

2.2. Electromagnetism and gravitation

Turning to physics, Weyl assumes that his "purely infinitesimal geometry" describes the structure of spacetime and consequently he requires that physical laws should satisfy a double-invariance: 1. They must be invariant with respect to arbitrary smooth coordinate transformations. 2. They must be *gauge invariant*, *i.e.*, invariant with respect to substitutions (9) for an arbitrary smooth function λ.

Nothing is more natural to Weyl, than identifying A_μ with the vector potential and $F_{\mu\nu}$ in Eq. (3) with the field strength of electromagnetism. In the absence of electromagnetic fields ($F_{\mu\nu} = 0$) the scale factor $\exp(-\int_\gamma A)$ in (2) for length transport becomes path independent (integrable) and one can find a gauge such that A_μ vanishes for simply connected spacetime regions. In this special case one is in the same situation as in GR.

Weyl proceeds to find an action which is generally invariant as well as gauge invariant and which would give the coupled field equations for g and A. We do not want to enter into this, except for the following remark. In his first paper [2] Weyl proposes what we call nowadays the Yang–Mills action

$$S(g, A) = -\frac{1}{4}\int \mathrm{Tr}(\Omega \wedge *\Omega)\,. \tag{10}$$

Here Ω denotes the curvature form and $*\Omega$ its Hodge dual[2]. Note that the latter is gauge invariant, *i.e.*, independent of the choice of $g \in [g]$. In Weyl's geometry the curvature form splits as $\Omega = \hat{\Omega} + F$, where $\hat{\Omega}$ is the metric piece [8]. Correspondingly, the action also splits,

$$\mathrm{Tr}(\Omega \wedge *\Omega) = \mathrm{Tr}(\hat{\Omega} \wedge *\hat{\Omega}) + F \wedge *F\,. \tag{11}$$

The second term is just the Maxwell action. Weyl's theory thus contains formally all aspects of a non-Abelian gauge theory.

Weyl emphasises, of course, that the Einstein–Hilbert action is not gauge invariant. Later work by Pauli [9] and by Weyl himself [1, 2] led soon to the conclusion that the action (10) could not be the correct one, and other possibilities were investigated (see the later editions of Weyl's classic treatise [1]).

[2] The integrand in (10) is in local coordinates indeed just the expression $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\sqrt{-g}dx^0 \wedge \ldots \wedge dx^3$ which is used by Weyl ($R_{\alpha\beta\gamma\delta}$= the curvature tensor of the Weyl connection).

Independent of the precise form of the action Weyl shows that in his theory gauge invariance implies the *conservation of electric charge* in much the same way as general coordinate invariance leads to the conservation of energy and momentum[3]. This beautiful connection pleased him particularly: "...[it] seems to me to be the strongest general argument in favour of the present theory — insofar as it is permissible to talk of justification in the context of pure speculation." The invariance principles imply five 'Bianchi type' identities. Correspondingly, the five conservation laws follow in two independent ways from the coupled field equations and may be "termed the eliminants" of the latter. These structural connections hold also in modern gauge theories.

