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DESIGN OF A MAPPING FOR HYPERBOLIC DIFFUSION*

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Dedicated to Professor Peter Talkner on the occasion of his 60th birthday

In this paper we investigate whether it is possible to obtain a hyperbolic diffusion from a chaotic mapping. We design an appropriate mapping, and explain which features are responsible for the hyperbolicity, together with a quantitative descriptions of the coefficients of the hyperbolic diffusion.

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1. Introduction

In a classical, phenomenological sense, diffusion is a process by which different substances mix as a result of the random motion of their component atoms, molecules and ions [1].

On this, macroscopic, level the process may be described by quite a wide class of partial differential equations, usually of the second order [2–5].

$$AU_{xx} + 2BU_{xt} + CU_{tt} + DU_x + EU_t + FU + G = 0\,. \tag{1}$$

According to the discriminant rule (which has its roots in conic curves classification) $d = AC - B^2$ we have the following types of equations

- elliptic if $d > 0$,
- parabolic if $d = 0$,
- hyperbolic if $d < 0$.

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Even if this classification is only local for equations with varying coefficients it is still useful. A sort of "finger print" of a parabolic type is the presence of the first order derivative with respect to time along with the second order with respect to position. Also, the presence of second order derivatives with respect to space and time with the proper signs, indicates the hyperbolic character of a given equation.

According to this classification "Diffusion Operator" (or Heat Operator) is parabolic, while d'Alambertian (wave operator) is hyperbolic.

When considering the microscopic view on diffusion through some correlated random walk one arrives at hyperbolic type of equation as a continuous limit [6–8].

So, as can be realized one has to be "more flexible" in using the classical ideas. The same holds for "randomness" introduced in a classical definition of diffusion [4, 7, 9–11, 13, 14]. It can be replaced formally by some deterministic entities [15–23]. In a sense it agrees with a general, and almost trivial, observation that a complex phenomenon can be described in a non-unique way *i.e.* by several different approaches each of them enlightening some aspects of it better than the others. So, the question of stochastic or deterministic nature of diffusion looks simply to be not "well posed" from the today's perspective. It, definitely, has rather complex character with many "faces" which can be shown either by stochastic or deterministic approaches. In this paper we will use the nonlinear chaotic iterated maps (NIM) to discuss certain aspects of the hyperbolic diffusion difficult to show otherwise.

2. Hyperbolic diffusion and the random walk: initial assumptions

To understand the idea behind hyperbolic diffusion out of a NIM system, we need to recall the derivation of the hyperbolic diffusion equation out of a correlated random walk. We will introduce it by extending the idea presented in the book of Zauderer [7].

In this approach we consider two populations of particles: one is the population of particles traveling to the left (concentration $\alpha(x,t)$), second is the population of particles traveling to the right (concentration $\beta(x,t)$). Now, p is the probability to retain the direction of motion, and q is the probability to change it. Therefore, the concentrations in a new time step τ can be described by the equation

$$\alpha(x,t+\tau) \; = \; p\alpha(x-\delta,t)+q\beta(x-\delta,t)\,, \tag{2}$$

$$\beta(x,t+\tau) \; = \; p\beta(x+\delta,t)+q\alpha(x+\delta,t)\,, \tag{3}$$

where δ is the grid spacing. Expanding these expressions to a Taylor series in time and position, we obtain:

$$\alpha(x,t)+\frac{\partial\alpha(x,t)}{\partial t}\tau = p\alpha(x,t)-p\delta\frac{\partial\alpha(x,t)}{\partial x}+q\beta(x,t)-q\delta\frac{\partial\beta(x,t)}{\partial x}, \quad (4)$$
$$\beta(x,t)+\frac{\partial\beta(x,t)}{\partial t}\tau = p\beta(x,t)+p\delta\frac{\partial\beta(x,t)}{\partial x}+q\alpha(x,t)+q\delta\frac{\partial\alpha(x,t)}{\partial x}. \quad (5)$$

Noticing that $p+q=1$, we can rewrite this equation as

$$q\alpha(x,t)+\frac{\partial\alpha(x,t)}{\partial t}\tau = -p\delta\frac{\partial\alpha(x,t)}{\partial x}+q\beta(x,t)-q\delta\frac{\partial\beta(x,t)}{\partial x}, \quad (6)$$
$$q\beta(x,t)+\frac{\partial\beta(x,t)}{\partial t}\tau = p\delta\frac{\partial\beta(x,t)}{\partial x}+q\alpha(x,t)+q\delta\frac{\partial\alpha(x,t)}{\partial x}. \quad (7)$$

