TWO CROSS-CORRELATED DICHOTOMIC NOISES: BARRIER CROSSING PROBLEM*

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Dedicated to Professor Peter Talkner on the occasion of his 60th birthday

Escape of an overdamped particle driven by two correlated dichotomic noises (DN) from a triangle potential well is studied. A general description of statistical properties of the noises is developed in terms of master equation and correlation functions. Using the kinetics of these noises, an equation for the mean first-passage times can be deduced, which enables us to investigate the impact of non-zero covariance on the barrier crossing rate. In various cases, both the acceleration and the slowing down of the escape process can be observed.

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1. Introduction

Due to their mathematical simplicity and many possibilities of experimental realizations, Markovian dichotomic processes have been extensively exploited in various models concerning the influence of noise on dynamical systems and discussing the role of fluctuations in the microscopic level phenomena (*cf. e.g.* [1, 2]). As some examples we mention the problems related to the action of molecular motors [3] or to the process of resonant activation [4, 5], in which random telegraph processes play a role of an external noise, which modifies the properties of equilibrium systems. In both cases it has been essential that this noise has finite memory. Because the features caused by other coloured noises are qualitatively very similar, so the importance of usage of DN seems indisputable.

On the other hand, it has been observed that even when only white noises are being considered, especially if one noise is additive and another one is multiplicative, the dynamics of a system may be significantly changed

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by cross-correlations between them. For instance, these correlations can modify properties of bistable systems (*cf. e.g.* [6]), influence the fluctuations of a single mode laser field (*cf. e.g.* [7]), allow for the control of stochastic flows in periodic potentials [8], *etc.* In [9] a quite realistic example of how such correlation might arise has been described and it has been found to be responsible for the phase locking of a laser. Most importantly for us, also the statistics of the first passage times for particles driven by two white noises which are not independent have been investigated, *cf. e.g.* [10–12]. Whereas the first passage times distribution moments for the model that was studied in [10] do not depend on the correlation, in [11, 12] it has been shown that non-zero covariance of white noises acting on some systems can lead to giant suppression or enhancement of activation rates.

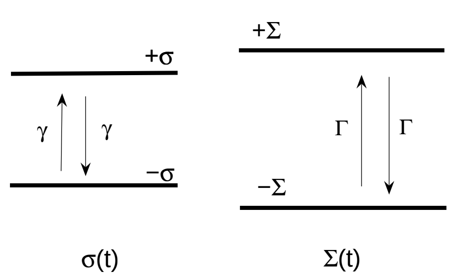
The aim of the present paper is to investigate whether the cross-correlation of coloured noises results also in any dramatic changes of activation process. Trying to answer this question, we study the influence of non-zero covariance of two DN on the behaviour of an overdamped particle in a very simple system. In the next section we present a mathematical description of two mutually correlated DNs. In the following part we study an escape process over a triangle potential barrier by means of the mean first-passage times.

2. Two dichotomic noises

Let us consider two symmetric Markovian dichotomic noises: $\Sigma(t)$ and $\sigma(t)$, switching between values $\pm\Sigma$ and $\pm\sigma$ with the rates Γ and γ, respectively. The joint process has four states: $(\pm\sigma, \pm\Sigma)$ (for simplicity in the following we miss amplitudes Σ and σ) with occupation probabilities $P_{\pm\pm}$. Let us construct a column vector $\underline{P} = [P_{++}\ P_{+-}\ P_{-+}\ P_{--}]^T$. Then the master equation for the four-state process takes the form:

$$\frac{\partial}{\partial t}\underline{P} = \underline{\underline{M_0}}\underline{P} = \left[\underline{\underline{1}} \otimes \begin{pmatrix} -\Gamma & \Gamma \\ \Gamma & -\Gamma \end{pmatrix} + \begin{pmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{pmatrix} \otimes \underline{\underline{1}}\right] \underline{P}, \qquad (1)$$

where $\underline{\underline{1}}$ is the 2×2 identity matrix. We can see that the matrix of kinetic coefficients on the r.h.s. can be decomposed into a sum of operators, which separately govern the evolution of each of the two dichotomic processes alone.



Now let us turn to the case when the noises are correlated. To the best of our knowledge there are no conclusive data concerning the form of correlation functions for dichotomic processes, neither composed nor even simple ones; for rather obvious reasons the exponential form is the most commonly used one, *cf.* [13]. Also when continuous noises acting on dynamical systems are concerned, various cross-correlations can be found in the literature. *E.g.*, let us consider two Ornstein–Uhlenbeck processes: $y_1(t)$ and $y_2(t)$, defined by the equation

$$\dot{y}_i(t) = -\frac{1}{\tau_i} y_i(t) + \sqrt{\frac{2}{\tau_i}}\, \eta_i(t)\,, \qquad i = 1,2\,, \tag{2}$$