2.3. Einstein's objection and reactions of other physicists

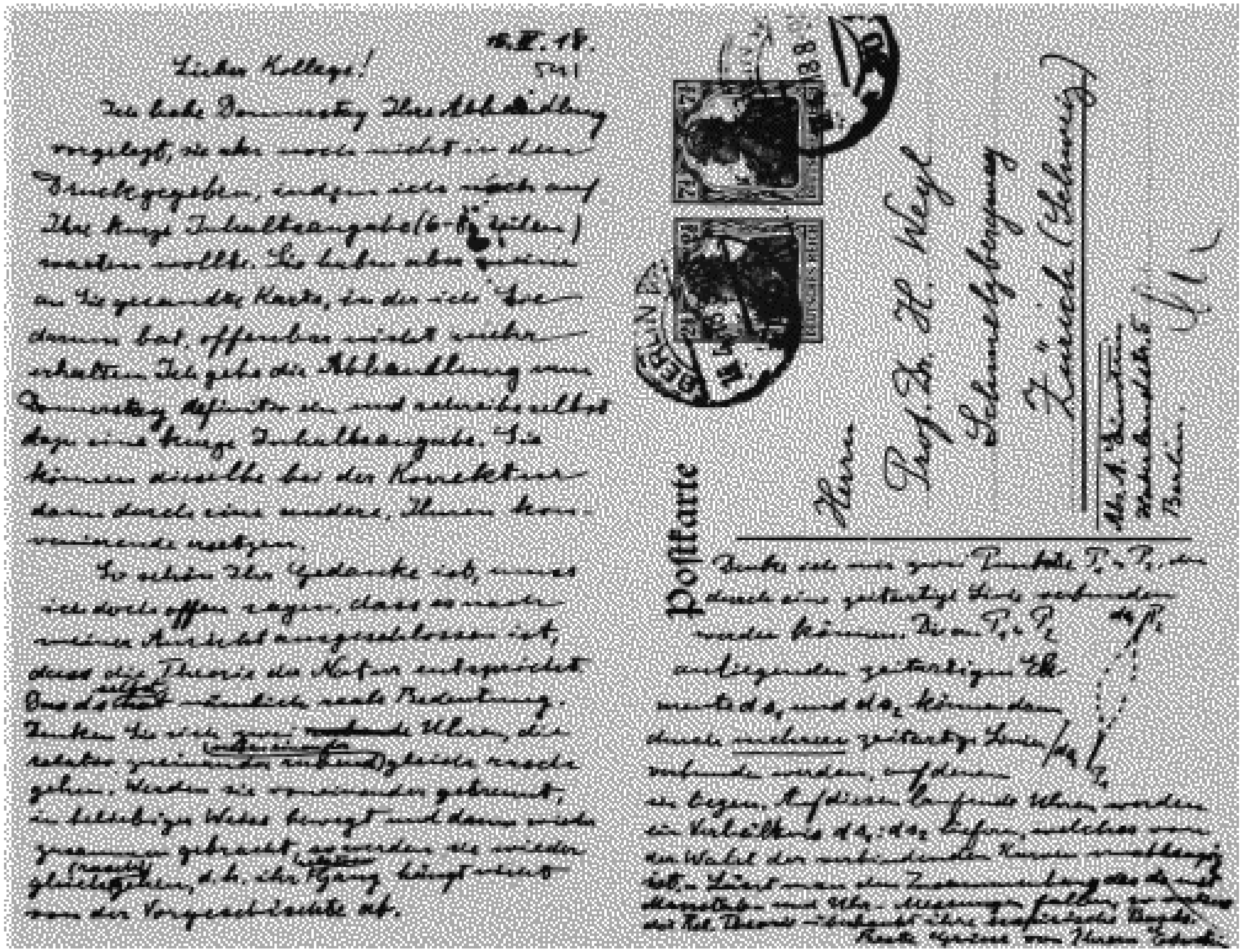
After this sketch of Weyl's theory we come to Einstein's striking counterargument which he first communicated to Weyl by postcard (see Fig. 3). The problem is that if the idea of a nonintegrable length connection (scale factor) is correct, then the behaviour of clocks would depend on their history. Consider two identical atomic clocks in adjacent world points and bring them along different world trajectories which meet again in adjacent world points. According to (2) their frequencies would then generally differ. This is in clear contradiction with empirical evidence, in particular with the existence of stable atomic spectra. Einstein therefore concludes (see [3], Vol. 8B, Doc. 507):

> *...(if) one drops the connection of the ds to the measurement of distance and time, then relativity looses all its empirical basis.*

Nernst shared Einstein's objection and demanded on behalf of the Berlin Academy that it should be printed in a short amendment to Weyl's article. Weyl had to accept this. One of us has described the intense and instructive subsequent correspondence between Weyl and Einstein elsewhere [4] (see also Vol. 8B of [3]). As an example, let us quote from one of the last letters of Weyl to Einstein ([3], Vol. 8B, Doc. 669):

> *This [insistence] irritates me of course, because experience has proven that one can rely on your intuition; so unconvincing as your counterarguments seem to me, as I have to admit ...*

[3] We adopt here the somewhat naive interpretation of energy-momentum conservation for generally invariant theories of the older literature.



15.IV.18.

Lieber Kollege!

