CASIMIR EFFECT IN EXTERNAL MAGNETIC FIELD*

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In this paper we examine the Casimir effect for charged fields in presence of external magnetic field. We consider scalar field (connected with spinless particles) and the Dirac field (connected with 1/2-spin particles). In both cases we describe quantum field using the canonical formalism. We obtain vacuum energy by direct solving field equations and using the mode summation method. In order to compute the renormalized vacuum energy we use the Abel–Plana formula.

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1. Introduction

The imposition of boundary condition on a quantum field leads to the modification of the vacuum energy level and can be observed as an associated vacuum pressure. This effect (called Casimir effect) has been predicted by Casimir in his original work in 1948 [1], and has been experimentally observed for electromagnetic field several years later. Until now, in many theoretical works Casimir energy has been computed for various types of boundary geometry and for fields other than electromagnetic one.

When we consider of charged fields another important question arise. How an external field, coupled to the charge, affect the vacuum energy of the field? Answer to this question is important for understanding of some aspects in particle physics. Within a hadron, for example, the vacuum energy of quark fields is affected by the electromagnetic field of the quarks and by the color field of gluons and quarks.

In order to investigate how charged fermionic and bosonic constrained vacuum fluctuations are affected by fields coupled to this charge, we consider

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vacuum energy of electrically charged fields under the influence of an external constant uniform magnetic field, and constrained by simple boundary conditions. In Section 2 we shall consider a complex scalar field confined between two infinite plates with Dirichlet boundary conditions with magnetic field in a direction perpendicular to the plates. In Section 3 we shall consider a Dirac quantum field under antiperiodic boundary conditions. This choice of geometry and external fields avoids technical difficulties and focuses our attention on the fundamental issue.

The fermionic Casimir effect was first calculated by Johnson [2] for applications in the MIT bag model. For a massless Dirac field, Johnson obtained an energy density 7/4 times the energy density of the electromagnetic field.

In both cases (bosonic and fermionic) we describe quantum field using the canonical formalism. We obtain the vacuum energy by direct solving the field equations and using the mode summation method. In order to compute the renormalized vacuum energy we use the Abel–Plana formula.

In order to obtain influence of magnetic field on the Casimir energy another methods can be used. Recently, the papers using Schwinger's proper time method have been published [3,4]. So, our purpose is to compare results obtained using our (canonical) method with the Schwinger's one (based on the covariant quantization).

In paper [5] the fermionic Casimir energy in external magnetic field is also investigate. However, in [5] authors consider Dirac field under different boundary conditions (MIT boundary conditions) and obtain different results.

2. Bosonic Casimir effect

In this section we consider scalar field $\phi(\vec{r},t)$, describing charged, spinless particles with mass m. It is under the influence of uniform magnetic field $\vec{B}=(0,0,B)$. We choose direction of z axis in such a way that B is positive.

Using gauge invariance, we choose electromagnetic potential corresponding to field $\vec{B}$ in the form: $A_\mu=(0,-yB,0,0)$.

The Lagrangian density of field ϕ takes the form:

$$\begin{aligned}\mathcal{L} &= (D_\mu\phi)^*(D^\mu\phi)-m^2\phi^*\phi \\ &= \partial_t\phi^*\partial_t\phi-(\partial_x+ieBy)\phi^*(\partial_x-ieBy)\phi \\ &\quad -\partial_y\phi^*\partial_y\phi-\partial_z\phi^*\partial_z\phi-m^2\phi^*\phi\,, \end{aligned}\tag{1}$$

and leads to the following equation of motion

$$\partial_t^2\phi-\partial_x^2\phi-\partial_y^2\phi-\partial_z^2\phi+2ieBy\partial_x\phi+\left(e^2B^2y^2+m^2\right)\phi=0\,,\tag{2}$$

where derivative $D_\mu=\partial_\mu+ieA_\mu$ in Eq. (1) is covariant derivative.

