CURVES AND THE PHOTON*

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The study of the number of photons leads to a new way of characterizing curves and to a novel integral invariant over curves.

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1. Introduction

It is always surprising how far one good idea can go. And while we are here celebrating the 100th birthday of the photon, I would like to add one more fillip on that good idea. This time verging into almost pure mathematics; the photon leads us to a new way of characterizing curves and to a novel integral invariant over curves.

In classical radiation theory practically all there is to calculate is the energy radiated; or perhaps occasionally a frequency spectrum or a polarization. But with the advent of the photon there is a new quantity in the radiation field: just the plain number of photons. This is an ordinary real number, evidently dimensionless. And independent of the Lorentz frame; while changing your velocity can change the energy of a photon, it cannot change one photon into two photons. If it is well-defined there ought to be some nice simple formula for it. "But aha", you will say, "that's just the trouble, it's not well defined. There's the notorious infra-red catastrophe, where as we all know, the number of photons radiated is infinite — even while the radiated energy remains finite."

Correct. But nevertheless, as I was surprised to realize, there is a general class of situations where the infra-red catastrophe is averted.

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2. Finite number of photons

Let n be the number of photons radiated by a charge following some given trajectory. Let the initial and final velocities be exactly the same — let us "identify" them. In this case n can in fact be finite. This is because in a scattering process the infra-red catastrophe arises from the infinite flight paths for the incoming and outgoing particle, as the infinite range field is shaken off or re-constituted. Or in terms of Fourier transforms, zero frequency results from infinite time. When the re-constituting is not necessary because the infinite paths are identical, or when the infinite times do not occur, the number of photons can be finite. Naturally the exact, perfect, identity of initial and final velocities will never occur in reality. But we can still imagine the idealized case and ask ourselves what the nice simple formula looks like. It looks like this [1]:

$$n = \frac{\alpha}{\pi} \iint dx_\mu \frac{1}{S_{i\epsilon}^2} dx'_\mu = \frac{2\alpha}{\pi} \iint dx_\mu \frac{\delta_{\mu\nu} - \frac{\Delta_\mu \Delta_\nu}{S^2}}{S^2} dx'_\nu = \frac{2\alpha}{\pi} \iint \frac{dx_\mu^{\mathrm{T}} dx_\mu^{\prime\mathrm{T}}}{S^2} \,. \tag{1}$$

In the first writing $S_{i\epsilon}$ is the four-distance between the points x, x' in the following way: $S_{i\epsilon}^2 = (t - t' + i\epsilon)^2 - (\boldsymbol{x} - \boldsymbol{x'})^2$. This formula follows from taking the text book formula for the energy radiated at frequency ω and dividing by ω to get n (Planck). This is the only place where quantum mechanics enters. The $i\epsilon$ arises in making sure the resulting integral is defined, but we get rid of it in the second writing by some manipulations which lead to the factor 2 and the "transverse tangent" expression in the last writing. The "transverse tangent" means to take the tangent vector dx and to remove the component of it along the vector $\Delta_\mu = (x_\mu - x'_\mu)$ connecting the two points x, x'. This construction cleverly avoids the threatened singularity as $x \to x'$, since as two points approach each other along a curve the tangents point at each other and $dx^{\mathrm{T}} \to 0$. The result refers of course to the average number of photons since the number fluctuates — n is not an integer.

Such a formula for n is quite amusing since it means that any curve satisfying the "identification" requirement has a real number belonging to it — its own "name". This is a simple number, an invariant, intrinsic property of the curve. The "identification" requirement is not really very restrictive since the curve can do almost anything it wants before it finally goes back out to infinity parallel (in the four-dimensional sense) to the direction it came in. "Almost" because to avoid that other lurking danger, the ultra-violet divergence, the curve must be smooth and not have any kinks or jumps. One may verify that Eq. (1) leads to the correct result for dipole radiation and yields the usual association between acceleration and radiation.

