APPROXIMATIONS OF THE SPECTRAL FUNCTION*

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The ICARUS and future liquid argon neutrino experiments generate demand for evaluating the spectral function of argon. In this paper we use oxygen nucleus as a testing ground for our phenomenological approach to the spectral function and probe the influence of momentum distribution and treatment of the mean field spectral function on the differential cross sections. The obtained model reproduces very well results of the exact spectral function of oxygen and can be applied to heavier nuclei, such as calcium or argon.

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1. Motivation and outline of the paper

Thanks to experience already gained with the ICARUS T600 TPC [1] one knows that liquid argon (LAr) has many advantages in neutrino experiments. They make LAr technology interesting for planned detectors, *e.g.* for T2K [2] and NuMI [3]. To use them fully, one has to reduce uncertainties by evaluating nuclear effects as precise as possible. From $(e,\, e')$ scattering it is clear that the Fermi gas model can only be the first approximation.

More elaborate approach is the spectral function (SF) formalism. Where the impulse approximation is valid (*i.e.* when neutrino energy E_ν is greater then a few hundred of MeV [4]), the SF describes nuclei most accurately. It was applied to $(l,\, l')$ scattering previously [5]. The problem is that the exact SFs exist only for a few double closed shell nuclei, ^{3}He, and nuclear matter [6–8]. For argon $^{40}_{18}$Ar exact computations cannot be performed, so one is forced to seek for the best possible approximation. Opportunity to verify the quality of given approximation is provided by oxygen nucleus,

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where the exact SF exists [7]: applying the same method to calculation of the oxygen SF and comparing the result to the exact SF can give a notion of discrepancies between them.

The first attempt at constructing the argon SF is Ref. [9]. Presented there approximation gives the differential cross sections which somewhat differ from the corresponding ones for the exact SF (compare Fig. 1). In the paper presumption was made that the discrepancies come from:

— oversimplified treatment of the mean field spectral function,

— different momentum distributions in the spectral functions.

This work is devoted to detailed study of the two effects. After a brief introduction into the SF approach in Sec. 2 and describing the simplest approximation of the SF in Sec. 3 (*i.e.* the approach of [9]), we consider in Sec. 4 an influence of NN-correlations on the mean field SF. Then in Sec. 5 the sensitivity of the SF on the momentum distribution is discussed.

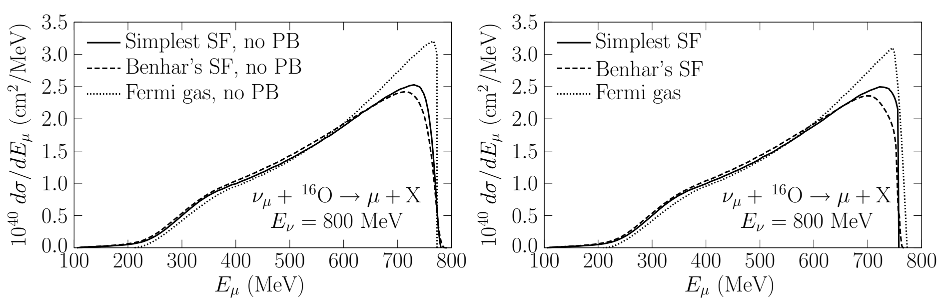

Fig. 1. Differential cross section $d\sigma/dE_\mu$ of quasielastic ν_μ scattering off ^{16}O obtained from the Fermi gas model (dotted line), the Benhar's spectral function (dashed line) and the simplest approximation of the spectral function [9] (solid line). Left: Pauli blocking is absent. Right: Pauli blocking included.

Conclusions allowed for working out in Sec. 6 a satisfactory approximation of the exact spectral function of oxygen, which can be applied to argon and other nuclei.