Solution of Eq. (2) is well known and is given in [6] in case of nonrelativistic quantum particle. We briefly remind the method of solving this kind of equation. The variables x, z and t do not occur in Eq. (2) explicity, therefore the solution of Eq. (2) takes the form:

$$\phi(x,y,z,t) = F(y)e^{i(kx+pz-\omega t)}\,. \tag{3}$$

After inserting of Eq. (3) into Eq. (2) the equation for function $F(y)$ is given by:

$$\partial_y^2 F + (\omega^2 - k^2 - p^2 - m^2 + 2eBky - e^2B^2y^2)F = 0\,, \tag{4}$$

which has a solution in the form:

$$F(y) = e^{-eB/2(y-y_0)^2}\mathrm{H}_n(\sqrt{eB}(y-y_0))\,, \tag{5}$$

where H_n are Hermite polynomials and $y_0 = k(eB)^{-1}$. The parameters n, p and ω fulfill the following relation

$$\omega = \sqrt{2eB(n+\tfrac{1}{2}) + p^2 + m^2}\,. \tag{6}$$

Let us introduce the Dirichlet boundary conditions in the form:

$$\phi(x,y,z=0,t) = \phi(x,y,z=a,t) = 0 \tag{7}$$

which implies that field ϕ is equal to zero on the two parallel planes (plates). The plates are perpendicular to the z-axis and the distance between them is equal to a.

In the area between plates, solution of Eq. (2) is given by:

$$\phi(\vec{r},t) = \int\limits_{-\infty}^{\infty} dk \sum_{l=1}^{\infty}\sum_{n=0}^{\infty}\big(a_{nlk}u_{nlk}(\vec{r})e^{-i\omega_{nl}t} + b_{nlk}^{+}u_{nlk}(\vec{r})e^{i\omega_{nl}t}\big). \tag{8}$$

Functions u_{nlk} are the special solutions of Eq. (2) in the form:

$$u_{nlk}(\vec{r}) = C_n e^{ikx}\sin(p_l z)e^{-eB/2(y-y_0)^2}\mathrm{H}_n(\sqrt{eB}(y-y_0))\,, \tag{9}$$

where $C_n = (eBa^2)^{1/4}(2^n n!\pi^{3/2})^{-1/2}$ is the normalization constant, and $p_l = \pi l/a$. Solutions of Eq. (9) fulfill relations of orthogonality in the form:

$$\int\limits_{\Gamma} d^3\vec{r}\, u_{n_1l_1k_1}(\vec{r})u^{*}_{n_2l_2k_2}(\vec{r}) = a^2\delta(k_1-k_2)\delta_{l_1l_2}\delta_{n_1n_2} \tag{10}$$

and relations of completeness in the form:

$$\int_{-\infty}^{\infty} dk \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} u_{nlk}(\vec{r}_1) u^*_{nlk}(\vec{r}_2) = a^2 \delta^3(\vec{r}_1 - \vec{r}_2), \tag{11}$$

where Γ is the area between plates.

Standard commutation relations for the field

$$\begin{aligned} [\phi(\vec{r}_1,t), \partial_t \phi^+(\vec{r}_2,t)] &= i\delta^3(\vec{r}_1 - \vec{r}_2) &(12)\\ [\phi(\vec{r}_1,t), \phi(\vec{r}_2,t)] &= [\partial_t \phi^+(\vec{r}_1,t), \partial_t \phi^+(\vec{r}_2,t)] = 0 &(13)\end{aligned}$$

lead to the following commutators for creation and annihilation operators

$$[a_{n_1 l_1}(k_1), a^+_{n_2 l_2}(k_2)] = [b_{n_1 l_1}(k_1), b^+_{n_2 l_2}(k_2)] = \tfrac{1}{2} a^{-2} \omega^{-1}_{n_1 l_1} \delta_{n_1 n_2} \delta_{l_1 l_2} \delta(k_1 - k_2)\,. \tag{14}$$

The Hamiltonian density

$$\mathcal{H} = \partial_t \phi^* \partial_t \phi + (\partial_x + ieBy)\phi^* (\partial_x - ieBy)\phi + \partial_y \phi^* \partial_y \phi + \partial_z \phi^* \partial_z \phi + m^2 \phi^* \phi \tag{15}$$

leads to the vacuum energy in the form:

$$E_{\rm vac} = L^2 \int_0^a dz \langle 0|\mathcal{H}|0\rangle = L^2 eB(2\pi)^{-1} \sum_{l=1}^{\infty} \sum_{n=0}^{\infty} \sqrt{\tfrac{\pi^2}{a^2} l^2 + eB(2n+1) + m^2}, \tag{16}$$

where L is the length of plates.

