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ON THE SCALAR PRODUCT OF SHORT AND LONG LIVING STATES OF NEUTRAL KAONS IN THE CPT INVARIANT SYSTEM

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This paper contains a detailed analysis of the properties of the scalar product of short and long living superpositions of neutral $|K_0\rangle, |\overline{K}_0\rangle$ mesons. It is shown for the exact effective Hamiltonian for neutral meson subsystem that the scalar product of its eigenvectors, which correspond with these short and long living superpositions, cannot be real under the assumption of CPT conserved and CP violated. The standard conclusion obtained within the Lee–Oehme–Yang theory of neutral kaons is that in this case such a product should be real. Also, the general and model independent proof that probabilities of transitions $|K_0\rangle \rightarrow |\overline{K}_0\rangle$ and $|\overline{K}_0\rangle \rightarrow |K_0\rangle$ are not equal in the CP non–invariant system is given.

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1. Introduction

Almost all properties of the neutral meson complex are described by solving the Schrödinger-like evolution equation [1–12] (we use $\hbar = c = 1$ units)

$$i\frac{\partial}{\partial t}\,|\psi;t\rangle_{\parallel} = H_{\parallel}\,|\psi;t\rangle_{\parallel}\,, \qquad (t \geq t_0)\,, \tag{1}$$

(where t_0 is the initial instant) for $|\psi;t\rangle_{\parallel}$ belonging to the subspace $\mathcal{H}_{\parallel} \subset \mathcal{H}$ (where $\mathcal{H}$ is the state space of the physical system under investigation), *e.g.*, spanned by orthonormal neutral kaons states $|K_0\rangle$, $|\overline{K}_0\rangle$, and so on, (then states corresponding to the decay products belong to $\mathcal{H} \ominus \mathcal{H}_{\parallel} \stackrel{\text{def}}{=} \mathcal{H}_{\perp}$), and the non-hermitian effective Hamiltonian $H_{\parallel}$ obtained usually by means of the Lee–Oehme–Yang (LOY) approach [1–12] (within the use of the Weisskopf–Wigner approximation (WW) [13]):

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$$H_{\parallel} \equiv M - \frac{i}{2}\,\Gamma\,, \tag{2}$$

where $M = M^{+},\ \Gamma = \Gamma^{+}$ are (2×2) matrices. In a general case $H_{||}$ can depend on time t, $H_{||} \equiv H_{||}(t)$, [14,15].

Usually, solutions of the evolution equation (1) are expressed in terms of the eigenvectors of $H_{||}$. Generally, in the case of two-dimensional subspace $\mathcal{H}_{||}$ the eigenvectors of $H_{\parallel}$ acting in this $\mathcal{H}_{||}$ will be denoted as $|l\rangle, |s\rangle$. In the general case solutions of the eigenvalue problem for $H_{||}$

$$H_{||}\ |\,l(s)\rangle = \mu_{l(s)}\ |\,l(s)\rangle\,, \tag{3}$$

have the following form [16,17]

$$|l(s)\rangle = N_{l(s)}\big(|\mathbf{1}\rangle - \alpha_{l(s)}|\mathbf{2}\rangle\big)\,, \tag{4}$$

where $|\mathbf{1}\rangle$ stands for the vectors of the $|K_0\rangle$, $|B_0\rangle$ type and $|\mathbf{2}\rangle$ denotes antiparticles of the particle "1": $|\overline{K}_0\rangle$, $\overline{B}_0\rangle$, $\langle \boldsymbol{j}|\boldsymbol{k}\rangle = \delta_{jk}$, $(j,k = 1,2)$,

$$N_{l(s)} = \frac{1}{\sqrt{1 + |\alpha_{l(s)}|^2}} = N^{*}_{l(s)}\,, \tag{5}$$

and

$$\alpha_{l(s)} = \frac{h_z - (+)h}{h_{12}}\,, \tag{6}$$

$$\mu_{l(s)} = h_0 + (-)h \equiv m_{l(s)} - \frac{i}{2}\gamma_{l(s)}\,. \tag{7}$$

Quantities $m_{l(s)}, \gamma_{l(s)}$ are real, and

$$\begin{aligned} h_0 &= \tfrac{1}{2}(h_{11} + h_{22})\,, && (8)\\ h &\equiv \sqrt{h_z^2 + h_{12}h_{21}}\,, && (9)\\ h_z &= \tfrac{1}{2}(h_{11} - h_{22})\,, && (10)\\ h_{jk} &= \langle \boldsymbol{j}|H_{\parallel}|\boldsymbol{k}\rangle\,, \quad (j,k = 1,2)\,. && (11) \end{aligned}$$

In the case of neutral kaons eigenvectors of $H_{||}$ are identified with the long, $|K_{\rm L}\rangle$, (vector $|l\rangle$) and short, $|K_{\rm S}\rangle$, (vector $|s\rangle$) living superpositions of K_0 and $\overline{K_0}$. This identification of vectors $|l(s)\rangle$ with states $|K_{\rm L(S)}\rangle$ corresponds to the standard phase convention for CP transformation: $\mathcal{CP}|\mathbf{1}\rangle = -|\mathbf{2}\rangle$, $\mathcal{CP}|\mathbf{2}\rangle = -|\mathbf{1}\rangle$.