Ich habe Donnerstag Ihre Abhandlung vorgelegt, sie aber noch nicht in den Druck gegeben, weil ich noch auf Ihre kurze Inhaltsangabe (6–8 Zeilen) warten wollte. Sie haben aber meine an Sie gesandte Karte, in der ich Sie darum bat, offenbar nicht mehr erhalten. Ich gebe die Abhandlung am Donnerstag definitiv ab und schreibe selbst dazu eine kurze Inhaltsangabe. Sie können dieselbe bei der Korrektur dann durch eine andere, Ihnen konvenierende ersetzen.

So schön Ihr Gedanke ist, muss ich doch offen sagen, dass es nach meiner Ansicht ausgeschlossen ist, dass die Theorie der Natur entspricht. Das ds selbst hat nämlich reale Bedeutung. Denken Sie sich zwei ~~ursprünglich~~ Uhren, die relativ zueinander ruhend gleich rasch gehen. Werden sie voneinander getrennt, in beliebiger Weise bewegt und dann wieder zusammen gebracht, so werden sie wieder gleich (rasch) gehen, d.h. ihr relativer Gang hängt nicht von der Vorgeschichte ab.

Denke ich mir zwei Punkte P_1 u. P_2, die durch eine zeitartige Linie verbunden werden können. Die an P_1 u. P_2 anliegenden zeitartigen Elemente $d\sigma_1$ und $d\sigma_2$ können dann durch mehrere zeitartige Linien verbunden werden, auf denen sie liegen. Auf diesen laufende Uhren werden ein Verhältnis $d\sigma_1 : d\sigma_2$ liefern, welches von der Wahl der verbindenden Kurven unabhängig ist. — Lässt man den Zusammenhang des ds mit Maßstab- und Uhr-Messungen fallen, so verliert die Rel. Theorie überhaupt ihre empirische Basis.

Beste Grüße von Ihrem Einstein.

Postkarte

Herrn
Prof. Dr. H. Weyl
Schmelzbergweg
Zürich (Schweiz)

Abs. A. Einstein
Haberlandstr. 5
Berlin.

Berlin W 15.4.18

Fig. 3. Postcard of Einstein to Weyl 15.4.1918 (Archives of ETH).

> *By the way, you should not believe that I was driven to introduce the linear differential form in addition to the quadratic one by physical reasons. I wanted, just to the contrary, to get rid of this 'methodological inconsistency (Inkonsequenz)' which has been a bone of contention to me already much earlier. And then, to my surprise, I realized that it looked as if it might explain electricity. You clap your hands above your head and shout: But physics is not made this way ! (Weyl to Einstein 10.12.1918).*

Weyl's reply to Einstein's criticism was, generally speaking, this: The real behaviour of measuring rods and clocks (atoms and atomic systems) in arbitrary electromagnetic and gravitational fields can be deduced only from a dynamical theory of matter.

Not all leading physicists reacted negatively. Einstein transmitted a very positive first reaction by Planck, and Sommerfeld wrote enthusiastically to Weyl that there was "... hardly doubt, that you are on the correct path and not on the wrong one."

In his encyclopedia article on relativity [10] Pauli gave a lucid and precise presentation of Weyl's theory, but commented on Weyl's point of view very critically. At the end he states:

> *... In summary one may say that Weyl's theory has not yet contributed to getting closer to the solution of the problem of matter.*

Also Eddington's reaction was at first very positive but he soon changed his mind and denied the physical relevance of Weyl's geometry.

The situation was later appropriately summarised by London in his 1927 paper [11] as follows:

> *In the face of such elementary experimental evidence, it must have been an unusually strong metaphysical conviction that prevented Weyl from abandoning the idea that Nature would have to make use of the beautiful geometrical possibility that was offered. He stuck to his conviction and evaded discussion of the abovementioned contradictions through a rather unclear re-interpretation of the concept of "real state", which, however, robbed his theory of its immediate physical meaning and attraction.*

In this remarkable paper, London suggested a reinterpretation of Weyl's principle of gauge invariance within the new quantum mechanics: The role of the metric is taken over by the wave function, and the rescaling of the metric has to be replaced by a phase change of the wave function.