where η_i are white Gaussian noises with unit intensities. The symbols τ_i denote correlation times, so that $\langle y_i(t) y_i(s)\rangle = \exp(-|t-s|/\tau_i)$. Let us assume that η_1 and η_2 have a common component, *i.e.* they have partially common origin. Then $\langle \eta_1(t)\,\eta_2(s)\rangle = \lambda\delta(t-s)$, $|\lambda| \leq 1$. Other, non-delta-type forms of $\langle \eta_1(t)\,\eta_2(s)\rangle$ have been proposed, which however, seem to have no simple justification. So, if the joint process is Markovian, the correlation function of $y_1(t)$ and $y_2(t)$ (in the stationary state) has the asymmetric exponential form

$$\langle y_1(t)\, y_2(s)\rangle = \begin{cases} 2\frac{\sqrt{\tau_1\tau_2}}{\tau_1+\tau_2}\,\lambda \exp\left(-\frac{|t-s|}{\tau_2}\right) & s \geq t\,, \\[2ex] 2\frac{\sqrt{\tau_1\tau_2}}{\tau_1+\tau_2}\,\lambda \exp\left(-\frac{|t-s|}{\tau_1}\right) & t \geq s\,. \end{cases} \tag{3}$$

Now, let us come back to the DN. If we again assume that the joint process $[\sigma(t), \Sigma(t)]$ is a Markov one, the $\underline{P}(t)$ vector must undergo a master equation. Its most general form follows from the conditions, which guarantee the conservation of positivity of probabilities and normalisation to unity. These conditions can be easily established: *(i)* negativity of diagonal kinetic coefficients, *(ii)* non-negativity of off-diagonal coefficients, *(iii)* columns of the matrix of evolution must sum up to zero [14, 15]. Obviously, when we average $P_{\pm\pm}$ over the first or over the second index, we must get the evolution equation for probabilities of single $\Sigma(t)$ or $\sigma(t)$, respectively. Thus, we obtain the general equation

$$\frac{\partial}{\partial t}\underline{P} = \underline{\underline{M}}\,\underline{P}\,, \tag{4}$$

where

$$\underline{\underline{M}} = \begin{bmatrix} -\gamma-\Gamma+a & \Gamma-b & \gamma-c & d \\ \Gamma-a & -\gamma-\Gamma+b & c & \gamma-d \\ \gamma-a & b & -\gamma-\Gamma+c & \Gamma-d \\ a & \gamma-b & \Gamma-c & -\gamma-\Gamma+d \end{bmatrix}, \tag{5}$$

with $0 \le a, b, c, d \le \min(\gamma, \Gamma)$. Now, when at least one of a, b, c, d does not vanish, the matrix of evolution cannot be decomposed as in (1). In the stationary state we have $P_{++} = P_{--} = \frac{1}{2}\frac{\gamma+\Gamma-b-c}{2\gamma+2\Gamma-a-b-c-d}$ and $P_{+-} = P_{-+} = \frac{1}{2}\frac{\gamma+\Gamma-a-d}{2\gamma+2\Gamma-a-b-c-d}$. Treating (4) with (5) as a differential Chapman–Kolmogorov equation for the two-point conditional probabilities (transition probabilities) one can calculate the stationary correlation function for the two DN. As in the case of Ornstein–Uhlenbeck processes, it is asymmetric

$$\langle\sigma(t)\Sigma(s)\rangle = \begin{cases} \frac{a+d-b-c}{2\gamma+2\Gamma-a-b-c-d}\,\sigma\Sigma\exp\left(-2\Gamma|t-s|\right) & s \ge t \\ \frac{a+d-b-c}{2\gamma+2\Gamma-a-b-c-d}\,\sigma\Sigma\exp\left(-2\gamma|t-s|\right) & t \ge s\,. \end{cases} \tag{6}$$

Incidentally, exponential correlations have been ad hoc imposed on two DN in some models concerning the stochastic resonance [16, 17].

One can see that the conditions *(i)–(ii)* yield an extra constriction for the correlation coefficient $C(\sigma(t), \Sigma(s)) = \langle\sigma(t)\Sigma(s)\rangle/\sigma\Sigma$:

$$\Big|C(\sigma(t), \Sigma(s))\Big| \le \frac{\min(\gamma, \Gamma)}{\max(\gamma, \Gamma)}\,. \tag{7}$$

The restriction (7) depends on the location in the (γ, Γ) space. Naturally, a very quickly fluctuating process cannot be highly correlated (nor anticorrelated) with a slowly varying one; we can see that $|C(\sigma(t), \Sigma(s))|$ can take values close to unity only when $\gamma \cong \Gamma$.

3. Crossing of a triangle barrier in the presence of correlated dichotomic noises

Mean first-passage times (MFPTs) for various processes proved to be a very useful tool in many branches of science and technology. In particular, when diffusion processes driven by white noises are concerned, calculating MFPTs over a potential barrier [18] is a possible way of determination of the activation rates. Problem of first-exit times for particles driven by dichotomic noises arose together with a need for extension of the developing reaction rate theory to the non-Markovian case. In general, this topic is very difficult and even models with simplest coloured noises disclosed some subtleties absent in the Markovian diffusive dynamics. Nonetheless, for processes driven by random-telegraph-signal some effective techniques have been developed and closed formulae for MFPTs in some cases have been found [19–27]. In the following part we study the MFPT for an overdamped particle moving in the interval $[0, 1]$ in a field of a linear potential of the slope h, subjected to the action of two symmetric correlated dichotomic noises described in the preceding paragraph. Motion is bounded from the left by an infinite potential barrier at $x = 0$ and at $x = 1$ the particle is absorbed (*cf.* Fig. 1). A similar problem was studied in [28], however in the presence of an additive Gaussian white noise and the DN were uncorrelated. What interests us mostly is to check how the cross-correlations influence the MFPT.