The following identities are true for μ_l and μ_s,

$$\mu_l + \mu_s \;=\; h_{11} + h_{22} \equiv \mathrm{Tr}\, H_{||}\,, \tag{12}$$

$$\mu_l - \mu_s \;=\; 2h \stackrel{\mathrm{def}}{\equiv} \Delta\mu = \Delta m - \frac{i}{2}\Delta\gamma\,, \tag{13}$$

$$\mu_l\, \mu_s \;=\; h_{11}h_{22} - h_{12}h_{21} \equiv \det\, H_{||}\,, \tag{14}$$

where

$$\begin{aligned} \Delta m \;&=\; m_l - m_s = (\Delta m)^*\,, \\ \Delta\gamma \;&=\; \gamma_l - \gamma_s = (\Delta\gamma)^*\,. \end{aligned} \tag{15}$$

In the standard approach to the description of properties of the neutral kaon complex many relations connecting parameters characterizing neutral kaons follow from the properties of the scalar product of state vectors $|K_{\mathrm{S}}\rangle, |K_{\mathrm{L}}\rangle$. The aim of this paper is to analyze in the general case detailed properties of the scalar product of eigenvectors $|l\rangle$ and $|s\rangle$ depending on CP and CPT transformations properties of the total system under considerations.

2. General properties of the product $\langle s|l\rangle$

Let us analyze the product $\langle s|l\rangle$ in the case of a general $H_{||}$ without any assumptions about CP- or CPT-symmetries of the system under considerations. From (4) one finds

$$\langle s|l\rangle = N_s\, N_l\, (1 + \alpha_s^*\, \alpha_l)\,. \tag{16}$$

The important question is whether the product $\langle s|l\rangle$ is real, $\langle s|l\rangle \equiv (\langle s|l\rangle)^*$, or not, $\langle s|l\rangle \neq (\langle s|l\rangle)^*$. It is obvious that the answer to this question depends on the properties of the product $\alpha_s^*\, \alpha_l$. From (6) it follows that

$$\alpha_s^*\, \alpha_l = \frac{1}{|h_{12}|^2}\Big[\Big(|h_z|^2 - |h|^2\Big) + 2i\, \Im\, (h_z\, h^*)\Big], \tag{17}$$

where $\Im\,(z)$ denotes the imaginary part of the complex number z ($\Re\,(z)$ is the real part of z). So the trivial conclusion is that

$$\langle s|l\rangle = (\langle s|l\rangle)^* \equiv \langle l|s\rangle \quad \Leftrightarrow \quad \Im\,(h_z\, h^*) = 0. \tag{18}$$

Taking into account the identity (13) one has $h = \frac{1}{2}\big(\Delta m - \frac{i}{2}\Delta\gamma\big)$ and thus it can be easily found that

$$\Im\,(h_z\, h^*) \;=\; \frac{1}{2}\,\Delta m\; \Im\,(h_z) \;+\; \frac{1}{4}\,\Delta\gamma\; \Re\,(h_z). \tag{19}$$

From this relation it is seen that if $h_z = 0$, that is, if $(h_{11} - h_{22}) = 0$ (see (10)) then $\Im\,(h_z\, h^*) \equiv 0$. This result does not depend on the values of Δm and $\Delta\gamma$. So, if $h_z = 0$ then the scalar product $\langle s|l\rangle$ must be real.

Now let us suppose that $h_z \neq 0$. In order to draw some conclusions about $\Im\,(h_z h^*)$ in this case one should rewrite $\Delta\mu$, $\Delta m, \Delta\gamma$ and $(h_{11} - h_{22})$ in a more convenient form. If the superweak phase $\phi_{\rm SW}$ [11, 12] is used,

$$\tan\,\phi_{\rm SW} = \frac{2(m_l - m_s)}{\gamma_s - \gamma_l} \equiv -\frac{2\,\Delta\mu}{\Delta\gamma}\,, \tag{20}$$

then one can find that

$$\Delta m = -|\Delta\mu|\sin\phi_{\rm SW}\,, \qquad \frac{\Delta\gamma}{2} = |\Delta\mu|\cos\phi_{\rm SW}\,. \tag{21}$$

Next one should find a similar expression for $\Re\,(h_z)$ and $\Im\,(h_z)$. One has

$$h_{jj} = \Re\,(h_{jj})\; + i\,\Im\,(h_{jj})\,, \tag{22}$$

$(j = 1, 2)$, where

$$\Re\,(h_{jj}) \equiv\; M_{jj}\,, \qquad \Im\,(h_{jj})\; \equiv -\tfrac{1}{2}\varGamma_{jj}\,. \tag{23}$$

Using the following definitions

$$\Delta M = M_{11} - M_{22}\,, \qquad \Delta\varGamma = \varGamma_{11} - \varGamma_{22}\,, \tag{24}$$

one can write that

$$h_{11} - h_{22} = \Delta M - \frac{i}{2}\Delta\varGamma \equiv 2h_z\,. \tag{25}$$

Next, if another phase ϕ_z is introduced analogously to the superweak phase $\phi_{\rm SW}$ by means of the relation

$$\tan\,\phi_z = -\frac{2\,\Delta M}{\Delta\varGamma}\,, \tag{26}$$

then one finds that

$$\Re\,(h_z) = -|h_z|\sin\,\phi_z, \quad \Im\,(h_z) = -|h_z|\,\cos\phi_z\,. \tag{27}$$