At this moment it is possible to take another assumption. This assumption will be used in the design of a NIM mapping and hence the need of presenting this derivation. We assume that $p=1-\lambda t$, $q=\lambda t$, which essentially says that the probability to change direction after short time period is small and increases with time (becoming quite probable after correlation time). Making use of this, we rewrite the previous equation into

$$\begin{aligned}\lambda\tau\alpha(x,t)+\frac{\partial\alpha(x,t)}{\partial t}\tau &= -\delta\frac{\partial\alpha(x,t)}{\partial x}+\lambda\tau\delta\frac{\partial\alpha(x,t)}{\partial x}\\ &\quad+\lambda\tau\beta(x,t)-\lambda\tau\delta\frac{\partial\beta(x,t)}{\partial x}, \qquad (8)\\ \lambda\tau\beta(x,t)+\frac{\partial\beta(x,t)}{\partial t}\tau &= \delta\frac{\partial\beta(x,t)}{\partial x}-\lambda\tau\delta\frac{\partial\beta(x,t)}{\partial x}\\ &\quad+\lambda\tau\alpha(x,t)+\lambda\tau\delta\frac{\partial\alpha(x,t)}{\partial x}. \qquad (9)\end{aligned}$$

Now we divide these equations by τ and assume $\delta/\tau=\gamma$:

$$\begin{aligned}\lambda\alpha(x,t)+\frac{\partial\alpha(x,t)}{\partial t} &= -\gamma\frac{\partial\alpha(x,t)}{\partial x}+\lambda\delta\frac{\partial\alpha(x,t)}{\partial x}\\ &\quad+\lambda\beta(x,t)-\lambda\delta\frac{\partial\beta(x,t)}{\partial x}, \qquad (10)\\ \lambda\beta(x,t)+\frac{\partial\beta(x,t)}{\partial t} &= \gamma\frac{\partial\beta(x,t)}{\partial x}-\lambda\delta\frac{\partial\beta(x,t)}{\partial x}\\ &\quad+\lambda\alpha(x,t)+\lambda\delta\frac{\partial\alpha(x,t)}{\partial x}, \qquad (11)\end{aligned}$$

Assuming now that $\delta,\tau\to 0$ in such way that γ exists, we can eliminate from the equation all terms containing δ. This is another important assumption. It means that after developing the NIM system, we need to consider long time scales and broad domain of position. The equations set changes under this assumption into

$$\lambda\alpha(x,t) + \frac{\partial\alpha(x,t)}{\partial t} = -\gamma\frac{\partial\alpha(x,t)}{\partial x} + \lambda\beta(x,t)\,, \tag{12}$$

$$\lambda\beta(x,t) + \frac{\partial\beta(x,t)}{\partial t} = \gamma\frac{\partial\beta(x,t)}{\partial x} + \lambda\alpha(x,t)\,. \tag{13}$$

Adding and subtracting these equations we obtain:

$$\frac{\partial(\alpha(x,t)+\beta(x,t))}{\partial t}+\gamma\frac{\partial(\alpha(x,t)-\beta(x,t))}{\partial x} = 0\,, \tag{14}$$

$$\frac{\partial(\alpha(x,t)-\beta(x,t))}{\partial t}+\gamma\frac{\partial(\alpha(x,t)+\beta(x,t))}{\partial x} = -2\lambda(\alpha(x,t)-\beta(x,t))\,. \tag{15}$$

Noticing that $\alpha(x,t) + \beta(x,t) = \rho(x,t)$ (the concentration of particles at point x in time t no matter the direction of motion), we would like to eliminate from the equations set the terms depending on $(\alpha(x,t) - \beta(x,t))$. Differentiating the upper equation by time, and the lower equation by position, we can readily eliminate the mixed derivative term, and we are able to substitute the RHS of lower equation by the LHS of upper equation before differentiation. This way we obtain the hyperbolic diffusion equation

$$2\lambda\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial^2\rho(x,t)}{\partial t^2} = \gamma^2\frac{\partial^2\rho}{\partial x^2}\,. \tag{16}$$

3. The chaotic mapping for hyperbolic diffusion

We have seen in the previous section that the key point of derivation was the assumption that the probability of turning changes with time as $p = 1 - \lambda t$.

Consider now a chaotic mapping, defined by the chart in Fig. 1 and the equation

$$x_{N+1} = [x_N] + F(x_N - [x_N])\,, \tag{17}$$

where the symbol $[\ldots]$ stands for the integer part of its argument. The construction of the mapping is according to traditional rules of symmetry, $F(x) = 1 - F(1-x)$ which assures that no direction of motion is preferred and we have no drift components. The function $F(x)$ itself on the domain (0,0.5) can be defined by:

$$x < x_1 \;:\; \frac{x}{x_1}\,, \tag{18}$$

$$x \geq x_1\,, \quad x < 0.5 - DX \;:\; 1 + x + DX\,, \tag{19}$$

$$x \geq 0.5 - DX \;:\; \frac{x}{DX} + \left(1 - \frac{0.5}{DX}\right)\,. \tag{20}$$