The energy (16) is infinite. Now, we define the renormalized vacuum energy $E_{\rm ren}$ as a difference between energy (16) and energy without boundary conditions:

$$\begin{aligned} E_{\rm ren} &= L^2 eB(2\pi)^{-1} \sum_{n=0}^{\infty} \Bigg(\sum_{l=1}^{\infty} \sqrt{\tfrac{\pi^2}{a^2} l^2 + eB(2n+1) + m^2} \\ &\quad - \int_{l=0}^{\infty} dl \sqrt{\tfrac{\pi^2}{a^2} l^2 + eB(2n+1) + m^2} \Bigg). \end{aligned} \tag{17}$$

In order to compute the renormalized energy we use the Abel–Plana formula in the form

$$\sum_{l=1}^{\infty} f(l) - \int_0^{\infty} f(l) dl = -\tfrac{1}{2} f(0) + i \int_0^{\infty} dt \frac{f(it) - f(-it)}{e^{2\pi t} - 1}. \tag{18}$$

This formula is often applied in renormalization problems. Its applications and generalizations can be found in many works [7,8].

The renormalized energy, after using formula (18), takes (in restored SI units) the final form:

$$E_{\text{ren}} = -L^2 ceBa^{-1} \sum_{n=0}^{\infty} \int_{\lambda_n}^{\infty} dt \sqrt{t^2 - \lambda_n^2} \left(e^{2\pi t} - 1\right)^{-1}, \tag{19}$$

where $\lambda_n = a\pi^{-1}\sqrt{eB\hbar^{-1}(2n+1) + m^2c^2\hbar^{-2}}$.

Eq. (19) may be written in the form:

$$E_{\text{ren}}(a, B, m) = E_0(a)\, f^B(\xi, \eta), \tag{20}$$

where

$$E_0(a) = -c\hbar\pi^2 L^2/(720\, a^3) \tag{21}$$

is the standard Casimir energy for $B = 0$ for charged, massless field [7] and

$$f^B(\xi, \eta) = 720\pi^{-2}\xi \sum_{n=0}^{\infty} \int_{\lambda_n}^{\infty} dt \sqrt{t^2 - \lambda_n^2}\, (e^{2\pi t} - 1)^{-1}, \tag{22}$$

where $\lambda_n = \pi^{-1}\sqrt{\xi(2n+1) + \eta^2)}$, $\xi = eBa^2\hbar^{-1}$ and $\eta = amc\hbar^{-1}$ are dimensionless parameters.

Decreasing n-times the distance between plates ($a \to a/n$) and increasing n^2-times the magnetic field ($B \to n^2 B$), we obtain the same value of ξ. So, for small a we need stronger magnetic field to obtain the same value of f^B. It is the reason why the curves in Figs. 2 and 5 coincide at small distances. (The same situation will be for fermions in the next section — Fig. 7.)

Numerical calculations of Eq. (19) are shown in Figs. 1–5. Main conclusion is that, external magnetic field decreases the Casimir energy and pressure of the scalar field (Fig. 1), and suppresses it completely in the limit of $B \to 0$. The decrease of the Casimir energy with a distance is faster for stronger magnetic field (Fig. 2).

Let us compare the behavior of the Casimir pressure for two different distances a (Figs. 3,4). In order to suppress the pressure noticeably, magnetic field should be of the order of 10 T for distance $a = 10$ nm, and of the order of 0.1 T for distance $a = 100$ nm.

Asymptotic behavior of energy (20) we obtain examining Eq. (22). For strong magnetic field ($\xi \to \infty$) we use the relation $\exp(2\pi t) - 1 \to \exp(2\pi t)$,

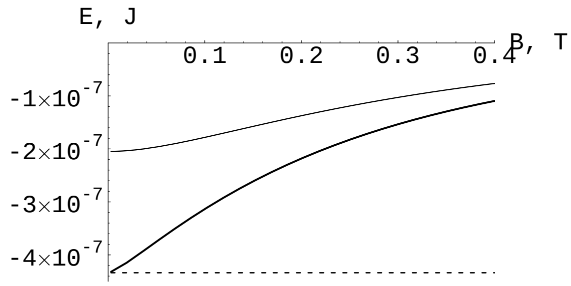

Fig. 1. Casimir energy for scalar field as a function of magnetic field B, corresponding to $a = 10^{-7}$ m and $L = 1$ m. Bold curve corresponds to mass $m = 0$ whereas thin curve corresponds to mass $m = 2$ eV. The dashed line corresponds to value $-\pi^2\hbar c/720a^3$. (It is standard value of energy for $B = 0$.)