Thus, using (21) and (27) relation (19) can be rewritten in a compact and convenient form

$$\Im\,(h_z\,h^*) = \tfrac{1}{2}\,|\Delta\mu|\,|h_z|\,\sin(\phi_{\rm SW} - \phi_z)\,. \tag{28}$$

Note that, *e.g.*., if $\Delta M \neq 0$ and $\Delta \mathit{\Gamma} = 0$ then $\phi_z = \frac{1}{2}\pi + n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$. On the other hand there is $\phi_{\rm SW} \approx 43,5^\circ$ in the case of neutral K mesons (see [10–12]) and thus $\sin(\phi_{\rm SW} - \phi_z) \neq 0$.

Let us now analyze the case $\Delta \mathit{\Gamma} \neq 0$ and $\Delta M = 0$. These assumptions yield $\phi_z = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$, which means that also $\sin(\phi_{\rm SW} - \phi_z) \neq 0$ in this case.

The last possibility is $\Delta \mathit{\Gamma} \neq 0$ and $\Delta M \neq 0$. There is

$$\delta \approxeq \frac{h_{11} - h_{22}}{2(\mu_l - \mu_s)} \equiv \delta_{||}\, e^{i\phi_{\rm SW}} + \delta_{\perp} e^{i(\phi_{\rm SW} + \pi/2)}\,, \tag{29}$$

in the case of neutral K system (see, *e.g.* [12], p. 623, formula (2)). Here

$$\delta_{||} = \frac{1}{4}\frac{\mathit{\Gamma}_{11} - \mathit{\Gamma}_{22}}{\sqrt{(\Delta m)^2 + (\frac{\Delta\gamma}{2})^2}}\,, \tag{30}$$

$$\delta_{\perp} = \frac{1}{2}\frac{M_{11} - M_{22}}{\sqrt{(\Delta m)^2 + (\frac{\Delta\gamma}{2})^2}}\,, \tag{31}$$

are the real parameters. Thus

$$\tan\,\phi_z \equiv -\frac{\delta_{\perp}}{\delta_{||}}\,. \tag{32}$$

From (29) one finds

$$\delta_{||} = \Re\,(\delta\, e^{-i\phi_{\rm SW}}) \equiv \Re\,(\delta)\,\cos\phi_{\rm SW} + \Im\,(\delta)\,\sin\phi_{\rm SW}, \tag{33}$$

$$\delta_{\perp} = \Im\,(\delta\, e^{-i\phi_{\rm SW}}) \equiv \Im\,(\delta)\,\cos\phi_{\rm SW} - \Re\,(\delta)\,\sin\phi_{\rm SW}\,, \tag{34}$$

which leads to the following formula for $\tan\,\phi_z$:

$$\tan\phi_z \equiv -\frac{\Im\,(\delta) - \Re\,(\delta)\,\tan\phi_{\rm SW}}{\Re\,(\delta) + \Im\,(\delta)\,\tan\phi_{\rm SW}}\,. \tag{35}$$

On the other hand

$$\Im(\delta) = \delta_{||}\,\sin\phi_{\rm SW} + \delta_{\perp}\cos\phi_{\rm SW}\,. \tag{36}$$

Now if $\Delta \mathit{\Gamma} \neq 0$ and $\Delta M \neq 0$, then using (30) and (31) one infers that $\delta_{||} \neq 0$ and $\delta_{\perp} \neq 0$. Thus it should be $\Im\,(\delta) \neq 0$ in such a case. Estimations of $\Im\,(\delta)$ and $\Re\,(\delta)$ obtained from tests with neutral kaons show that $\Im\,(\delta) \neq \Re\,(\delta)$ (see, *e.g.* [12]). Taking into account all these properties it can be easily verified that neither $\phi_z = \phi_{\rm SW}$ nor $\phi_z = \phi_{\rm SW} \pm n\pi$, $(n = 1, 2, 3, \ldots)$ fulfills the relation (35) in the case of $\phi_{\rm SW} \approx 43,5^{\circ}$. Therefore, the conclusion that there must be $\phi_z \neq \phi_{\rm SW}$ and $\phi_z \neq \phi_{\rm SW} \pm n\pi$ in the neutral meson complexes seems to be obvious. So in the general case $\Delta\Gamma \neq 0$, $\Delta M \neq 0$ the condition $h_z \neq 0$ causes also that $\Im\,(h_z\,h^*) \neq 0$ and thus by (16) and (17) that $\langle s|l\rangle \neq \langle l|s\rangle \equiv (\langle s|l\rangle)^*$.