In this context an astonishing early paper by Schrödinger [12] has to be mentioned, which also used Weyl's "*World Geometry*" and is related to Schrödinger's later invention of wave mechanics. This relation was discovered by Raman and Forman [13]. (See also the discussion by Yang in [14].)

Even earlier than London, Fock [15] arrived along a completely different line at the principle of gauge invariance in the framework of wave mechanics. His approach was similar to the one by Klein [16].

The contributions by Schrödinger [12], London [11] and Fock [15] are commented in [7], where also English translations of the original papers can be found. Here, we concentrate on Weyl's seminal paper "*Electron and Gravitation*".

3. Weyl's 1929 Classic: "Electron and Gravitation"

Shortly before his death late in 1955, Weyl wrote for his *Selecta* [17] a postscript to his early attempt in 1918 to construct a 'unified field theory'. There he expressed his deep attachment to the gauge idea and adds (p.192):

> *Later the quantum-theory introduced the Schrödinger-Dirac potential ψ of the electron-positron field; it carried with it an experimentally-based principle of gauge-invariance which guaranteed the conservation of charge, and connected the ψ with the electromagnetic potentials A_μ in the same way that my speculative theory had connected the gravitational potentials $g_{\mu\nu}$ with the A_μ, and measured the A_μ in known atomic, rather than unknown cosmological units. I have no doubt but that the correct context for the principle of gauge-invariance is here and not, as I believed in 1918, in the intertwining of electromagnetism and gravity.*

This re-interpretation was developed by Weyl in one of the great papers of this century [5]. Weyl's classic does not only give a very clear formulation of the gauge principle, but contains, in addition, several other important concepts and results — in particular his two-component spinor theory.

The modern version of the gauge principle is already spelled out in the introduction:

> *The Dirac field-equations for ψ together with the Maxwell equations for the four potentials f_p of the electromagnetic field have an invariance property which is formally similar to the one which I called gauge-invariance in my 1918 theory of gravitation and electromagnetism; the equations remain invariant when one makes the simultaneous substitutions*
>
> $$\psi \quad \text{by} \quad e^{i\lambda}\psi \qquad \text{and} \quad f_p \quad \text{by} \quad f_p - \frac{\partial\lambda}{\partial x^p}\,,$$

> *where λ is understood to be an arbitrary function of position in four-space. Here the factor $\frac{e}{ch}$, where $-e$ is the charge of the electron, c is the speed of light, and $\frac{h}{2\pi}$ is the quantum of action, has been absorbed in f_p. The connection of this "gauge invariance" to the conservation of electric charge remains untouched. But a fundamental difference, which is important to obtain agreement with observation, is that the exponent of the factor multiplying ψ is not real but pure imaginary. ψ now plays the role that Einstein's ds played before. It seems to me that this new principle of gauge-invariance, which follows not from speculation but from experiment, tells us that the electromagnetic field is a necessary accompanying phenomenon, not of gravitation, but of the material wave-field represented by ψ. Since gauge-invariance involves an arbitrary function λ it has the character of "general" relativity and can naturally only be understood in that context.*

We shall soon enter into Weyl's justification which is, not surprisingly, strongly associated with general relativity. Before this we have to describe his incorporation of the Dirac theory into GR which he achieved with the help of the tetrad formalism.

One of the reasons for adapting the Dirac theory of the spinning electron to gravitation had to do with Einstein's recent unified theory which invoked a distant parallelism with torsion. Wigner [18] and others had noticed a connection between this theory and the spin theory of the electron. Weyl did not like this and wanted to dispense with teleparallelism. In the introduction he says:

> *I prefer not to believe in distant parallelism for a number of reasons. First my mathematical intuition objects to accepting such an artificial geometry; I find it difficult to understand the force that would keep the local tetrads at different points and in rotated positions in a rigid relationship. There are, I believe, two important physical reasons as well. The loosening of the rigid relationship between the tetrads at different points converts the gauge-factor $e^{i\lambda}$, which remains arbitrary with respect to ψ, from a constant to an arbitrary function of space-time. In other words, only through the loosening the rigidity does the established gauge-invariance become understandable.*

This thought is carried out in detail after Weyl has set up his two-component theory in special relativity, including a discussion of P and T invariance. He emphasises thereby that the two-component theory excludes a linear implementation of parity and remarks: "It is only the fact that the left–right symmetry actually appears in Nature that forces us to introduce a second pair of ψ-components." To Weyl the mass-problem is thus not relevant for this[4]. Indeed he says: "Mass, however, is a gravitational effect; thus there is hope of finding a substitute in the theory of gravitation that would produce the required corrections."