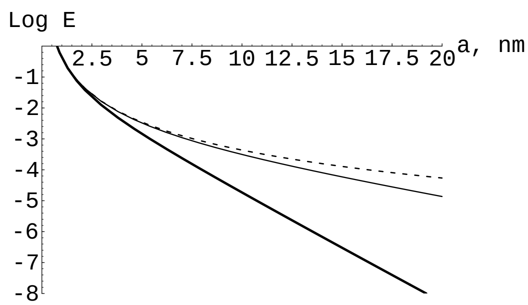

Fig. 2. Casimir energy for scalar field as a function of distance a, for massless field $m = 0$ and $L = 1$ m. Bold curve corresponds to magnetic field $B = 100$ T whereas the thin one corresponds to $eB = 10$ T. The dashed curve corresponds to function $-\pi^2\hbar c a^{-3}/720$ (standard function of distance a for $B = 0$). Values of energy are shown in logarithmic scale.

which gives:

$$\begin{aligned} f^B_\infty(\xi,\eta) &= 720\pi^{-2}\xi\sum_{n=0}^{\infty}\int_{\lambda_n}^{\infty} dt\sqrt{t^2-\lambda_n^2}\,e^{-2\pi t} \\ &= 360\pi^{-4}\xi\sum_{n=0}^{\infty}\sqrt{\xi(2n+1)+\eta^2}\,\mathrm{K}_1\left(\sqrt{\xi(2n+1)+\eta^2}\right), \quad (23) \end{aligned}$$

where $K_1(x)$ is the Bessel function. Numerical calculations show that error defined by $|f^B_\infty/f^B - 1|$ is less than 10^{-2} for $\xi > 3$ and $\eta = 0$ ($\eta = 0$ is the worst case).

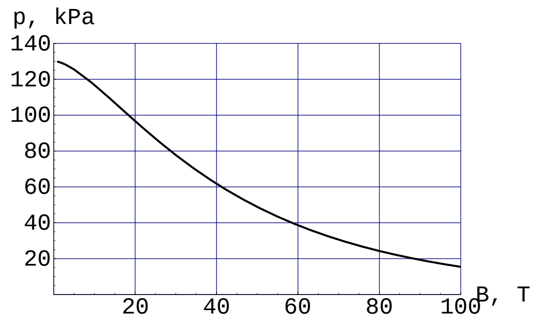

Fig. 3. Casimir pressure for scalar field as a function of magnetic field B, corresponding to distance $a = 10$ nm. The diagram is made in SI units for massless field.

Fig. 4. Casimir pressure for scalar field as a function of magnetic field B, corresponding to distance $a = 100$ nm. The diagram is made in SI units for massless field.

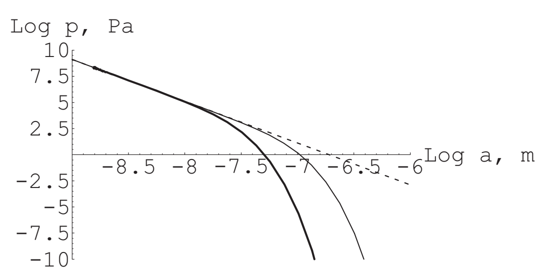

Fig. 5. Casimir pressure for massless, scalar field as a function of distance a. Diagram is made in SI units, using logarithmic scales. Bold curve corresponds to $B = 10$ T, thin to $B = 1$ T, and dashed to $B = 0$ T.

For weak magnetic field ($\xi \to 0$) we change sum into integration, which gives:

$$
\begin{aligned}
f_0^B(\xi,\eta) &= 720\pi^{-2}\xi \int\limits_0^\infty dn \int\limits_{\lambda(n)}^\infty dt \sqrt{t^2-\lambda^2(n)}\left(e^{2\pi t}-1\right)^{-1} \\
&= 720\pi^{-4} \int\limits_1^\infty dy \sqrt{y^2-1} \int\limits_M^\infty dx\, x^3 \left(e^{2xy}-1\right)^{-1}, \qquad (24)
\end{aligned}
$$

where $M=\sqrt{\xi+\eta^2}$. Numerical calculations show that error $|f_0^B/f^B-1|$ is less than 5×10^{-2} for $\xi<0.04$ and $\eta=0$ (the worst case). For small ξ formula (24) is better for numerical calculation than formula (22), because it leads to more convergent numerical algorithms.