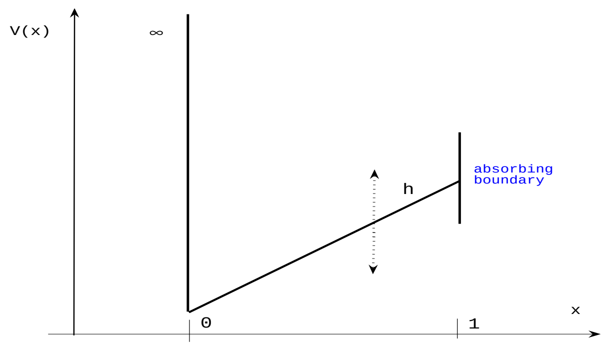


Fig. 1. Schematic of the considered fluctuating barrier problem.

3.1. Model

The Langevin-type equation has a form:

$$\dot{x} = -h - \sigma(t) - \Sigma(t) \,. \tag{8}$$

Clearly, since DN are coloured, the process $x(t)$ is not memoryless. However, performing extension to a three-dimensional process $[x(t),\ \sigma(t),\ \Sigma(t)]$ and employing the backward Chapman–Kolmogorov equation (see *e.g.* [18]; *cf.* (4), (5)) one can write down a set of linear differential equations for the MFPT (*i.e.* averaged over all realizations of the noises $\sigma(t)$ and $\Sigma(t)$):

$$\begin{aligned}
-(h+\sigma+\Sigma)T'_{++} - (\gamma+\Gamma-a)T_{++} + (\Gamma-a)T_{+-} + (\gamma-a)T_{-+} + aT_{--} &= -1\,,\\
-(h+\sigma-\Sigma)T'_{+-} + (\Gamma-b)T_{++} - (\Gamma+\gamma-b)T_{+-} + bT_{-+} + (\gamma-b)T_{--} &= -1\,,\\
-(h-\sigma+\Sigma)T'_{-+} + (\gamma-c)T_{++} + cT_{+-} - (\gamma+\Gamma-c)T_{-+} + (\Gamma-c)T_{--} &= -1\,,\\
-(h-\sigma-\Sigma)T'_{--} + dT_{++} + (\gamma-d)T_{+-} + (\Gamma-d)T_{-+} - (\gamma+\Gamma-d)T_{--} &= -1\,.
\end{aligned} \tag{9}$$

The symbols T_{ij}, $i,j = +,-$ denote conditional mean escape times given the initial state (i,j). Assuming the stationarity of the process $[\sigma(t), \Sigma(t)]$ the complete MFPT $T(x)$ is given by the average of T_{ij} over probabilities of the states $(\pm,\pm)$:

$$T(x) = \sum_{i,j} P_{ij} T_{ij}(x) \,.$$

A transition over the barrier is possible only when at least in one (of four) configuration the slope is negative. Assuming that $\Sigma > \sigma$ and since $h > 0$ we distinguish two cases: *I.* $\Sigma + \sigma > h > \Sigma - \sigma$ (one "channel of escape"), *II.* $\Sigma - \sigma > h$ (two "channels").

In their general form equations (9) contain many independent parameters. To simplify our investigations, we consider two particular cases, only:

(A) $b = c = 0$ and $d = a$, for which $\langle \Sigma(t)\sigma(s)\rangle > 0$ (positive correlation),

(B) $d = a = 0$ and $b = c$, for which $\langle \Sigma(t)\sigma(s)\rangle < 0$ (anticorrelation).

The way of calculation is similar for both (A) and (B) cases, as well as for other choices of the parameters a, b, c, d. Thus, in the following we will present some details only for the case (A).

3.2. Boundary conditions

This topic needs some care. In the diffusive counterpart of the problem considered here one used to employ absorbing and reflecting boundary conditions understood as vanishing probability density or its gradient, respectively. Similar conditions for the MFPT are derived from the backward Fokker–Planck equation, *cf.* [18]. They obviously can not hold for dichotomous flows, where only first-order partial differential equations appear. It is important, that when the driving process has finite number of states, then in some cases the instantaneous probability distribution can be decomposed into an absolutely continuous and a discrete part, even when the initial probability density was purely continous. There was some discussion concerning boundary conditions for dichotomic flows, in particular various interpretations of the term "reflecting boundary" (whose meaning is not clear for overdamped particles in the non-diffusive case) were given, *cf.* [21–23, 26, 29, 30]. Some of them seem to be rather unphysical, furthermore, some of the proposed boundary conditions just turned out to be wrong. It is worth mentioning, that in some sense these problems can be avoided, if methods involving integral rather than differential equations are applied, like the stochastic trajectory analysis technique developed by West, Lindenberg and Masoliver [19, 20].