All the above analysis leads to the conclusion that in the case on neutral mesons the following theorem holds:

Theorem For the values of Δm and $\Delta\gamma$ which are typical for neutral meson complexes

$$\langle s|l\rangle = (\langle s|l\rangle)^* \equiv \langle l|s\rangle \;\Leftrightarrow\; (h_{11} - h_{22}) = 0\,. \tag{37}$$

It is interesting to confront this observation with the properties of matrix elements, h_{jk}, of the approximate as well as the exact effective Hamiltonians for neutral meson complex following from the CP- or CPT-symmetries of the total system under considerations. The standard approach to the description of properties of neutral mesons is based on the LOY effective Hamiltonian, $H_{\rm LOY}$. Taking $H_{||} = H_{\rm LOY}$ and assuming that the CPT invariance holds in the system considered one easily finds the standard result of the LOY approach

$$h_{11}^{\rm LOY} = h_{22}^{\rm LOY}\,, \tag{38}$$

where $h_{jk}^{\rm LOY} = \langle \boldsymbol{j}|H_{\rm LOY}|\boldsymbol{k}\rangle$, $(j,k = 1,2)$. Therefore, within the LOY theory the property that in a CPT invariant system $\langle K_{\rm S}|K_{\rm L}\rangle = (\langle K_{\rm S}|K_{\rm L}\rangle)^* \equiv \langle K_{\rm L}|K_{\rm S}\rangle$, is considered as quite obvious and unquestionable. This is one of the standard results of the LOY theory of neutral meson complexes. The question is whether such a property of the scalar product under considerations holds in the case of the exact effective Hamiltonian for the neutral mesons complex or not.

3. CP and CPT transformations and the exact $\boldsymbol{H_{||}}$

Solutions of the Schrödinger-like equation (1) can be written in the matrix form and such a matrix defines the evolution operator (which is usually non-unitary) $U_{||}(t)$ acting in $\mathcal{H}_{||}$:

$$|\psi;t\rangle_{||} \stackrel{\rm def}{=} U_{||}(t)|\psi\rangle_{||}\,, \tag{39}$$

where

$$|\psi\rangle_{||} \equiv q_1|\mathbf{1}\rangle + q_2|\mathbf{2}\rangle\,, \tag{40}$$

is the initial state of the system, $|\psi\rangle_{||} \equiv |\psi, t = t_0\rangle_{||} \in \mathcal{H}_{||}$. CP and CPT transformation properties of the matrix elements, h_{jk}, of the exact effective Hamiltonian, $H_{||}$, can be extracted from the suitable properties of the exact evolution operator $U_{||}(t)$. The exact evolution operator $U_{||}(t)$ has the following form [18]

$$U_{||}(t) = PU(t)P\,, \tag{41}$$

where P is the projection operator onto subspace $\mathcal{H}_{||}$ and $U(t)$ is the total unitary evolution operator, which solves the Schrödinger equation

$$i\frac{\partial}{\partial t}U(t)|\psi\rangle_{||} = HU(t)|\psi\rangle_{||}\,, \qquad U(t=t_0) = I\,, \tag{42}$$

where I is the unit operator in $\mathcal{H}$ and H is the total (selfadjoint) Hamiltonian acting in $\mathcal{H}$. In the considered case the projector P can be defined as follows [17,18]

$$P = |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}|\,. \tag{43}$$

One has

$$\mathcal{H}_{||} = P\mathcal{H}\,, \quad \mathcal{H}_{\perp} = (I-P)\mathcal{H} \stackrel{\text{def}}{=} Q\mathcal{H}. \tag{44}$$

The evolution operator $U_{||}(t)$ has a nontrivial form only if

$$[P,H] \neq 0\,, \tag{45}$$

and only then transitions of states from $\mathcal{H}_{||}$ into $\mathcal{H}_{\perp}$ and *vice versa*, *i.e.*, decay and regeneration processes, are allowed.

Within the matrix representation one can write [18]

$$U_{||}(t) \equiv \begin{pmatrix} \boldsymbol{A}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{46}$$

where $\mathbf{0}$ denotes the suitable zero submatrices and a submatrix $\boldsymbol{A}(t)$ is the (2×2) matrix acting in $\mathcal{H}_{||}$,

$$\boldsymbol{A}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \tag{47}$$

and $A_{jk}(t) = \langle\boldsymbol{j}|U_{||}(t)|\boldsymbol{k}\rangle \equiv \langle\boldsymbol{j}|U(t)|\boldsymbol{k}\rangle$, $(j,k=1,2)$.

Now if we assume that

$$[\Theta, H] = 0, \tag{48}$$

(where $\Theta = \mathcal{CPT}$ is an antiunitary operator with unitary $\mathcal{C},\mathcal{P}$ and antiunitary $\mathcal{T}$ denoting operators realizing the charge conjugation, the parity and time reversal for vectors in $\mathcal{H}$, respectively), then one easily finds that [6,18–22]

$$A_{11}(t) = A_{22}(t)\,. \tag{49}$$

The assumption (48) gives no relations between $A_{12}(t)$ and $A_{21}(t)$.

If the system under considerations is assumed to be CP invariant,

$$[\mathcal{CP}, H] = 0\,, \tag{50}$$

then using the following, most general, phase convention

$$\mathcal{CP}|\mathbf{1}\rangle = e^{-i\alpha}|\mathbf{2}\rangle, \quad \mathcal{CP}|\mathbf{2}\rangle = e^{+i\alpha}|\mathbf{1}\rangle, \tag{51}$$

