PT-SYMMETRY, PSEUDO-HERMITICITY: THE REAL SPECTRA OF NON-HERMITIAN HAMILTONIANS

Sanjib Meyur, S. Debnath†

Department of Mathematics, Jadavpur University, Kolkata-700032, India

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PT-symmetry gives rise to a new class of complex Hamiltonians with real spectrum. The pseudo-Hermicity of Hamiltonians are discussed and PT-symmetric Hamiltonians are shown to belong to class of pseudo-Hermitian Hamiltonians. We have studied this for a general potential with an emphasis on a particular type.

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The Hermiticity of the Hamiltonian had been accepted as the necessary condition for the real spectrum since 1998 [1]. Recently a more physical alternative axiom called PT-symmetry has been investigated [1]. The condition $H = H^\dagger$ is being replaced by weaker and more physical requirement $H = H^{\mathrm{PT}}$ where P and T are respectively space reflexion and time reversal, and one obtains new classes of complex Hamiltonian whose spectra are still real and positive. The PT-symmetry Hamiltonian obeys the following properties:

(i) PT-symmetry is exact : The spectrum is real.

PT-symmetry is broken : There are complex eigen values.

(ii) The indefinite innerproduct $\ll | \gg$ is defined by $\ll \psi_1|\psi_2 \gg = \langle\psi_1|P|\psi_2\rangle$ $\forall\ |\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert Space.

Mostafazadeh [5] in his very noteworthy work introduces the concept of pseudo-Hermiticity in which he has pointed out that all the PT-symmetric Hamiltonians regarded so far [1–5]are actually P-pseudo Hermitian, namely $PHP^{-1} = H^\dagger$. Again it is claimed that it is nothing but η-pseudo Hermiticity *i.e.* $\eta H \eta^{-1} = H^\dagger$ [5]. By highlighting the concept of pseudo-Hermiticity

† Corresponding author. E-mail: debnath_swapna@yahoo.co.in

he has addressed that pseudo-Hermitian is a generalisation of Hermiticity. It can now be found that the real eigen values of the PT-symmetric potential are to be connected to the concept of η-distorted inner product $\langle\psi|\eta\psi\rangle$ [6]. It is observed [7] that the distinct real eigen values have η-orthogonal eigen vectors. It is also known that the complex eigen values have zero η-norm. Some other classes of non-Hermitian Hamiltonians both PT-symmetric and non PT-symmetric Hamiltonians have been accepted as pseudo-Hermitian under $\eta = e^{-\phi(x)}$ and $\eta = e^{-\theta p}$ [8].

In this paper, we consider a general model of a non-Hermitian Hamiltonians which have real spectrum and will show that pseudo-Hermiticity is more consistent than PT-symmetry. Let the Hamiltonian be of the form

$$H_\epsilon = \frac{[p + i\varepsilon g(x)]^2}{2m} + V(x)\,, \quad (m = 1 = \hbar)\,. \tag{1}$$

By applying gauge-like transformation [8]

$$e^{f(x)}[p + i\varepsilon g(x)]e^{-f(x)} = p - i\varepsilon g(x)\,,$$

where $f(x) = -2\varepsilon \int g(x)dx\,.$

Again

$$\begin{aligned}
e^{f(x)}[p + i\varepsilon g(x)]^2 e^{-f(x)} &= e^{f(x)}[p + i\varepsilon g(x)][p + i\varepsilon g(x)]e^{-f(x)} \\
&= e^{f(x)}[p + i\varepsilon g(x)]e^{-f(x)}e^{f(x)}[p + i\varepsilon g(x)]e^{-f(x)} \\
&= [p - i\varepsilon g(x)][p - i\varepsilon g(x)] \\
&= [p - i\varepsilon g(x)]^2\,.
\end{aligned}$$

Similarly $e^{f(x)}[p + i\varepsilon g(x)]^n e^{f(x)} = [p - i\varepsilon g(x)]^n$.

Hence $e^{f(x)}H_\varepsilon e^{-f(x)} = H_\varepsilon^\dagger$ and this implies H_ε to be pseudo-Hermitian under this transformation.