When the particle is initially located at the instantaneous equilibrium point $x = 0$, then it must stay there until the potential flips to such a configuration in which there is a force allowing it to leave this position. If the particle starts at the absorbing boundary $x = 1$, then it immediately leaves the region [0,1] if the r.h.s. of (8) is positive. We apply these requirements in our model. The boundary conditions for the case (A) when $\Sigma - \sigma < h$ are:

$$
\begin{aligned}
&\text{(a)}\ \ T_{--}(1) \;=\; 0\,,\\
&\text{(b)}\ \ T_{+-}(0) \;=\; \frac{\varGamma}{\gamma+\varGamma}\,T_{++}(0)+\frac{\gamma}{\gamma+\varGamma}\,T_{--}(0)+\frac{1}{\gamma+\varGamma}\,,\\
&\text{(c)}\ \ T_{-+}(0) \;=\; \frac{\gamma}{\gamma+\varGamma}\,T_{++}(0)+\frac{\varGamma}{\gamma+\varGamma}\,T_{--}(0)+\frac{1}{\gamma+\varGamma}\,,\\
&\text{(d)}\ \ T_{++}(0) \;=\; \frac{\varGamma-a}{\gamma+\varGamma-a}\,T_{+-}(0)+\frac{\gamma-a}{\gamma+\varGamma-a}\,T_{-+}(0)\\
&\qquad\qquad\qquad +\frac{a}{\gamma+\varGamma-a}\,T_{--}(0)+\frac{1}{\gamma+\varGamma-a}\,.
\end{aligned}
\tag{10}
$$

For $\varSigma-\sigma>h$ instead of Eq. (10) case (b) we have:

$$
\text{(b}'\text{)}\ \ T_{+-}(1)=0\,. \tag{11}
$$

The construction of these equations follows the arguments of [31], which allowed the authors to recover the appropriate zero white noise limit of the well-known Bier–Astumian model [5]. In this model as well as in our case the potential is not differentiable (at $x=0$). Though, if the sharp shape in this region is as usual understood as a limit of continuous deformation of a smooth potential, then for uncorrelated noises ($a=0$) the conditions written above fully agree with those introduced in [23] for multiple dichotomic processes. In general, however, the correlation is not zero and we argue that it must enter the boundary conditions. The extension is yet rather obvious.

The Eq. (10) case (a), means that in the $(-,-)$ configuration, in which the slope of the potential is negative, the particle starting at $x=1$ immediately crosses the boundary. When $\varSigma-\sigma<h$ the other "channels" are closed because the slope is positive and if the particle starts at $x=0$ then it waits there till both noises will take on their negative values. This can be expressed as follows. When the particle is at $x=0$ and the initial state is $(+,-)$, nothing happens at least until the potential flips to another configuration. The state $(+,-)$ depopulates with the rate $\varGamma$ to $(+,+)$ and with the rate γ to $(-,-)$, *cf.* Eqs. (4), (5). So, making use of the Markovian character of the process $[\varSigma(t),\sigma(t)]$, the conditional MFPT T_{+-} must be equal to the mean residence time in the $(+,-)$ state, that reads $1/(\gamma+\varGamma)$, plus the weighted average of the two conditional MFPTs for the states to which the potential can flip. This is exactly Eq. (10) case (b). An analogous condition (Eq. (10) case (c)) can be written, when the initial state is $(-,+)$. The last equation is introduced in a similar way. Because of the non-zero correlation, the $(+,+)$ state depopulates with the modified rate $\gamma+\varGamma-a$. As can be seen from (4) and (5), the probabilities of transition to $(+,-)$, $(-,+)$, $(-,-)$ are proportional to $\varGamma-a$, $\gamma-a$ and a, respectively, and thus Eq. (10) case (d), follows.

When $\Sigma-\sigma<h$, the $(+,-)$ channel is open and we must simply replace (10) case (b) by an additional absorbing boundary condition (11). Let us note that in each case our boundary conditions can be recovered by reducing (from eight to four) corresponding relations valid for the diffusive case [28], *i.e.* putting $T'_{++}(0)=0$, $T'_{-+}(0)=0$, $T_{--}(1)=0$ and depending on the situation either $T'_{+-}(0)=0$ or $T_{+-}(1)=0$, as can be seen from (9).

3.3. Solution

The problem can be solved using a method similar to that presented in [5]. The general form of the solution reads:

$$T(x)=A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}+A_3e^{\lambda_3 x}+A_4+\frac{1}{h}x\,, \tag{12}$$

where $A_i, i=1\ldots4$ are some constants, which must be found using boundary conditions. Eigenvalues $\lambda_i, i=1,2,3$ are the roots of the third-order algebraic equation:

$$\begin{aligned}&[h^2-(\Sigma+\sigma)^2][h^2-(\Sigma-\sigma)^2]\lambda^3+\{4h(\gamma+\Gamma)(h^2-\Sigma^2-\sigma^2)+2ha[(\Sigma-\sigma)^2-h^2]\}\lambda^2\\&+4\{(3h^2-\sigma^2-\Sigma^2)\gamma\Gamma+(h^2-\Sigma^2)\gamma^2+(h^2-\sigma^2)\Gamma^2\\&+a[\sigma^2\Gamma+\Sigma^2\gamma-(h^2+\sigma\Sigma)(\gamma+\Gamma)]\}\lambda+8h\gamma\Gamma(\gamma+\Gamma-a)=0\,.\end{aligned} \tag{13}$$

One can show that all they are real, provided a is small enough. Proof is given in the Appendix.