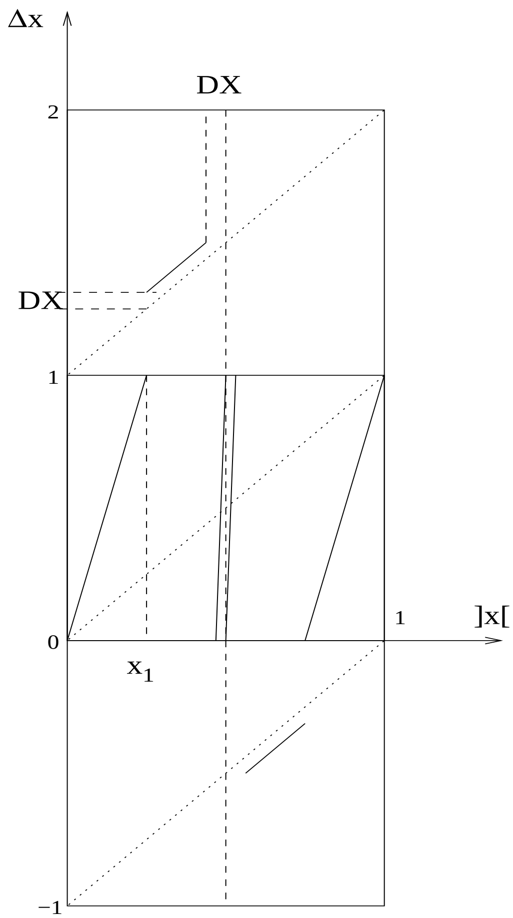

Fig. 1. The chart of the mapping that generates hyperbolic diffusion.

One can easily see that the only way to escape from the unit cell is through the region where $]x[= x - [x] \in (x_1, 0.5 - DX) \cup (0.5 + DX, 1 - x_1)$. This region, however, presents a sort of intermittency. The intermittency is not of standard type, where the trajectory sticks in some point of space (which corresponds to a sub-diffusive behavior [24]), but it is an intermittency modulo 1, where the particle "stroboscopically" travels through subsequent cells (*cf.* Fig. 2).

In this chart we can see that the particle, once it falls into the upper region (which corresponds to travel in the right direction), it remains there for certain time that depends on the point of injection. If the particle is injected closer to $0.5 - DX$, then it will remain there for a shorter time, if closer to x_1, it will remain there longer. Similar analysis can be done for the region $(0.5 + DX, 1 - x_1)$.

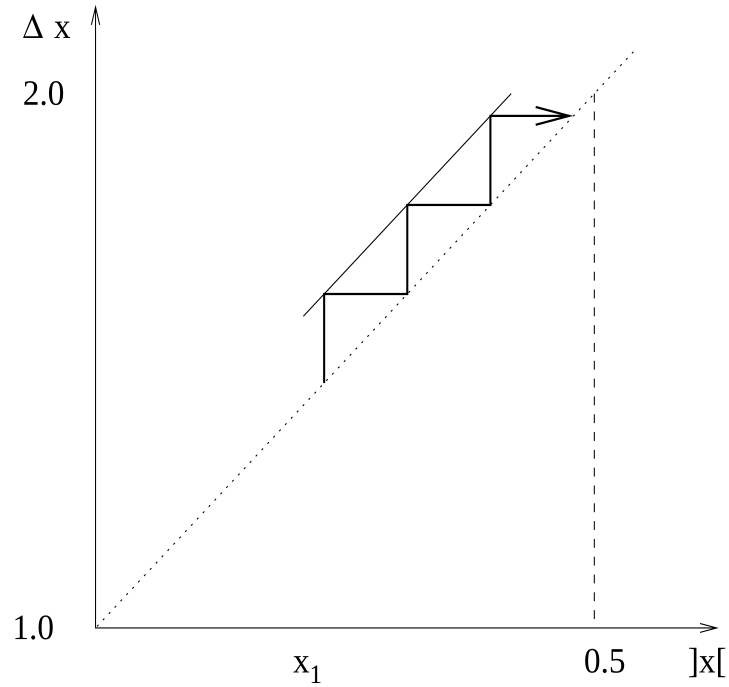

Fig. 2. Magnification of the upper square from Fig. 1 and the idea of intermittency modulo 1.

If x_1 is close to 0.5, the mixing in the region $\{(0,1)-(x_1,1-x_1)\} \cup (0.5-DX, 0.5+DX)$ can take several iterations before the particle hits the escaping region $(x_1, 1-x_1)-(0.5-DX, 0.5+DX)$. Therefore, the point of injection can be considered quite random, and because of its small width and the form of invariant measure [23, 25] for a linear chaotic mapping with constant slope (flat one), we assume equal probability distribution for these injections.