(instead of the standard one: $\mathcal{CP}|\mathbf{1}\rangle = -|\mathbf{2}\rangle$, $\mathcal{CP}|\mathbf{2}\rangle = -|\mathbf{1}\rangle$) one easily finds that for the diagonal matrix elements of the matrix $\boldsymbol{A}(t)$ the relation (49) holds in this case also, and that for the off-diagonal matrix elements

$$A_{12}(t) = e^{2i\alpha} A_{21}(t)\,. \tag{52}$$

This means that if the CP symmetry is conserved in the system containing the subsystem of neutral mesons, then for every $t > 0$ there must be

$$\left| \frac{A_{12}(t)}{A_{21}(t)} \right| = 1 \,\equiv\, \text{const}\,. \tag{53}$$

Now let us consider the case when CP symmetry is violated,

$$[\mathcal{CP}, H] \neq 0\,. \tag{54}$$

For our considerations it is convenient to decompose the total Hamiltonian H into two parts [8, 9],

$$H \equiv H_{+} + H_{-}\,, \tag{55}$$

where

$$H_{\pm} = \tfrac{1}{2}\,[H \pm (\mathcal{CP})H(\mathcal{CP})^{+}]\,. \tag{56}$$

Under $\mathcal{CP}$, H_{+} is even and H_{-} is odd,

$$(\mathcal{CP})H_{\pm}(\mathcal{CP})^{+} = \pm H_{\pm}\,. \tag{57}$$

(Note that if relation (50) holds then $H_{-} \equiv 0$.) Now using relations (55)–(57) one can easily conclude that

$$(\mathcal{CP})H(\mathcal{CP})^{+} \equiv H - 2H_{-}\,. \tag{58}$$

This result helps one to solve the problem of how the solution, $U(t)$, to the Schrödniger equation (42) transforms under $\mathcal{CP}$. So let us define $U_{\text{CP}}(t) \stackrel{\text{def}}{=} (\mathcal{CP})U(t)(\mathcal{CP})^{+}$, where $U(t)$ solves Eq. (42). Starting from Eq. (42) one obtains

$$i\frac{\partial}{\partial t}U_{\text{CP}}(t) \;=\; (H - 2H_{-})U_{\text{CP}}(t) \tag{59}$$

$$\equiv\; HU_{\text{CP}}(t) - 2H_{-}U_{\text{CP}}(t)\,, \tag{60}$$

with the initial condition $U_{\rm CP}(0) = I$. The solution, $U(t)$, of Eq. (42) is the "free" solution for Eq. (60) and thus the solution of this last equation can be expressed as follows [23]

$$U_{\rm CP}(t) = U(t) + 2i \int_0^t U(t-\tau)\, H_- \, U_{\rm CP}(\tau)\, d\tau \,. \tag{61}$$

Of course, from (59) it follows that $U_{\rm CP}(t) = \exp[-it(H - 2H_-)t]$ but this formula is much less convenient than (61).

Assuming that the system under consideration is not CP invariant and using (51) it is easy to find that

$$A_{12}(t) \equiv e^{2i\alpha} \langle \mathbf{2} | U_{\rm CP}(t) | \mathbf{1} \rangle \,. \tag{62}$$

Next, inserting there $U_{\rm CP}(t)$ given by (61) yields

$$A_{12}(t) = e^{2i\alpha} A_{21}(t) + 2i e^{2i\alpha} \langle \mathbf{2} | \int_0^t U(t-\tau)\, H_- \, U_{\rm CP}(\tau)\, d\tau | \mathbf{1} \rangle \,. \tag{63}$$

From this last relation one infers that when CP symmetry is violated then for $t > 0$ there must be

$$\left| \frac{A_{12}(t)}{A_{21}(t)} \right|^2 \equiv 1 + |r_{21}(t)|^2 + 2\Re\left(r_{21}(t) \right), \qquad (t > 0)\,, \tag{64}$$

where

$$r_{21}(t) = \frac{2i}{A_{21}(t)} \int_0^t \langle \mathbf{2} | U(t-\tau)\, H_- \, U_{\rm CP}(\tau) | \mathbf{1} \rangle \, d\tau \,, \tag{65}$$

and $r_{21}(t) \neq 0$ for $t > 0$.

Let us analyze the simplest case when t is very short, $t \approx 0$, but still $t > 0$ and $\langle \mathbf{2} | H | \mathbf{1} \rangle \neq 0$. Then after some algebra one finds

$$\left. \left| \frac{A_{12}(t)}{A_{21}(t)} \right|^2 \right|_{0<t\approx 0} \simeq 1 + 4 \left| \frac{\langle \mathbf{2} | H_- | \mathbf{1} \rangle}{\langle \mathbf{2} | H | \mathbf{1} \rangle} \right|^2 - 4\Re\left(\frac{\langle \mathbf{2} | H_- | \mathbf{1} \rangle}{\langle \mathbf{2} | H | \mathbf{1} \rangle} \right) \neq 1 \,. \tag{66}$$

If $\langle \mathbf{2} | H | \mathbf{1} \rangle = 0$ but $\langle \mathbf{2} | H^2 | \mathbf{1} \rangle \neq 0$ then for very short t one has

$$\begin{aligned} \left. \left| \frac{A_{12}(t)}{A_{21}(t)} \right|^2 \right|_{0<t\approx 0} \simeq\ & 1 + 4 \left| \frac{\langle \mathbf{2} | (H_+ H_- + H_- H_+) | \mathbf{1} \rangle}{\langle \mathbf{2} | H^2 | \mathbf{1} \rangle} \right|^2 \\ & - 4\,\Re\left(\frac{\langle \mathbf{2} | (H_+ H_- + H_- H_+) | \mathbf{1} \rangle}{\langle \mathbf{2} | H^2 | \mathbf{1} \rangle} \right) \neq 1 \,. \end{aligned} \tag{67}$$