3.1. Tetrad formalism

In order to incorporate his two-component spinors into GR, Weyl was forced to make use of local tetrads (Vierbeine). In section 2 of his paper he develops the tetrad formalism in a systematic manner. This was presumably independent work, since he does not give any reference to other authors. It was, however, mainly Cartan who demonstrated with his work [20] the usefulness of locally defined orthonormal bases — also called moving frames — for the study of Riemannian geometry.

In the tetrad formalism the metric is described by an arbitrary basis of orthonormal vector fields $\{e_\alpha(x); \alpha = 0,1,2,3\}$. If $\{e^\alpha(x)\}$ denotes the dual basis of 1-forms, the metric is given by

$$g = \eta_{\mu\nu} e^\mu(x) \otimes e^\nu(x)\,, \qquad (\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)\,. \tag{12}$$

Weyl emphasises, of course, that only a class of such local tetrads is determined by the metric: the metric is not changed if the tetrad fields are subject to spacetime-dependent Lorentz transformations:

$$e^\alpha(x) \to \Lambda^\alpha{}_\beta(x) e^\beta(x). \tag{13}$$

With respect to a tetrad, the connection forms $\omega = (\omega^\alpha{}_\beta)$ have values in the Lie algebra of the homogeneous Lorentz group:

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\,. \tag{14}$$

(Indices are raised and lowered with $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, respectively.) They are determined (in terms of the tetrad) by the first structure equation of Cartan:

$$de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0\,. \tag{15}$$

(For a textbook derivation see, *e.g.*, [21], especially Sects. 2.6 and 8.5.) Under local Lorentz transformations (13) the connection forms transform in the same way as the gauge potential of a non-Abelian gauge theory:

$$\omega(x) \to \Lambda(x)\omega(x)\Lambda^{-1}(x) - d\Lambda(x)\Lambda^{-1}(x)\,. \tag{16}$$

[4] At the time it was thought by Weyl, and indeed by all physicists, that the 2-component theory requires a zero mass. In 1957, after the discovery of parity non-conservation, it was found that the 2-component theory could be consistent with a finite mass. See Case, [19].

The curvature forms $\Omega = (\Omega^{\mu}{}_{\nu})$ are obtained from ω in exactly the same way as the Yang–Mills field strength from the gauge potential:

$$\Omega = d\omega + \omega \wedge \omega \tag{17}$$

(second structure equation).

For a vector field V, with components V^{α} relative to $\{e_{\alpha}\}$, the covariant derivative DV is given by

$$DV^{\alpha} = dV^{\alpha} + \omega^{\alpha}_{\ \beta} V^{\beta} \,. \tag{18}$$

Weyl generalises this in a unique manner to spinor fields ψ:

$$D\psi = d\psi + \frac{1}{4}\omega_{\alpha\beta}\sigma^{\alpha\beta}\psi \,. \tag{19}$$

Here, the $\sigma^{\alpha\beta}$ describe infinitesimal Lorentz transformations (in the representation of ψ). For a Dirac field these are the familiar matrices

$$\sigma^{\alpha\beta} = \frac{1}{2}[\gamma^{\alpha}, \gamma^{\beta}] \,. \tag{20}$$

(For 2-component Weyl fields one has similar expressions in terms of the Pauli matrices.)

With these tools the action principle for the coupled Einstein–Dirac system can be set up. In the massless case the Lagrangian is

$$\mathcal{L} = \frac{1}{16\pi G}R - i\bar{\psi}\gamma^{\mu}D_{\mu}\psi \,, \tag{21}$$

where the first term is just the Einstein–Hilbert Lagrangian (which is linear in Ω). Weyl discusses, of course, immediately the consequences of the following two symmetries:

(i) local Lorentz invariance,

(ii) general coordinate invariance.