3.4. Results

Generally, when the noises are uncorrelated, the dependence of the MFPT on the switching rates is represented by a smooth surface over the (γ,Γ) plane with a global minimum, *cf.* Fig. 2. Thus we may state that a resonant activation appears with respect to both rates γ and Γ. This behaviour is robust against a small amount of thermal noise in (8), though then $T(x)$ does not tend to infinity when $\gamma,\Gamma\longrightarrow 0,\infty$, since the particle can leave the region diffusively, *cf.* [28].

What happens when the noises are correlated? Positive correlation (case (A), $0<a<\min(\gamma,\Gamma)\equiv a_{\max}$, $C(\Sigma(t)\sigma(t))=a/(\gamma+\Gamma-a)$) means that the configurations $(+,+)$ and $(-,-)$ appear more frequently than the remaining two. The $(-,-)$ state is the most efficient or even the only one (when $\Sigma-\sigma<h$) possible channel of escape. Longer sojourn time in the $(+,+)$ state is not very important, since after reaching $x=0$ the particle does not move. Thus, we can expect an enhancement of the barrier crossing rate, what can indeed be seen in Fig. 3, in which logarithm of the relative

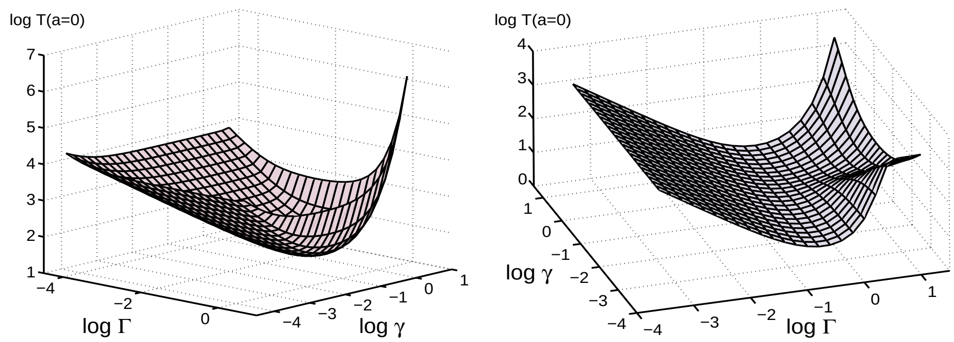


Fig. 2. MFPT from $x = 0$ to $x = 1$ as a function of γ and Γ for $a = 0$, $h = 1$, $\sigma = 0.6$ and $\Sigma = 0.7$ (left-hand side) or $\Sigma = 2$ (right-hand side).

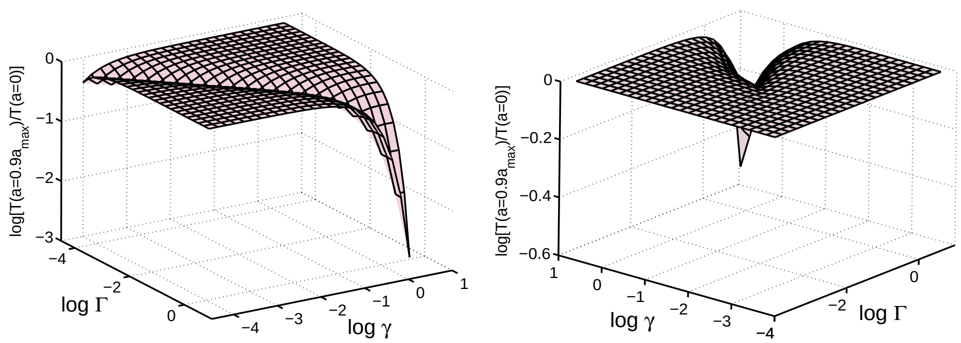


Fig. 3. Relative MFPT from $x = 0$ to $x = 1$ as a function of γ and Γ for $h = 1$, $\sigma = 0.6$ and $\Sigma = 0.7$ (left-hand side) or $\Sigma = 2$ (right-hand side) and for positive correlation (case (A), $a = 0.9\min(\gamma, \Gamma)$, $b = 0$).

MFPT is plotted *versus* γ and Γ. Naturally, the effect is mostly visible when $\gamma \sim \Gamma$, since only in this region the covariance can take values significantly different from zero. The edge at $\gamma = \Gamma$ has no special meaning. It is just a consequence of the way the correlation coefficient varies in the (γ, Γ) plane, *cf.* (7). When the dichotomic noises switch very quickly, *i.e.* when their intensities and in consequence their influence on the particle is small, then the effect of the correlation is very significant; the MFPT may be changed by few orders of magnitude.