Note that these results and (64) are the quite general and that they do not depend on any model or approximation used. Relations (64), (66) and (67) prove that if the property (54) holds in the system, that is if the CP symmetry is violated, then in such a system the modulus of the ratio $A_{12}(t)/A_{21}(t)$ must be different from 1 for every $t > 0$,

$$[\mathcal{CP}, H] \neq 0 \quad \Rightarrow \quad \left|\frac{A_{12}(t)}{A_{21}(t)}\right| \neq 1\,, \quad (t > 0)\,. \tag{68}$$

The importance of this result consists in the fact that it is the rigorous consequence of only two assumptions. The first is that the real properties of the system follow from the solutions of the Schrödinger Equation (42). The second one is that the total selfadjoint Hamiltonian H does not commute with the $\mathcal{CP}$ operator. Apart from these two assumptions no additional model assumptions or approximations were used in order to prove (68). In particular, no properties of the eigenvectors $|l\rangle, |s\rangle$ for the effective Hamiltonian $H_{||}$ and no assumptions about their form were used in the above considerations leading to the conclusion (68).

So, we already have all the necessary CP- and CPT-transformation properties of the matrix elements of the exact evolution operator $U_{||}(t)$ for the subspace of neutral mesons,$\mathcal{H}_{||}$, and now we can extract from them the suitable properties of the matrix elements of the exact effective Hamiltonian for this subspace. One can find the necessary properties of the matrix elements of $H_{||}$ by analyzing the following identity [14, 15, 18, 24, 25]

$$H_{||} \equiv H_{||}(t) = i\frac{\partial U_{||}(t)}{\partial t}\,[U_{||}(t)]^{-1}\,, \tag{69}$$

where $[U_{||}(t)]^{-1}$ is defined as follows

$$U_{||}(t)\,[U_{||}(t)]^{-1} = [U_{||}(t)]^{-1}\,U_{||}(t) = P\,. \tag{70}$$

(Note that the identity (69) holds, independent of whether $[P, H] \neq 0$ or $[P, H] = 0$.) The expression (69) can be rewritten using the matrix $\boldsymbol{A}(t)$

$$H_{||}(t) \equiv i\frac{\partial \boldsymbol{A}(t)}{\partial t}[\boldsymbol{A}(t)]^{-1}. \tag{71}$$

Relations (69), (71) must be fulfilled by the exact as well as by every approximate effective Hamiltonian governing the time evolution in every two dimensional subspace $\mathcal{H}_{||}$ of states $\mathcal{H}$ [14, 16, 24, 25].

It is easy to find from (71) the general formulae for the diagonal matrix elements, h_{jj}, of $H_{||}(t)$, in which we are interested. We have [18]

$$h_{11}(t) = \frac{i}{\det \boldsymbol{A}(t)} \left(\frac{\partial A_{11}(t)}{\partial t} A_{22}(t) - \frac{\partial A_{12}(t)}{\partial t} A_{21}(t) \right), \tag{72}$$

$$h_{22}(t) = \frac{i}{\det \boldsymbol{A}(t)} \left(- \frac{\partial A_{21}(t)}{\partial t} A_{12}(t) + \frac{\partial A_{22}(t)}{\partial t} A_{11}(t) \right). \tag{73}$$

Using (72), (73) the difference $(h_{11} - h_{22}) = 2h_z$, whose properties are crucial for the question whether the product $\langle s|l\rangle$ is real or not, can be expressed as follows [18]

$$\begin{aligned} h_{11}(t) - h_{22}(t) &= i \, \frac{1}{\det \boldsymbol{A}(t)} \Bigg\{ A_{11}(t)\, A_{22}(t) \frac{\partial}{\partial t} \ln \left(\frac{A_{11}(t)}{A_{22}(t)} \right) \\ &\quad - A_{12}(t)\, A_{21}(t) \frac{\partial}{\partial t} \ln \left(\frac{A_{12}(t)}{A_{21}(t)} \right) \Bigg\}. \end{aligned} \tag{74}$$

At this point one should use the fact that an important relation between amplitudes $A_{12}(t)$ and $A_{21}(t)$ is described by the famous Khalfin's Theorem [19–21, 26, 27]. This Theorem states that in the case of unstable states, if amplitudes $A_{12}(t)$ and $A_{21}(t)$ have the same time dependence

$$r(t) \stackrel{\text{def}}{=} \frac{A_{12}(t)}{A_{21}(t)} = \text{const.} \equiv r\,, \tag{75}$$

then there must be $|r| = 1$.

The proof of this theorem is rigorous and it does not use the CP- or CPT-transformation properties of the system considered.