2. Basic information on the spectral function

The spectral function (SF) of a given nucleus $P(\boldsymbol{p}, E)$ is the probability distribution of finding a nucleon with momentum $\boldsymbol{p}$ and removal energy E. Formal definition can be expressed as [10]

$$P(\boldsymbol{p}, E) \equiv \langle i(M_A)|a^\dagger(\boldsymbol{p})\delta(\hat{H} - M_A + M - E)a(\boldsymbol{p})|i(M_A)\rangle, \tag{1}$$

where $\hat{H}$ is the intrinsic Hamiltonian of the $(A-1)$-nucleon system, $|i(M_A)\rangle$ denotes the state of the initial nucleus of mass M_A (assumed to be at rest), and M is nucleon mass.

Oxygen nucleus is isoscalar ($N = Z$), therefore its SF consists of only two parts: the mean field and the short-range correlation one:

$$P(\boldsymbol{p}, E) = \frac{N+Z}{2} \left[P_{\mathrm{MF}}(\boldsymbol{p}, E) + P_{\mathrm{corr}}(\boldsymbol{p}, E) \right]. \tag{2}$$

All the knowledge about low-energy and low-momentum nucleons (*i.e.* the shell model information) is included in the mean field term $P_{\mathrm{MF}}(\boldsymbol{p}, E)$. Corrections to the independent-particle behavior of nucleons are described by the correlation part $P_{\mathrm{corr}}(\boldsymbol{p}, E)$. From the nuclear matter calculations one knows that at high energy and momentum such correlations are dominated by the two-nucleon interactions [11].

The mean field SF consists of contributions of every shell model state α below the Fermi level α_{F} [12]:

$$P_{\mathrm{MF}}(\boldsymbol{p}, E) \equiv \frac{1}{A} \sum_{\alpha < \alpha_{\mathrm{F}}} c_\alpha A_\alpha |\phi_\alpha(\boldsymbol{p})|^2 \delta\big(E_\alpha + E_{\mathrm{R}}(\boldsymbol{p}) - E\big). \tag{3}$$

In the above equation E_α is the energy of the state α described by the single-nucleon wave function $\phi_\alpha(\boldsymbol{p})$ (normalized to 1) with the occupation probability c_α, and the number of particles A_α, whereas $E_{\mathrm{R}}(\boldsymbol{p}) = \boldsymbol{p}^2/(2M_{A-1})$ is the recoil energy of the residual nucleus.

Interactions between nucleons cause their partial redistribution from the states below the Fermi level to the levels of higher energy, and occurrence of non-zero width of each level. It means that when NN-correlations in the MF part are taken into account, delta function in Eq. (3) should be replaced by appropriate distribution [7]. We shall return to this issue in Sec. 4.

Due to Eq. (1) the momentum distribution of nucleons reads

$$n(\boldsymbol{p}) \equiv \langle i(M_A) | a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) | i(M_A) \rangle = \int P(\boldsymbol{p}, E)\, dE \tag{4}$$

and as a consequence of Eq. (2) consists of two contributions [12]:

$$n(\boldsymbol{p}) = \frac{N+Z}{2} \left[n_{\mathrm{MF}}(\boldsymbol{p}) + n_{\mathrm{corr}}(\boldsymbol{p}) \right]. \tag{5}$$

3. Simplest approximation of the SF

Firstly note that Eq. (3) simplifies significantly when one replaces E_α in the argument of the delta function by the average separation energy $E^{(1)} = \sum_{\alpha<\alpha_{\rm F}} c_\alpha E_\alpha A_\alpha / A$ [14]:

$$P_{\rm MF}^{\rm SSF}(\boldsymbol{p}, E) = n_{\rm MF}(\boldsymbol{p})\,\delta\big(E^{(1)} + E_{\rm R}(\boldsymbol{p}) - E\big)\,, \tag{6}$$

because then only the momentum distribution

$$n_{\rm MF}(\boldsymbol{p}) = \sum_{\alpha<\alpha_{\rm F}} c_\alpha |\phi_\alpha(\boldsymbol{p})|^2 \frac{A_\alpha}{A} \tag{7}$$

occurs, which is averaged over single-particle levels.