Conversely, the anticorrelation (case (B), $0 < b < \min(\gamma, \Gamma) \equiv b_{\max}$, $C(\Sigma(t)\sigma(t)) = (-b)/(\gamma + \Gamma - b))$ means that the $(+,-)$ and $(-,+)$ states are most probable, what delays the escape process (Fig. 4), and when $\Sigma - \sigma < h$ for $C(\Sigma(t), \sigma(t)) \approx -1$, makes it practically impossible. The reason is that the only channel of escape, the $(-,-)$ state, is being very seldom occupied.

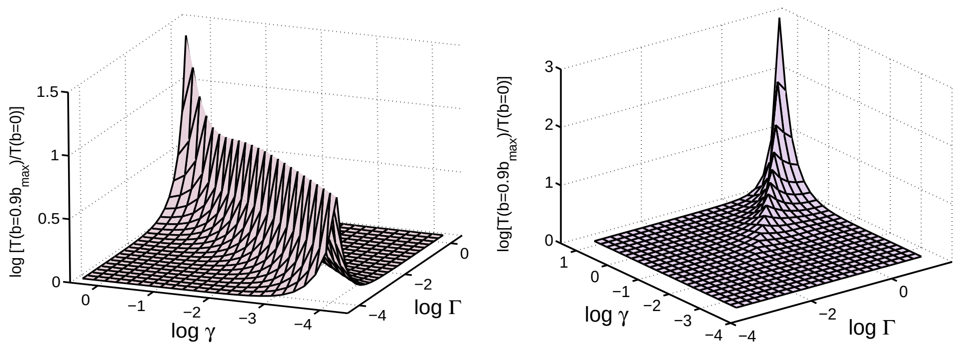


Fig. 4. Same as Fig. 3 but for negative correlation (case (B), $b = 0.9\min(\gamma, \Gamma)$, $a = 0$).

The above effects caused by the correlated DN are quite similar to the giant suppression and enhancement of the activation rate in bistable systems in the presence of correlated white noises, first reported in [11]. In the model considered in that paper an additive noise was delta-correlated with a multiplicative one. When the sign of the cross-correlation intensity was such that tilting in the same direction was preferred, then the acceleration of the activation was found (and the slowing down in the opposite case). In our simple model there is no difference between addittive and multiplicative noises and acceleration is observed just when both noises prefer the same sign, *i.e.* are positively correlated.

One might expect, that such effects will be present for systems driven by any pair of coloured driving noises.

3.5. Influence of white noise

When a small amount of white noise is added on the r.h.s. of (8), the situation is slightly more complicated. We have

$$\dot{x} = -h - \sigma(t) - \Sigma(t) + \sqrt{2q}\zeta(t) \tag{14}$$

with $\zeta(t)\zeta(s) = \delta(t-s)$. For small γ and Γ the thermal activation becomes more efficient way of barrier crossing than just sliding down after flipping to a configuration with a negative slope. Longer sojourn times in the state with the steepest positive slope enlarge the MFPT. Thus, the positive correlation delays the escape, while the negative one accelerates it, see Fig. 5, where we display some results of numerical calculations with reflecting and absorbing boundary conditions similar to those of [18]. So, for the same correlation in different regions of (γ, Γ) plane, we can observe slower or faster barrier crossing. Unlike in the absence of ζ, for very large gammas non-zero correlation cannot influence significantly the process $x(t)$, since in this limit the

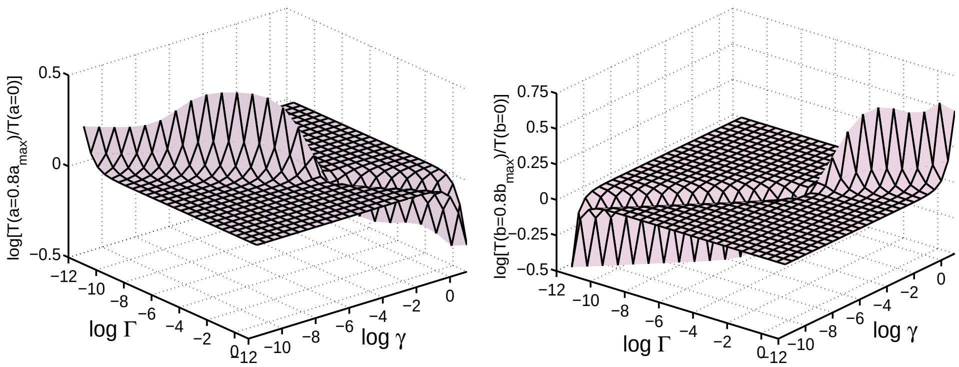


Fig. 5. Relative MFPT from $x = 0$ to $x = 1$ as a function of γ and Γ for $h = 1$, $\sigma = 0.6$, $\Sigma = 0.7$ and positive (left-hand side), case (A), $a = 0.8\min(\gamma, \Gamma)$, $b = 0$) or negative (right-hand side), case (B), $b = 0.8\min(\gamma, \Gamma)$, $a = 0$) correlation in the presence of white noise with intensity $q = 0.1$.

coloured noises practically vanish and escape event is governed by Gaussian fluctuations. We can see this on Fig. 6, where the intersection of the relative MFPT surface for different correlation coefficients along the line $\gamma = \Gamma$ is shown. In this case there is one characteristic time scale $1/\gamma$ for both processes. We can observe further peculiarities of the relation between the relative MFPT and the cross-correlation. Few extrema can occur along the