We shall now discuss a Hamiltonian of the type

$$H_\varepsilon = \frac{[p + i\varepsilon x^m]^2}{2} + \frac{a^2 + \varepsilon^2}{2}x^2\,, \tag{2}$$

where m is some positive integer.

Let us remember that

$$\begin{aligned}
PxP^{-1} &= -x\,; \quad PpP^{-1} = -p = TpT^{-1}\,; \quad TiIT^{-1} = -iI\,, \\
x^\dagger &= x\,, \quad i^\dagger = -i\,, \quad p^* = -p\,, \quad p^\dagger = p\,.
\end{aligned}$$

Then H_ε satisfies

(i) H_ε is non-Hermitian.

(ii) H_ε is real.

(iii) H_ε is PT-symmetric if m is odd and H_ε is PT-pseudo-Hermitian if m is even.

(iv) H_ε is not T-pseudo-Hermitian.

(v) H_ε is P-pseudo-Hermitian if m is even.

The eigen value equation is

$$H_\varepsilon \psi_n(x) = E_n \psi_n(x)\,. \tag{3}$$

Applying a suitable transformation H_ε becomes

$$e^{f(x)/2} H_\varepsilon e^{-f(x)/2} = H_{\mathrm{SHO}} = \frac{p^2}{2} + \frac{a^2+\varepsilon^2}{2} x^2\,, \tag{4}$$

where $f(x) = -\frac{2\varepsilon}{m+1} x^{m+1}$ and ψ_n becomes

$$\psi_n(x) = \exp\left[\frac{\varepsilon}{(m+1)} x^{m+1}\right] \phi_n(x)\,,$$

where $\phi_n(x)$ are well known eigen functions of simple Harmonic oscillator, one therefore obtains

$$E_n = \left(n + \frac{1}{2}\right)\left(a^2+\varepsilon^2\right)^{1/2}\,, \quad n = 0, 1, 2\ldots \tag{5}$$

$$\psi_n(x) = N_n \exp\left[-\left(a^2+\varepsilon^2\right)\frac{x^2}{2} + \frac{\varepsilon x^{m+1}}{m+1}\right] \times H_n\left[x\left(a^2+\varepsilon^2\right)^{1/4}\right] \tag{6}$$

$$N_n = \text{normalization constant} = \frac{\left(a^2+\varepsilon^2\right)^{1/8}}{\left(2^n n!(\pi)^{1/2}\right)^{1/2}}\,.$$

Hence the eigen values are real and eigen functions are real and normalizable.

By replacing ε by $i\delta$ in equation (2) we have

$$H_\delta = \frac{(p-\delta x^m)^2}{2} + \frac{a^2-\delta^2}{2} x^2\,. \tag{7}$$

The results (5) and (6) consequently change to

$$E_n = \left(n + \frac{1}{2}\right)\left(a^2 - \delta^2\right)^{1/2}, \quad a > \delta, n = 0, 1, 2\ldots$$

and

$$\psi_n(x) = M_n \exp\left[-\left(a^2 - \delta^2\right)^{1/2}\frac{x^2}{2} + \frac{i\delta x^m}{m+1}\right] H_n\left[x\left(a^2 - \delta^2\right)^{1/4}\right], \quad (8)$$

where

$$M_n = \text{normalization constant} = \frac{\left(a^2 - \delta^2\right)^{1/8}}{\left(2^n n! (\pi)^{1/2}\right)^{1/2}}.$$

Hence for the Hamiltonian (7) we have real eigen values with complex eigen functions. However, the eigen values become complex as eigen functions are non-localised. This transition of eigen values from real to complex takes place when $\delta > a = \delta_{\text{critical}}$.

The exactness of PT-symmetry implies the reality of the spectrum. More specifically, if an eigen vector $|\phi\rangle$ is PT-invariant, $\mathrm{PT}|\phi\rangle = \phi$, then the corresponding eigen value is real. We have noticed that real Hamiltonians with real eigen values can have real eigen functions but non-real Hamiltonians with real eigen values cannot have real eigen functions, and that pseudo-Hermiticity is only a necessary condition for the reality of the spectrum, not a sufficient condition.

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