We compared Eq. (19) with Eq. (15) from [3] (based on Schwinger's method) using *Mathematica 4.1*. We obtained the same numerical results. However, we are not able to prove analytically that these two equations are equivalent.

3. Fermionic Casimir effect

The Lagrangian density of the Dirac field in external electromagnetic field takes the form:

$$
\mathcal{L} = \Psi^+\gamma^0(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\Psi, \qquad (25)
$$

and leads to the following equation of motion

$$
(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\Psi^a = 0, \qquad (26)
$$

where $a=1,2,3,4$ are numbers of bispinor components.

We take γ matrices in Dirac representation:

$$
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \qquad (27)
$$

Analogously as in previous section, we solve Eq. (26) in presence of an external magnetic field $\vec{B}=(0,0,B)$, but now we impose antiperiodic boundary conditions in the form:

$$
\Psi(x,y,z=0,t) = -\Psi(x,y,z=a,t). \qquad (28)
$$

Solution of Eq. (26) can be predicted as follows:

$$
\Psi^a(\vec{r},t) = \exp(i(\omega t + kx + pz))\eta^a(y), \qquad (29)
$$

where bispinors $\eta^a(y)$ take the form:

$$\eta^a(y) = \begin{pmatrix} c_1 u_{n-1}(y) \\ c_2 u_n(y) \\ c_3 u_{n-1}(y) \\ c_4 u_n(y) \end{pmatrix}, \tag{30}$$

where c_a are number parameters independent of space co-ordinates, and functions $u_n(y)$ are given by:

$$u_n(y) = e^{-eB/2(y-y_0)^2}\mathrm{H}_n(\sqrt{eB}(y-y_0)), \tag{31}$$

where $y_0 = k/(eB)$.

After inserting Eq. (31) into Eq. (26) we obtain matrix equation

$$\begin{pmatrix} -(\omega+m) & 0 & -p & 2n\sqrt{eB} \\ 0 & -(\omega+m) & \sqrt{eB} & p \\ p & -2n\sqrt{eB} & \omega-m & 0 \\ -\sqrt{eB} & -p & 0 & \omega-m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \vec{0}\,. \tag{32}$$

Nontrivial solutions of Eq. (32) (*i.e.* different from null vector) exist if rank of matrix is different than 4. This situation occurs when

$$\omega^2 = p^2 + 2eBn + m^2. \tag{33}$$

After inserting solutions of Eq. (32) into Eq. (29) we obtain general solution of Eq. (26) in the following form:

$$\begin{aligned} \Psi(\vec{r},t) \; = \; & \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\int_{-\infty}^{\infty} dk \sum_s \Big(e^{-i(Et+kx+p_l z)} a_{n,l,s}(k)\vec{\xi}_{n,l,s}(k,y) \\ & + e^{i(Et+kx+p_l z)} b^+_{n,l,s}(k)\vec{\eta}_{n,l,s}(k,y)\Big), \end{aligned} \tag{34}$$

where $p_l = \pi/a(2l+1)$, $E = |\omega|$, and variable s takes the values -1, 1.

For $n>0$ bispinors $\vec{\xi}$, $\vec{\eta}$ from Eq. (34) are given by:

$$\vec{\xi}_{n,s=1}(k,p,y) = \frac{C_n}{\sqrt{2^{n-1}(n-1)!}}\begin{pmatrix} -\frac{p}{E+m}u_{n-1}(y) \\ \frac{\sqrt{eB}}{E+m}u_n(y) \\ u_{n-1}(y) \\ 0 \end{pmatrix}, \tag{35}$$

$$\vec{\xi}_{n,s=-1}(k,p,y) = \frac{C_n}{\sqrt{2^n n!}}\begin{pmatrix} \frac{2n\sqrt{eB}}{E+m}u_{n-1}(y) \\ \frac{p}{E+m}u_n(y) \\ 0 \\ u_n(y) \end{pmatrix}, \tag{36}$$