Secondly, as it was mentioned in the previous section, major contribution to the correlations between nucleons comes from the two-nucleon process. It consists in forming a cluster by two nucleons with high relative momentum while the other $(A-2)$ nucleons remain soft [13]. When we restrict ourself to such an interactions, the correlation SF can be expressed as [14]:

$$P_{\rm corr}(\boldsymbol{p}, E) = n_{\rm corr}(\boldsymbol{p}) \frac{M}{|\boldsymbol{p}|}\sqrt{\frac{\alpha}{\pi}}\left[\exp(-\alpha \boldsymbol{p}_{\rm min}^2) - \exp(-\alpha \boldsymbol{p}_{\rm max}^2)\right]. \tag{8}$$

Occurring here $\alpha = 3/(4\langle \boldsymbol{p}^2\rangle \beta)$ is inversely proportional to the mean value of the MF momentum squared $\langle \boldsymbol{p}^2\rangle$ times $\beta = (A-2)/(A-1)$ and

$$\begin{aligned} \boldsymbol{p}_{\rm min}^2 &= \left[\beta|\boldsymbol{p}| - \sqrt{2M\beta[E - E^{(2)} - E_{\rm R}(\boldsymbol{p})]}\right]^2, \\ \boldsymbol{p}_{\rm max}^2 &= \left[\beta|\boldsymbol{p}| + \sqrt{2M\beta[E - E^{(2)} - E_{\rm R}(\boldsymbol{p})]}\right]^2. \end{aligned} \tag{9}$$

The threshold value $E^{(2)}$ is interpreted as the two-nucleon separation energy averaged over low-energy configurations of the $(A-2)$-nucleon system and can be approximated by $E^{(2)} = M_{A-2} + 2M - M_A$.

We will refer to the described approach as "the simplest approximation of the spectral function" or "the simplest spectral function" (SSF).

Figure 1 illustrates that the cross section $d\sigma/dE_\mu$ of both SFs clearly differs from the one of the Fermi gas. As long as Pauli blocking (PB) is not included, the SSF reproduces the result of the Benhar's (*i.e.* exact) SF quite well. Presence of PB enlarges difference between them, because it changes the shape of the peak in other way.

In the next section we investigate whether more elaborate approximation of the mean field SF removes this discrepancy.

4. Improved treatment of the mean field SF

In Sec. 2 we mentioned that NN-correlations broaden energy levels, so in Eq. (3) instead of delta function an appropriate distribution should occur [7]:

$$P_{\mathrm{MF}}(\boldsymbol{p}, E) \equiv \frac{1}{A} \sum_{\alpha<\alpha_F} c_\alpha A_\alpha |\phi_\alpha(\boldsymbol{p})|^2 F_\alpha\big(E_\alpha + E_{\mathrm{R}}(\boldsymbol{p}) - E\big) \,. \tag{10}$$

In order to simplify it, let us replace the single-particle momentum distribution $|\phi_\alpha(\boldsymbol{p})|^2$ by the average $n_{\mathrm{MF}}(\boldsymbol{p})$ defined in Eq. (7):

$$P_{\mathrm{MF}}(\boldsymbol{p}, E) = n_{\mathrm{MF}}(\boldsymbol{p}) \frac{1}{A} \sum_{\alpha<\alpha_F} c_\alpha A_\alpha F_\alpha(E_\alpha + E_{\mathrm{R}}(\boldsymbol{p}) - E) \,. \tag{11}$$

We decided to use the Gaussian distribution

$$F_\alpha(x) = \frac{1}{\sqrt{\pi} D_\alpha} \exp\left[-\left(\frac{x}{D_\alpha}\right)^2\right] . \tag{12}$$

To describe a level α we need to now its diffuseness D_α and its "mean field" occupation probability c_α. If one used the "raw" c_α's (given for example in [15]), partial double counting would occur due to contribution of the correlation term of the SF (raw value says only that the nucleon can be found in given state with certain probability, not how it get there: whether it happened due to "natural placement" or due to "correlation kick").