3.2. The new form of the gauge-principle

All this is a kind of a preparation for the final section of Weyl's paper, which has the title "electric field". Weyl says:

> *We come now to the critical part of the theory. In my opinion the origin and necessity for the electromagnetic field is in the following. The components ψ_1 ψ_2 are, in fact, not uniquely*

> *determined by the tetrad but only to the extent that they can still be multiplied by an arbitrary "gauge-factor" $e^{i\lambda}$. The transformation of the ψ induced by a rotation of the tetrad is determined only up to such a factor. In special relativity one must regard this gauge-factor as a constant because here we have only a single point-independent tetrad. Not so in general relativity; every point has its own tetrad and hence its own arbitrary gauge-factor; because by the removal of the rigid connection between tetrads at different points the gauge-factor necessarily becomes an arbitrary function of position.*

In this manner Weyl arrives at the gauge-principle in its modern form and emphasises: "From the arbitrariness of the gauge-factor in ψ appears the necessity of introducing the electromagnetic potential." The first term $d\psi$ in (19) has now to be replaced by the covariant gauge derivative $(d - ieA)\psi$ and the nonintegrable scale factor (1) of the old theory is now replaced by a phase factor:

$$\exp\left(-\int\limits_{\gamma} A\right) \rightarrow \exp\left(-i\int\limits_{\gamma} A\right) ,$$

which corresponds to the replacement of the original gauge group $\mathbb{R}$ by the compact group U(1). Accordingly, the original Gedankenexperiment of Einstein translates now to the Aharonov–Bohm effect, as was first pointed out by Yang in [22]. The close connection between gauge invariance and conservation of charge is again uncovered. The current conservation follows, as in the original theory, in two independent ways: On the one hand it is a consequence of the field equations for matter plus gauge invariance, at the same time, however, also of the field equations for the electromagnetic field plus gauge invariance. This corresponds to an identity in the coupled system of field equations which has to exist as a result of gauge invariance. All this is nowadays familiar to students of physics and does not need to be explained in more detail.

Much of Weyl's paper penetrated also into his classic book "The Theory of Groups and Quantum Mechanics" [23]. There he mentions also the transformation of his early gauge-theoretic ideas: "This principle of gauge invariance is quite analogous to that previously set up by the author, on speculative grounds, in order to arrive at a unified theory of gravitation and electricity. But I now believe that this gauge invariance does not tie together electricity and gravitation, but rather electricity and matter."

When Pauli saw the full version of Weyl's paper he became more friendly and wrote [24]:

> *In contrast to the nasty things I said, the essential part of my last letter has since been overtaken, particularly by your paper in Z. f. Physik. For this reason I have afterward even regretted that I wrote to you. After studying your paper I believe that I have really understood what you wanted to do (this was not the case in respect of the little note in the Proc. Nat. Acad.). First let me emphasise that side of the matter concerning which I am in full agreement with you: your incorporation of spinor theory into gravitational theory. I am as dissatisfied as you are with distant parallelism and your proposal to let the tetrads rotate independently at different space-points is a true solution.*

In brackets Pauli adds:

> *Here I must admit your ability in Physics. Your earlier theory with $g'_{ik} = \lambda g_{ik}$ was pure mathematics and unphysical. Einstein was justified in criticising and scolding. Now the hour of your revenge has arrived.*

Then he remarks in connection with the mass-problem:

> *Your method is valid even for the massive* [Dirac] *case. I thereby come to the other side of the matter, namely the unsolved difficulties of the Dirac theory (two signs of m_0) and the question of the 2-component theory. In my opinion these problems will not be solved by gravitation … the gravitational effects will always be much too small.*

Many years later, Weyl summarised this early tortuous history of gauge theory in an instructive letter [25] to the Swiss writer and Einstein biographer Seelig, which we reproduce in an English translation.