Now one is ready to examine consequences of the assumptions that $(h_{11}(t) - h_{22}(t)) = 0$ is admissible for $t > 0$. In such a case an analysis of the expression (74), relations (49), (53), (64), (68) and the Khalfin's Theorem (75) allows one to conclude that

Conclusion 1

If $(h_{11}(t) - h_{22}(t)) = 0$ for $t > 0$ then there must be

(a) $$\frac{A_{11}(t)}{A_{22}(t)} = \text{const.} \;\text{ and }\; \frac{A_{12}(t)}{A_{21}(t)} = \text{const.} \;\; (\text{for } \; t > 0)\,,$$

or

(b) $$\frac{A_{11}(t)}{A_{22}(t)} \neq \text{const.} \;\text{ and }\; \frac{A_{12}(t)}{A_{21}(t)} \neq \text{const.} \;\; (\text{for } \; t > 0)\,.$$

The following interpretation of (a) and (b) follows from (49), (53), (64), (68) and from the Khalfin's Theorem (75). The case (a) means that the CP-symmetry is conserved and there is no information about the CPT invariance. The case (b) denotes that the system under considerations is neither CP-invariant nor CPT-invariant.

In our discussion the CPT Theorem [28–33] cannot be neglected. The CPT Theorem is a fundamental theorem of axiomatic quantum field theory. It follows from locality, Lorentz invariance and unitarity. One should also take into account another fact that there is no experimental evidence that CPT symmetry is violated [12, 34]. Therefore, the assumption that any quantum theory of elementary particles should be CPT invariant seems to be obvious. So let us assume that CPT symmetry is the exact symmetry of the system under considerations, that is that the condition (48) holds. In such a case the relation (49) holds. The consequence of this is that the expression (74) becomes simpler and it is easy to prove that the following property must hold [18]

$$h_{11}(t) - h_{22}(t) = 0 \;\Leftrightarrow\; \frac{A_{12}(t)}{A_{21}(t)} = \text{const.}, \quad (t > 0)\,. \tag{76}$$

Now let us go on to analyze the conclusions following from the Khalfin's Theorem. CP noninvariance requires that $|r| \neq 1$ (see (53), (64), (68) and also [1–12, 19–22]). This means that in such a case there must be $r = r(t) \neq$ const. So, if in the system considered the properties (48) and (54) hold then, as it follows from (76), at $t > 0$ there must be $(h_{11}(t) - h_{22}(t)) \neq 0$ in this system [18]. Thus, keeping in mind results (28) and (37) one can state that following conclusion must be true

Conclusion 2
If the CPT symmetry is the real symmetry of the system containing neutral meson subsystem and the CP symmetry is violated in this system (*i.e.*, if (48) and (54) hold) then there must be

$$\langle s|l\rangle \neq (\langle s|l\rangle)^* \equiv \langle l|s\rangle\,. \tag{77}$$

4. Discussion

As it was mentioned, the CPT Theorem follows from basic principles of quantum theory. Simply it is the mathematical consequence of basic assumptions of quantum theory. There is no evidence that the basic principles of the quantum mechanics are violated. There is also no evidence of the CPT violation. In contrast to the lack of evidence of the CPT noninvariance, the CP violation is an experimental fact [4, 12, 34]. This means (by (68)) that in the real system there must be $\left|\frac{A_{12}(t)}{A_{21}(t)}\right| \neq 1$ for $(t > 0)$

and, therefore, due to the Khalfin's theorem (75) and the relation (76), there must be $(h_{11}(t) - h_{22}(t)) \neq 0$ for $(t > 0)$ in real systems. Thus the real property of the system containing neutral mesons is that it must be $\langle s|l\rangle \neq (\langle s|l\rangle)^* \equiv \langle l|s\rangle$ (rather than $\langle s|l\rangle = (\langle s|l\rangle)^* \equiv \langle l|s\rangle$). This means that one of the standard results of the LOY theory that in a CPT invariant system $\langle K_{\rm S}|K_{\rm L}\rangle = (\langle K_{\rm S}|K_{\rm L}\rangle)^* \equiv \langle K_{\rm L}|K_{\rm S}\rangle$, (where $|K_{\rm S}\rangle$, $|K_{\rm L}\rangle$ correspond to $|s\rangle, |l\rangle$), is wrong.

On the other hand, the assumption that the inner product $\langle K_{\rm S}|K_{\rm L}\rangle$ should be real (or equivalently that there should be $\Im\,\langle K_{\rm S}|K_{\rm L}\rangle = 0)$ when the CPT symmetry holds was considered in the literature as the fundamental property of CPT invariant system allowing one to derive many relations connecting parameters characterizing the neutral K system and some constrains on these parameters [2–12, 35–40], *etc.*

The result (77) means that all relations and constrains obtained in this way need not reflect real properties of systems under consideration. Simply, they may lead to wrong conclusions obfuscating the real properties of the neutral meson systems and thus our opinion about the interactions causing decay process of these mesons. So within the standard LOY theory of neutral meson complexes one should be very careful interpreting the results of experiments with neutral mesons and in such a case one can never be sure that this interpretation corresponds to the real properties of the system under investigations. These reservations also concern relations derived within the use of the Bell–Steinberger relation. Properties of the inner product $\langle K_{\rm S}|K_{\rm L}\rangle$ are crucial for the interpretation of such relations [35, 40]. It seems that while performing an analysis of the results of such experiments, only relations connecting the parameters characterizing neutral meson complexes which do not depend on any approximations and which follow directly from the general principles of the quantum theory should be taken into account. The mentioned above Khalfin's Theorem is an example of such relations.