For ^{16}O we shall determine the value of D_α from the Benhar's SF, for other nuclei one should get them from direct calculations (*e.g.* for $^{40}_{20}$Ca the data is available in [16]). Oxygen is a quite heavy nucleus in the sense that its recoil energy for the mean value of momentum $|\boldsymbol{p}| = 180$ MeV/c [9] is ~ 1 MeV. When we neglect it, the separation

$$P_{\mathrm{MF}}(\boldsymbol{p}, E) = n_{\mathrm{MF}}(\boldsymbol{p}) S_{\mathrm{MF}}(E) \,. \tag{13}$$

holds, and we easily get that

$$S_{\mathrm{MF}}(E) \propto \int P_{\mathrm{MF}}(\boldsymbol{p}, E)\, d^3p = \int \left[P(\boldsymbol{p}, E) - P_{\mathrm{corr}}(\boldsymbol{p}, E)\right] \, d^3p. \tag{14}$$

We inserted in the above relation the Benhar's SF and calculated the energy distribution $S_{\mathrm{MF}}(E)$. By fitting Eq. (12) successively to every peak corresponding to the energy level, we determined all D_α's, see Figure 2.

Note that $S_{\mathrm{MF}}(E)$ shown in Fig. 2 vanishes for $E \gtrsim 87$ MeV. For oxygen $P_{\mathrm{corr}}(\boldsymbol{p}, E)$ appears above the threshold energy $E^{(2)} = 26.33$ MeV higher then the energy of quite sharp levels $1p\frac{3}{2}$ and $1p\frac{1}{2}$, hence the correlations

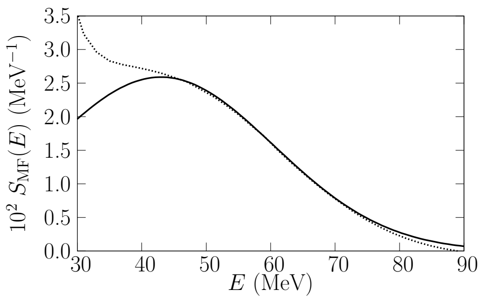

Fig. 2. Our fit of the Gaussian distribution of the level $1s\frac{1}{2}$ (solid line) to appropriate region of the energy distribution $S_{\mathrm{MF}}(E)$ calculated from the Benhar's spectral function of ^{16}O (dotted line).

admixes only to c_α of $1s\frac{1}{2}$ level. After subtracting calculated value of $\int S_{\mathrm{corr}}(E)dE$ over $E \in [26.33; 87]$ MeV from c_α of $1s\frac{1}{2}$, we get the proper "mean field" value. Now we have all the indispensable parameters to obtain predictions for the model.

The cross sections for this approximation are shown in Fig. 3: the momentum distribution is the same as in the simplest SF and so is the correlation part of the SF, the only change is the different treatment of the mean field SF. This approach causes that the shape of the cross section is similar to one of the exact SF, regardless of including PB or not. The height of the peak is also slightly reduced, therefore the agreement between the approximation and the Benhar's SF is now better.

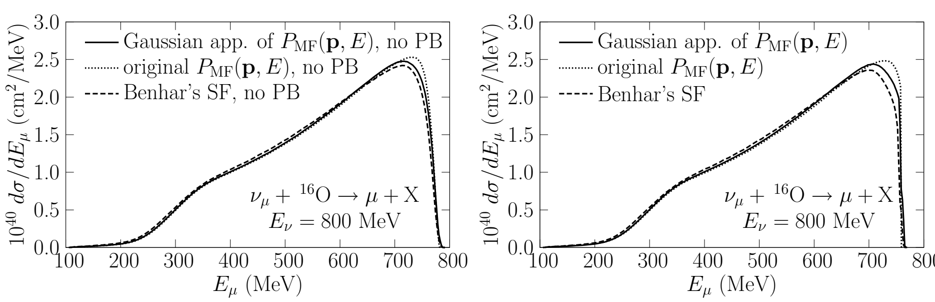

Fig. 3. Sensitivity of the quasielastic differential cross section $d\sigma/dE_\mu$ on treatment of the mean field spectral function. The simplest (dotted line) and the Gaussian approximation (solid line) compared to the Benhar's spectral function. Left: Results without Pauli blocking. Right: Pauli blocking included.