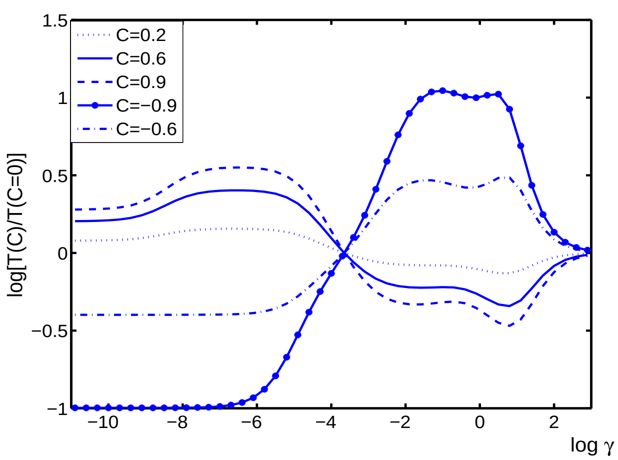


Fig. 6. Relative MFPT from $x = 0$ to $x = 1$ as a function of $\gamma = \Gamma$ for $h = 1$, $\sigma = 0.6$, $\Sigma = 0.7$ and various correlation coefficients $C(\Sigma(t), \sigma(t))$ in the presence of white noise with intensity $q = 0.1$.

γ line. Next, for this combination of amplitudes there is a value of γ for which MFPT remains unchanged by the correlation in the whole range from -1 to 1 (all the lines intersect in this point)! One cannot observe this effect when $\Sigma - \sigma > h$.

4. Summary

The goal of this work has been to analyse properties of mutually correlated symmetric Markovian dichotomic noises and to study their influence on the escape process from a potential well. We have been able to introduce the correlation in a rather straightforward way by means of the master equation and to find the general form of the correlation functions. We have applied our results to an investigation of mean first-passage times over a triangle barrier. Generally speaking, the rules found by Madureira, Hänggi and Wio [11] for activation in the presence of correlated white noises have been confirmed, *i.e.* depending on the sign of the correlation coefficient we have observed acceleration or delay of the escape event.

This simple conclusion is, however, not true when the thermal fluctuations are present. In fact, the same value of the correlation coefficient can result in a decrease or an increase of the MFPT for different switching rates of dichotomic noises. Moreover, in some cases one can choose these rates in such a way that any variation of the correlation coefficient has no influence on the first exit time.

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Appendix A

First we are going to show that when there is no correlation ($a = 0$), then Eq. (13) has real roots. In this case the roots of the secular equation

$$
\begin{aligned}
&[h^2 - (\Sigma + \sigma)^2][h^2 - (\Sigma - \sigma)^2]\lambda^3 + 4h(\gamma + \Gamma)(h^2 - \Sigma^2 - \sigma^2)\lambda^2 \\
&+ 4\,[(3h^2 - \sigma^2 - \Sigma^2)\gamma\Gamma + (h^2 - \Sigma^2)\gamma^2 + (h^2 - \sigma^2)\Gamma^2]\lambda \\
&+ 8h\gamma\Gamma(\gamma + \Gamma) = 0
\end{aligned}
\tag{A.1}
$$

coincide with the non-zero eigenvalues of the matrix:

$$\underline{\underline{A}} = \begin{bmatrix} -\frac{\gamma+\Gamma}{h+\sigma+\Sigma} & \frac{\Gamma}{h+\sigma+\Sigma} & \frac{\gamma}{h+\sigma+\Sigma} & 0 \\ \frac{\Gamma}{h+\sigma-\Sigma} & -\frac{\gamma+\Gamma}{h+\sigma-\Sigma} & 0 & \frac{\gamma}{h+\sigma-\Sigma} \\ \frac{\gamma}{h-\sigma+\Sigma} & 0 & -\frac{\gamma+\Gamma}{h-\sigma+\Sigma} & \frac{\Gamma}{h-\sigma+\Sigma} \\ 0 & \frac{\gamma}{h-\sigma-\Sigma} & \frac{\Gamma}{h-\sigma-\Sigma} & -\frac{\gamma+\Gamma}{h-\sigma-\Sigma} \end{bmatrix}, \tag{A.2}$$

what can be infered from the discussion of Sec. 3. Naturally, the transposed matrix $\underline{\underline{G}} \equiv \underline{\underline{A^T}}$ has the same eigenvalues and it constitutes a master-type equation:

$$\underline{H}' = \underline{\underline{G}}\,\underline{H}, \tag{A.3}$$

with $\underline{H} = [H_{++}\ H_{+-}\ H_{-+}\ H_{--}]^T$ being an auxiliary variable. $\underline{\underline{G}}$ is singular and we have one degree of freedom in choice of the "stationary" solution:

$$\underline{\underline{G}}\,\underline{H}^s = 0.$$

One can easily see that the solution is $H^s_{++} = (h+\sigma+\Sigma)l$, $H^s_{++} = (h+\sigma-\Sigma)l$, $H^s_{++} = (h-\sigma+\Sigma)l$, $H^s_{++} = (h-\sigma-\Sigma)l$. We may take $l=1$. In this way we obtain a system with "detailed balance", *i.e.*:

$$G_{xy,\,pq}H^s_{pq} = G_{pq,\,xy}H^s_{xy}\,, \tag{A.4}$$

where p, q, x, $y = +,-$. H^s_{pq} can be both positive and negative. Moreover, $\underline{\underline{G}}$ is an asymmetric matrix. One can, however, show that its eigenvalues equal the eigenvalues of an auxiliary hermitian matrix.