Note that if we assume that real properties of the system are described by the solutions of the Schrödinger Equation (42) (with $H = H^+$) then there can be $\langle s|l\rangle = \langle l|s\rangle \equiv (\langle s|l\rangle)^*$ for $h_z \neq 0$ only if $\phi_z = \phi_{\rm SW}$, that is if

$$\frac{2\,\Delta M}{\Delta\mathit{\Gamma}} \equiv -\,\frac{\Re\,(h_{11} - h_{22})}{\Im\,(h_{11} - h_{22})} = \frac{2\,(\mu_l - \mu_s)}{\gamma_l - \gamma_s}. \tag{78}$$

The result of the Fridrichs–Lee model [19] calculations performed in [17] with the assumption that values of parameters of this model correspond to the parameters of neutral K complex is the following [18]

$$\Re(h_{11}^{\rm FL} - h_{22}^{\rm FL}) = \Delta M^{\rm FL} \sim 1.7 \times 10^{-13}\,\Im\,(\langle \mathbf{1}|H|\mathbf{2}\rangle) \neq 0\,, \tag{79}$$

and

$$\Im(h_{11}^{\rm FL} - h_{22}^{\rm FL}) \equiv -\tfrac{1}{2}\,\Delta\mathit{\Gamma}^{\rm FL} = 0\,. \tag{80}$$

So, the relation (78) does not take place in the case of the Fridrichs–Lee model. The same conclusion one can draw analyzing the experimentally obtained values $\Delta M = m_{K_0} - m_{\bar{K}_0}$ and $\Delta\mu = m_{K_{\rm L}} - m_{K_{\rm S}}$, $\Delta\gamma = \Gamma_{K_{\rm L}} - \Gamma_{K_{\rm S}}$ [12].

The result (77) is the consequence of the property (68) that for $t > 0$ modulus of the ratio $A_{12}(t)/A_{21}(t)$ must be different from unity if CP symmetry is violated. Within the LOY theory such a property follows from the properties of matrix elements of the approximate LOY effective Hamiltonian. On the other hand, taking into account the conclusions derived in [42], the property (68) cannot considered to be the obvious.

In [42] it was found that within the LOY theory the modulus of the similar ratio can be equal to one for some models of interactions. Note that the conclusion (68) does not depend on any approximation. It also does not depend on any model of interactions. As it was mentioned it is the simple implication of two general assumptions: that the total Hamiltonian, H, does not commute with the $\mathcal{CP}$ operator and that the Schrödinger equation describes correctly the system we are interested in. This means that this relation reflects real properties of system with CP violated.

All evolution equations for neutral meson complex have the form of Eq. (1) (see [1–12, 43, 44]). Solutions of this equation are used to describe time evolution of neutral mesons and mixing processes. Solving this equation one can find amplitudes $A_{jk}(t)$, $(j, k = 1, 2)$. An important property of the ratio $A_{12}(t)/A_{21}(t) = r(t)$ follows from the Khalfin's Theorem (75). The main result of this paper (*i.e.*, the property (77)) and the earlier result that there must be $h_{11}(t) - h_{22}(t)) \neq 0$ for $t > 0$ in the CPT invariant system [18] (see also [45]) is the consequence of this Theorem. One may want to confront the Khalfin's Theorem with the experimental results, which give $|1 - |r(t)|\,| \sim 10^{-3} = \text{const.}$ with some limited accuracy (see, *e.g.*, [12, 44]). This does not mean that the Khalfin's Theorem is wrong. Simply effects connected with the Khalfin's Theorem are very tiny and they seem to be beyond the accuracy of recent experiments (see also [27]). In the light of the detailed model analysis given in [19] the conclusion that for $t \gg t_0$, $|r_{\rm max}(t) - r_{\rm min}(t)| \stackrel{\rm def}{=} \Delta r < 10^{-11}$, seems to be acceptable. Within the LOY approximation physical states, $|l\rangle, |s\rangle$, decays exponentially. In general there are tiny corrections to the exponential decay laws at very short and very long times [46]. The amplitudes $A_{jk}(t)$ calculated within the LOY, that is in fact within the WW approximation give the result $r(t) = r_{\rm LOY} = \text{const.}$ These amplitudes calculated more accurately contains non-exponential and non-oscillatory tiny corrections (see [19, 22]) leading to varying in time $r(t)$ with the spectrum of changes limited by $\Delta r < 10^{-11}$. In other words there is

$$r(t) = r_{\rm LOY} + d(t)\,, \tag{81}$$

where $d(t)$ varies in time t and $|d(t)| \leq \Delta r$ for $t \gg t_0$. (Note that the Khalfin's Theorem does not require $d(t)$ to be large.) These corrections seem to be irrelevant for many parameters describing neutral meson complex but they and, therefore, the consequences of the Khalfin's Theorem, must be taken into account in high precision CPT symmetry tests.

The last remark. Within the standard theory of neutral meson complexes all evolution equations are derived from the Schrödinger Equation (42) using more or less accurate approximations (see [1–12, 35–40] and so on). This means that there is a consensus that the Schrödinger Equation describes correctly time evolution in such systems. So, if we adopt this opinion and assume that the Schrödinger Equation describes correctly real properties of the varying in time processes in the systems, *e.g.*, containing neutral meson complex as a subsystem, then we must also accept all rigorous consequences of such an assumption. The main conclusions of this paper are consequences of this type.

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