> *The first attempt to develop a unified field theory of gravitation and electromagnetism dates to my first attempt in 1918, in which I added the principle of gauge-invariance to that of coordinate invariance. I myself have long since abandoned this theory in favour of its correct interpretation: gauge-invariance as a principle that connects electromagnetism not with gravitation but with the wave-field of the electron. — Einstein was against it* [the original theory] *from the beginning, and this led to many discussions. I thought that I could answer his concrete objections. In the end he said "Well, Weyl, let us leave it at that! In such a speculative manner, without any guiding physical principle, one cannot make Physics." Today one could say that in this respect*

> *we have exchanged our points of view. Einstein believes that in this field* [Gravitation and Electromagnetism] *the gap between ideas and experience is so wide that only the path of mathematical speculation, whose consequences must, of course, be developed and confronted with experiment, has a chance of success. Meanwhile my own confidence in pure speculation has diminished, and I see a need for a closer connection with quantum-physics experiments, since in my opinion it is not sufficient to unify Electromagnetism and Gravity. The wave-fields of the electron and whatever other irreducible elementary particles may appear must also be included.*

Independently of Weyl, Fock [26] also incorporated the Dirac equation into GR by using the same method. On the other hand, Tetrode [27], Schrödinger [28] and Bargmann [29] reached this goal by starting with space-time dependent γ-matrices, satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$. A somewhat later work by Infeld and van der Waerden [30] is based on spinor analysis.

4. Concluding remarks

Gauge invariance became a serious problem when Heisenberg and Pauli began to work on a relativistically invariant QED that eventually resulted in two important papers "On the Quantum Dynamics of Wave Fields" [33], [34]. Straightforward application of the canonical formalism led, already for the free electromagnetic field, to nonsensical results. Jordan and Pauli on the other hand, proceeded to show how to quantise the theory of the *free field* case by dealing only with the field strengths $F_{\mu\nu}(x)$. For these they found commutation relations at different space-time points in terms of the now famous invariant Jordan–Pauli distribution that are manifestly Lorentz invariant.

The difficulties concerned with applying the canonical formalism to the electromagnetic field continued to plaque Heisenberg and Pauli for quite some time. By mid-1928 both were very pessimistic, and Heisenberg began to work on ferromagnetism[5]. In fall of 1928 Heisenberg discovered a way to bypass the difficulties. He added the term $-\frac{1}{2}\varepsilon(\partial_\mu A^\mu)^2$ to the Lagrangian,

[5] Pauli turned to literature. In a letter of 18 February 1929 he wrote from Zürich to Oskar Klein: "For my proper amusement I then made a short sketch of a utopian novel which was supposed to have the title 'Gulivers journey to Urania' and was intended as a political satire in the style of Swift against present-day democracy. [...] Caught in such dreams, suddenly in January, news from Heisenberg reached me that he is able, with the aid of a trick ... to get rid of the formal difficulties that stood against the execution of our quantum electrodynamics." [6]

in which case the component π_0 of the canonical momenta

$$\pi_\mu = \frac{\partial L}{\partial_0 A_\mu}$$

does no more vanish identically ($\pi_0 = -\varepsilon \partial_\mu A^\mu$). The standard canonical quantisation scheme can then be applied. At the end of all calculations one could then take the limit $\varepsilon \to 0$.

In their second paper, Heisenberg and Pauli stressed that the Lorentz condition cannot be imposed as an operator identity but only as a supplementary condition selecting admissible states. This discussion was strongly influenced by a paper of Fermi from May 1929.

As in Weyl's work GR also played a crucial role in Pauli's discovery of non-Abelian gauge theories. (See Pauli's letters to Pais and Yang in Vol. 4 of [6]). He arrived at all basic equations through dimensional reduction of a generalisation of Kaluza–Klein theory, in which the internal space becomes a two-sphere. (For a description in modern language, see [31].)

On the other hand, in the work of Yang and Mills GR played no role. In an interview Yang said on this in 1991:

> "*It happened that one semester [around 1970] I was teaching GR, and I noticed that the formula in gauge theory for the field strength and the formula in Riemannian geometry for the Riemann tensor are not just similar – they are, in fact, the same if one makes the right identification of symbols! It is hard to describe the thrill I felt at understanding this point.*"

The developments after 1958 consisted in the gradual recognition that — contrary to phenomenological appearances — Yang–Mills gauge theory could describe weak and strong interactions. This important step was again very difficult, with many hurdles to overcome.

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