The feature of mixing time, however, does not correspond to the hyperbolic model that we have derived for a random walk, since in that model the particle could not remain at rest in a place. Therefore, the mixing time should be short comparing to the travel time. This can be done by decreasing the value of DX which controls the length of jump in intermittent state.

Another question concerning the presented mapping, which can arise is the following: what happens when the particle escapes the intermittent region? Then the particle lands randomly in the region $(0.5-DX, 0.5)$. In this region we have defined a mapping that spreads the trajectory all over the unit cell. Thus, the particle will undergo a new mixing step inside a unit cell.

The next question is whether we can say something about the probability to change direction of motion. When the trajectory is in the middle of intermittency, there is no such possibility. It arises only when it escapes it. Thus, we ask the question what is the probability to escape the intermittent region, assuming the trajectory is inside it.

The probability of retaining the direction in step zero is equal to probability of being injected in the region of intermittency or to the probability of being injected to intermittent region K steps before, and remaining in the intermittent region after K iterations. Thus taking N for the maximum number of iterations within intermittent region ($N = (0.5 - x_1 - DX)/DX$), we have the following number of combinations

$$N + (N-1) + (N-2) + \ldots + 1 = \frac{N}{2}\,. \tag{21}$$

After one step the situation changes because one point of injection is no longer a good candidate (the furthest point escapes the intermittent region) and the number of realizations from previous injections also decreases (for example one can preform $1 \ldots N-2$ iterations from the point of injection being most to the left instead of $1 \ldots N-1$), giving the number of trajectories as

$$(N-1) + (N-2) + (N-3) + \ldots + 1 = \frac{N-1}{2}\,. \tag{22}$$

In K steps we have the expression

$$(N-K) + (N-K-1) + (N-K-2) + \ldots + 1 = \frac{N-K}{2}\,. \tag{23}$$

Normalizing equation (23) with (21), we obtain the probability to keep the direction of motion:

$$p = \frac{N-K}{N} = 1 - \frac{K}{N}\,, \tag{24}$$

which corresponds to the probability model assumed in the random walk derivation and which gives the interpretation to the λ coefficient, giving $\lambda = 1/N = DX/(0.5 - DX - x_1)$.

4. Concluding remarks

Hyperbolic diffusion is an important extension of a classical diffusion process. It corrects the major drawback of classical diffusion which is the infinite speed of establishing any probability density distribution (concentration profile). Rewriting the hyperbolic diffusion equation (16) in a new form, with no coefficient standing next to the first derivative in time (as in classical diffusion equation), we can relate some physical parameters describing the transport process, obtaining

$$\frac{\partial \rho(x,t)}{\partial t} + \tau \frac{\partial^2 \rho(x,t)}{\partial t^2} = D \frac{\partial^2 \rho(x,t)}{\partial x^2}\,, \tag{25}$$

where $\tau = 1/(2\lambda)$, $D = \gamma^2/(2\lambda)$. τ in this equation is the inverse of λ and has the interpretation of the relaxation time for the motion in given direction (recall that the probability of changing direction equals to $p = 1 - \lambda t = 1 - t/(2\tau)$).

Notice that for small relaxation times (or high λ), the probability to retain direction reduces to 1 at $t \to 0$, and to zero elsewhere. This is a return to classical random walk.

Knowing the relations to diffusion coefficient, we can also derive the equation for γ, *i.e.* for the velocity of propagation for the probability density. It equals to

$$v = \gamma = \sqrt{\frac{D}{\tau}} = \sqrt{2D\lambda}\,. \tag{26}$$

In the deterministic map, velocity can be considered given, since it is just the ratio of the grid spacing to time step of the mapping, and can be adjusted to any physical situation of particular interest. Knowing it, and having the equation for λ which depends upon the maps parameters, we can estimate the diffusion coefficient for such map as

$$D = \gamma^2 \tau = \frac{\gamma^2}{2\lambda} = \frac{\gamma^2(0.5 - DX - x_1)}{2DX}\,. \tag{27}$$

A useful conclusion can be drown from this paper, *i.e.* that the intermittency causes not only a sub-diffusion (which is widely known *cf.* Geisel map [22]) but it can also generate the hyperbolic diffusion.

It is worth to mention a modification of the mapping which gets beyond the idea of hyperbolic diffusion. In some physical phenomena it may appear that the motion in given direction is certain in first few time steps. It is possible to implement such behavior in the mapping very easily. This can be done by limiting the range of mapping in the area $(0, x_1) \cup (1 - x_1, 1)$ from $(0, 1)$ to $(0, 0.5 - (K + 1)DX) \cup (0.5 + (K + 1)DX, 1)$.

Finally, it is worth noting that presented approach to hyperbolic diffusion provides also a new way to deal with numerical analysis of hyperbolic diffusion. It is quite fast and always stable. The drawback is that the criteria used in derivation should be met, *i.e.* small DX coefficient.

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