3. Euclidean space

In Minkowski space then, we have a nice way of associating a number to a curve. We may now even forget photons and physics for a moment and wonder if this applies to ordinary curves, plain garden-variety curves in Euclidean space. Indeed, if we take the last form in Eq. (1) as our starting point, there appears to be nothing against this. The argument that the transverse tangent construction is finite as $x \to x'$ still holds. So we write

$$n = -2 \int \int \frac{\boldsymbol{dx}^{\mathrm{T}} \boldsymbol{dx'}^{\mathrm{T}}}{(\boldsymbol{x} - \boldsymbol{x'})^2} \tag{2}$$

and n is still an invariant dimensionless quantity, an intrinsic property of the curve — basically characterizing its "wiggliness". We must of course retain our requirement concerning the infra-red problem. As the curve goes to $\pm\infty$ it has the same tangent — becomes the same straight line (remember that in four-space the tangent was the velocity). But otherwise, as long as it is smooth, the curve can do anything on its way from $-\infty$ to $+\infty$. We have dropped the α/π since we are now only interested in the purely mathematical structure, but have kept the $(-)$ sign since this makes things come out positive.

Fig. 1 shows an example, with n evaluated according to Eq. (2). According to a little Fortran program, the number for this curve is 38.8. Playing with the end points indicates that the number has essentially gone asymptotic in the picture. For the straight line Eq. (2) gives of course zero.

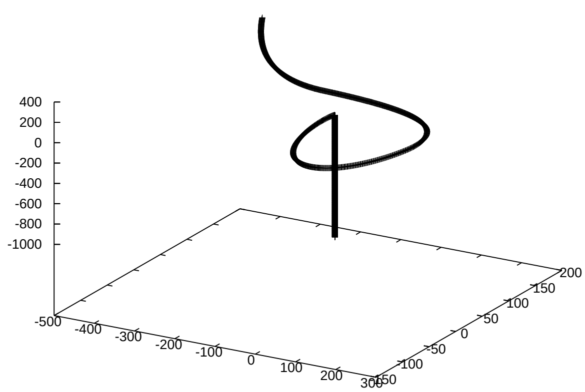

Fig. 1. A curve whose number is 38.8.

Furthermore, in Euclidean space a new opportunity presents itself: the possibility of closed curves. In Minkowski space a curve could not go "backwards" since it could not have a tangent $> 45°$, outside the light cone. But now with no light cone to cross, there is no reason not to consider closed curves. Furthermore, note that for a closed curve the "beginning" and "end" of a trajectory are the same, so the identity of initial and final tangents is automatic.

If all this is true then what is the number of that most basic of closed curves, the circle? It is:

$$n_{\text{circle}} = 2\pi^2 \,, \tag{3}$$

as just follows from integrating Eq. (2) directly. The value is of course a result of the normalization we chose for our integral, but once we have chosen it, it is the same for all circles, big ones and small ones; n depends only the shape of the curve. In addition, we also might have the strong suspicion that among all plane curves the circle has the smallest n. I believe this is indeed true and hope to present a proof shortly. To exemplify this, here is a little table for the ellipse with different eccentricities ϵ. The first entry, for the circle, is close to $\pi^2 = 9.87$, and as would be expected, the more eccentric the ellipse the larger n.

TABLE

Numerical evaluation of Eq. (2) for ellipses of increasing eccentricity.

Eccentricity	$n/2$
0	9.83
0.5	9.93
0.7	10.4
0.9	13.4
0.95	17.2
0.99	35.2

4. Inversion and a new integral invariant

Given a curve, what other curves have the same n? We might suspect, because of the dimensionless character of Eq. (2), that in addition to the usual invariances under translation, rotation and so forth, that n is conformally invariant [2]. The essential part of conformal invariance is the inversion $x_i \to \frac{x_i}{x^2}$. The inversion carries closed curves into closed curves except when the center of inversion is on the curve itself, in which case the circle, for example, becomes the straight line going to infinity. It then turns out that Eq. (2) is invariant under inversions, except for these exceptional cases. In the exceptional cases one must universally add $2\pi^2$. This is just right to get Eq. (3) when inverting the straight line, for which n is zero, to a circle.

The fact that this extra addition is universal — in this case is the same for any closed curve — is suggestive [3] of the "anomaly". That is we have symmetry breaking — here for the inversion — but in a universal manner. The analysis of this situation leads to the study of a quantity called I [2]. I has the same value for all curves of a given class, of which there are four, and represents a novel kind of integral invariant.

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