$$\vec{\eta}_{n,s=1}(k,p,y) = \frac{C_n}{\sqrt{2^n n!}}\begin{pmatrix} 0 \\ u_n(y,-k) \\ -\frac{2n\sqrt{eB}}{m+E}u_{n-1}(y,-k) \\ \frac{p}{m+E}u_n(y,-k) \end{pmatrix}, \tag{37}$$

$$\vec{\eta}_{n,s=-1}(k,p,y) = \frac{C_n}{\sqrt{2^{n-1}(n-1)!}}\begin{pmatrix} u_{n-1}(y,-k) \\ 0 \\ -\frac{p}{m+E}u_{n-1}(y,-k) \\ -\frac{\sqrt{eB}}{m+E}u_n(y,-k) \end{pmatrix}, \tag{38}$$

where $C_n = \left((eB/\pi)^{1/2}(E+m)/2E\right)^{1/2}$.

For $n=0$ bispinors $\vec{\xi}_{0,s=-1}$ and $\vec{\eta}_{0,s=1}$ are also given by these equations, if we take $u_{-1}(y)=0$. Whereas spinors $\vec{\eta}_{0,s=-1}$, $\vec{\xi}_{0,s=1}$ are equal to zero.

The spinors $\vec{\eta}$, $\vec{\xi}$ are orthonormal in the sense of the norm defined by:

$$\|\vec{\xi}_{n,s}(k,p,y)\|^2 = \int\limits_{-\infty}^{\infty} dy \vec{\xi}_{n,s}(k,p,y)\vec{\xi}^{*}_{n,s}(k,p,y). \tag{39}$$

Anticommutation relations for the field operators Ψ, $\Pi = i\Psi^+$ in the form:

$$[\Psi_a(\vec{x},t), \Pi_b(\vec{y},t)]_+ = i\delta(\vec{x}-\vec{y})\delta_{ab} \tag{40}$$

lead to the following anticommutators for creation and annihilation operators

$$\begin{aligned}[a_{n_1l_1s_1}(k_1), a^+_{n_2l_2s_2}(k_2)]_+ &= [b_{n_1l_1s_1}(k_1), b^+_{n_2l_2s_2}(k_2)]_+ \\ &= (2\sqrt{\pi}a)^{-1}\delta_{s_1s_2}\delta_{l_1l_2}\delta_{n_1n_2}\delta(k_1-k_2)\,. \end{aligned} \tag{41}$$

In the Dirac theory, the vacuum state is defined as the state with filled up negative energy levels. This situation corresponds to:

$$a_{nls}|0\rangle = 0\,, \qquad b_{nls}|0\rangle = 0\,. \tag{42}$$

The Hamiltonian density of the field

$$\mathcal{H} = -\Psi^{+}\gamma^{0}(i\gamma^{i}\partial_{i} - e\gamma^{\mu}A_{\mu} - m)\Psi = i\Psi^{+}\partial_{t}\Psi \qquad (43)$$

(where in the second step we use the fact that Ψ fulfills the Dirac equation) leads to the vacuum energy density as follows:

$$\begin{aligned}\mathcal{E}_{\text{vac}}(a) &= \langle 0|\mathcal{H}|0\rangle \\ &= \tfrac{-1}{2\sqrt{\pi}a}\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\sum_{s}\int_{-\infty}^{\infty} dkE\big(\vec{\eta}^{*}_{nl1}(k,y)\vec{\eta}_{nl1}(k,y) + \vec{\eta}^{*}_{nl,-1}(k,y)\vec{\eta}_{nl,-1}(k,y)\big) \\ &= -eB(\sqrt{\pi}a)^{-1}\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}\alpha_n\sqrt{\tfrac{\pi^2}{a^2}(2l+1)^2 + 2eBn + m^2}\,, \qquad (44)\end{aligned}$$

where $\alpha_0 = 1/2$ and $\alpha_n = 1$ for $n > 0$.