The eigenvalue problem can be written as:

$$\underline{\underline{G}}\,\underline{\phi}_\alpha = \lambda_\alpha \underline{\phi}_\alpha\,. \tag{A.5}$$

In general it is not equivalent to the adjoint problem, since $\underline{\underline{\mathrm{G}}}$ is asymmetric. $\underline{\underline{G}}$ can be written in such a way that it will constitute a Pauli equation (*cf.* [32]), namely:

$$G_{xy,\,pq} = \mathcal{W}(pq \to xy) - \delta_{pq,xy} \sum_{m,n=+,-} \mathcal{W}(mn \to xy)\,, \tag{A.6}$$

or $pq \neq xy$ we define:

$$G^h_{xy,\,pq} = \begin{cases} \mathcal{W}(pq \to xy)\frac{\sqrt{H^s_{pq}}}{\sqrt{H^s_{xy}}}, & H^s_{pq}, H^s_{xy} > 0\,, \\ \mathcal{W}(pq \to xy)\frac{\sqrt{-H^s_{pq}}}{\sqrt{-H^s_{xy}}}, & H^s_{pq}, H^s_{xy} < 0\,, \\ \mathcal{W}(pq \to xy)\frac{i\sqrt{-H^s_{pq}}}{\sqrt{H^s_{xy}}}, & H^s_{pq} < 0, H^s_{xy} > 0\,, \\ \mathcal{W}(pq \to xy)\frac{\sqrt{H^s_{pq}}}{-i\sqrt{-H^s_{xy}}}, & H^s_{pq} > 0, H^s_{xy} < 0\,. \end{cases} \tag{A.7}$$

When $pq = xy$ we left $G^h_{xy,\,pq} = G_{xy,\,pq}$. Now we put:

$$\phi^{xy}_\alpha = r_{xy}\tilde{\phi}^{xy}_\alpha = \begin{cases} \sqrt{H^s_{xy}}\tilde{\phi}^{xy}_\alpha, & H^s_{xy} > 0\,, \\ -i\sqrt{-H^s_{xy}}\tilde{\phi}^{xy}_\alpha, & H^s_{xy} < 0\,. \end{cases} \tag{A.8}$$

$\underline{\underline{G^h}}$ is Hermitian. We have:

$$\sum_{pq} G_{xy,pq} r_{pq} \tilde{\phi}^{xy}_\alpha = \lambda_\alpha r_{xy} \phi^{xy}_\alpha \tag{A.9}$$

and dividing by r_{xy}:

$$\underline{\underline{G^h}}\,\underline{\tilde{\phi}_\alpha} = \lambda_\alpha \underline{\tilde{\phi}_\alpha}. \tag{A.10}$$

So, $\underline{\underline{G}}$ has the same eigenvalues as $\underline{\underline{G^h}}$, what ends the proof.

Furthermore, if we rewrite (A.1) as:

$$w\lambda^3 + v\lambda^2 + u\lambda + z = 0 \tag{A.11}$$

and denote its roots by x_1, x_2, x_3, then

$$x_1 x_2 x_3 = -\frac{z}{w}\,, \qquad \text{and} \qquad x_1 + x_2 + x_3 = -\frac{v}{w}\,.$$

When $\Sigma - \sigma > h$, then $z > 0$ and $w > 0$ and the number of negative roots must be odd. But $v < 0$ and at least one root must be positive. From these facts one can infer that two roots are positive and one is negative.

Conversely, when $\Sigma - \sigma < h$, $z > 0$ and $w < 0$ and the number of negative roots must be even. But when $h^2 < \Sigma^2 + \sigma^2$, then $v < 0$ and $w < 0$ so that there is one positive root and two negative ones. On the other hand, when $h^2 < \Sigma^2 + \sigma^2$, we can once again employ Viete formulae to obtain

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = -\frac{u}{z}\,.$$

Taking into account that $u > 0$ we also come to the conclusion that only one root is positive. Now let us turn to the case with correlation. The secular equation (13) can be rewritten as:

$$\begin{aligned} \text{l.h.s. of (A.1)} \;=\; & a\{[h^3 - 2h(\Sigma - \sigma)^2]\lambda^2 \\ & + 4[(h^2 + \Sigma\sigma)(\gamma + \Gamma) - \sigma^2\Gamma - \Sigma^2\gamma]\lambda + 8h\gamma\Gamma\}. \end{aligned} \tag{A.12}$$

Thus, when a tends to 0, the quadratic function on the r.h.s. converges to 0. It is clear that for small enough a the curve corresponding to this function must cross the graph of the third order polynomial on the l.h.s. of (A.12) in three points, so all roots of (A.12) are real as well.

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