The energy density (44) is infinite. Renormalized vacuum energy is the difference between energy density (44) and vacuum energy without boundary conditions, and is given by:

$$\begin{aligned}\mathcal{E}_{\text{ren}}(a) \;=\; -2eB(\sqrt{\pi}a)^{-1}\sum_{n=0}^{\infty}\alpha_n\Bigg(&\sum_{l=0}^{\infty}\sqrt{4\pi^2a^{-2}(l+1/2)^2 + 2eBn + m^2} \\ &-\int_0^{\infty} dl\sqrt{4\pi^2a^{-2}l^2 + 2eBn + m^2}\Bigg). \qquad (45)\end{aligned}$$

In order to calculate Eq. (45) we use Abel–Plana formula in the useful form:

$$\sum_{t=0}^{\infty}F(t+1/2) - \int_0^{\infty} dtF(t) = -i\int_0^{\infty}\frac{dt}{e^{2\pi t}+1}(F(it) - F(-it))\,, \qquad (46)$$

and finally we obtain renormalized energy density (in restored SI units) in the form:

$$\mathcal{E}_{\text{ren}}(a) = -8ceBa^{-2}\sum_{n=0}^{\infty}\alpha_n\int_{\lambda_n}^{\infty}\frac{dt}{e^{2\pi t}+1}\sqrt{t^2 - \lambda_n^2}\,, \qquad (47)$$

where $\lambda_n = a(2\pi)^{-1}\sqrt{2\hbar^{-1}eBn + m^2c^2\hbar^{-2}}$.

Eq. (47) may be written in the form:

$$\mathcal{E}_{\text{ren}}(a,B,m) = \mathcal{E}_0(a)\,f^F(\xi,\eta)\,, \qquad (48)$$

where

$$\mathcal{E}_0(a) = -7c\hbar\pi^2/(180a^4) \tag{49}$$

is the standard Casimir energy density for $B = 0$ for massless field [2] and

$$f^F(\xi, \eta) = \tfrac{1440}{7}\pi^{-2}\xi \sum_{n=0}^{\infty} \alpha_n \int_{\lambda_n}^{\infty} dt \sqrt{t^2 - \lambda_n^2}\,(e^{2\pi t} + 1)^{-1}, \tag{50}$$

where $\lambda_n = (2\pi)^{-1}\sqrt{2n\xi + \eta^2}$, $\xi = eBa^2\hbar^{-1}$ and $\eta = amc\hbar^{-1}$.

Numerical calculations of Eq. (47) are shown in Figs. 6,7. In contrast to the bosonic case, for the Dirac field the Casimir energy is enhanced by the external magnetic field (Fig. 6). So, the Dirac vacuum behaves like paramagnetic medium. Decrease of the Casimir energy with a distance is slower in strong magnetic field (Fig. 7).

Vacuum pressure is shown in Figs. 8–9. For massless field (and strong B) it is increasing linear function of magnetic field B (Fig. 8). For massive field, pressure depends on the field B very weakly — two curves in Fig. 9 (for $B = 1$ T and $B = 100$ T) coincide. For electron field values of Casimir pressure are measurable for distances smaller then 0.1 nm (Fig. 9). In practice it excludes possibilities of experimental tests. It can play a role only in subatomic distances, for example in some particle models.

Analogously as for bosons, we compared Eq. (47) with Eq. (9) from [4] (based on Schwinger's method) using *Mathematica 4.1*. We obtained the same numerical results. However, we are not able to prove analytically that these two equations are equivalent.

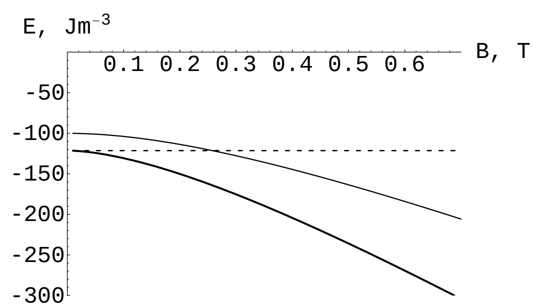

Fig. 6. Casimir energy density for the Dirac field as a function of magnetic field B, corresponding to distance $a = 1$ m. Bold curve corresponds to mass $m = 0$ whereas the thin one corresponds to mass $m = 2$ eV. Dashed line corresponds to the value equal to $-7\pi^2\hbar c a^{-4}/180$ (standard value of energy for $B = 0$).

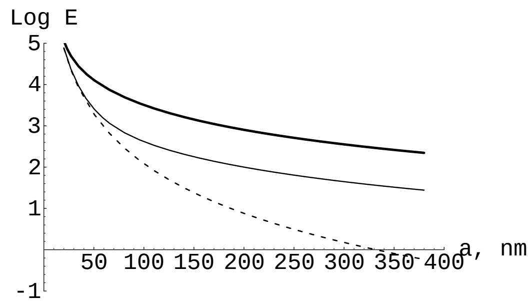

Fig. 7. Casimir energy density for the Dirac field as a function of distance a, for massless field $m = 0$. Bold curve corresponds to magnetic field $B = 8$ T whereas the thin one corresponds to $B = 1$ T. Dashed curve corresponds to function $-7\pi^2\hbar c a^{-4}/180$ (standard function of energy for $B = 0$). Values of energy are shown in logarithmic scale.

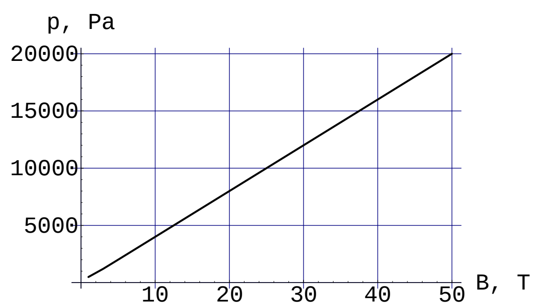

Fig. 8. Casimir pressure for massless, the Dirac field as a function of magnetic field B, corresponding to distance $a = 100$ nm. The diagram is made in SI units.

We examine the asymptotic behavior of energy density (48) using the same methods as in previous section (for bosons). For strong magnetic field ($\xi \to \infty$) and for $\eta \to \infty$ we use the relation $\exp(2\pi t) - 1 \to \exp(2\pi t)$, what gives:

$$\begin{aligned} f_\infty^F(\xi,\eta) &= \tfrac{1440}{7}\pi^{-2}\xi \sum_{n=0}^{\infty} \alpha_n \int_{\lambda_n}^{\infty} dt \sqrt{t^2 - \lambda_n^2}\, e^{-2\pi t} \\ &= \tfrac{360}{7}\pi^{-4}\xi \sum_{n=0}^{\infty} \alpha_n \sqrt{2\xi n + \eta^2}\, \mathrm{K}_1(\sqrt{2\xi n + \eta^2})\,. \end{aligned} \tag{51}$$

Using formula (51), we receive good numerical results (error $|f_\infty^F/f^F - 1| < 1\%$) for $\eta > 2.6$ and $\xi > 0.5$.

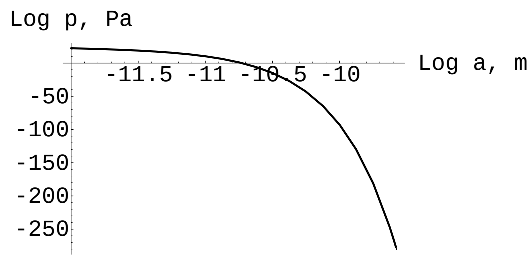

Fig. 9. Casimir pressure for the Dirac field of electrons ($m = m_e$) as a function of distance a. The diagram is made in SI units for two values of field $B = 1$ T and $B = 100$ T. Both curves coincide.

For weak magnetic field ($\xi \to 0$) we change sum into integration, which gives:

$$\begin{aligned} f_0^F(\xi,\eta) &= \tfrac{1440}{7}\pi^{-2}\xi \int\limits_0^\infty dn \int\limits_{\lambda(n)}^\infty dt\sqrt{t^2-\lambda^2(n)}\left(e^{2\pi t}+1\right)^{-1} \\ &= \tfrac{360}{7}\pi^{-4}\int\limits_1^\infty dy\sqrt{y^2-1}\int\limits_\eta^\infty dx\, x^3(e^{xy}+1)^{-1}\,. \end{aligned} \tag{52}$$

Unfortunately, this method is not suitable in this case. It is "too brutal" and we lose the ξ parameter in the expression (52). It suggests that $\mathcal{E}_{B\to 0} = \mathcal{E}_{(B=0)} + \mathcal{O}(B^2)$. In Fig. 6 we can see that both curves ale locally horizontal for $B \to 0$.

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