5. Dependence on the momentum distribution

To study how the difference between the momentum distributions affects difference between the cross sections, we divided the Benhar's distribution into MF and correlated part as in Fig. 4: we kept the same value of $n_{\mathrm{corr}}(\boldsymbol{p})$ for $\boldsymbol{p} = 0$ and took care of its smooth transition into the original distribution for high values of $\boldsymbol{p}$. The calculated components of the total distribution were applied to the simplest approximation of the spectral function (SSF).

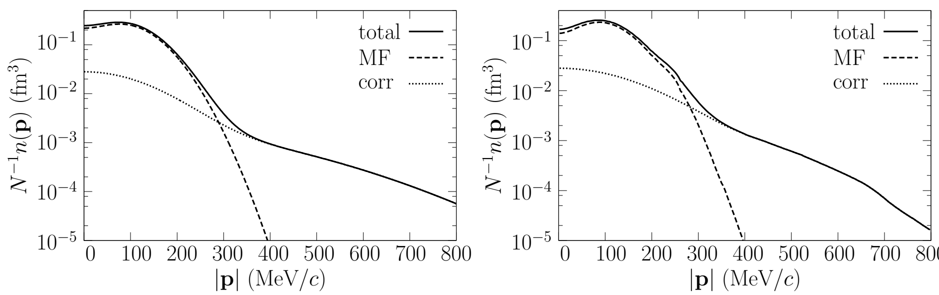

Fig. 4. Contributions to the momentum distribution of nucleons: the mean field (dashed) and correlation part (dotted line) sum up to the total momentum distribution (solid line). Left: Distribution from [12]. Right: Corresponding one calculated from the Benhar's SF, by analogy divided into two parts (details in text).

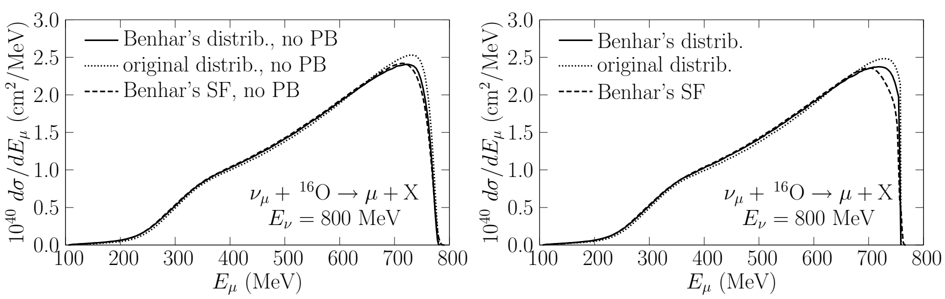

Fig. 5. Sensitivity of the quasielastic differential cross section $d\sigma/dE_\mu$ on applied momentum distribution. The simplest approximation with the momentum distribution used previously [9] (from [12]; dotted line) and with "the Benhar's distribution" (see Fig. 4; solid line) compared to the Benhar's spectral function (dashed line). Left: Without Pauli blocking. Right: With Pauli blocking included.

As it follows from Fig. 5, nearly whole discrepancy between the cross section for SSF obtained in this way and the corresponding one for the Benhar's SF disappeared when Pauli blocking (PB) is absent. Inclusion of PB reveals different behavior of the two approaches.

We conclude that the momentum distribution is responsible for the height of the peak, but not for its shape (compare solid and dotted line in Fig. 5). Therefore the SSF diverges from the exact SF when PB is present.

Another important information is the high sensitivity of $d\sigma/dE_\mu$ on $n(\boldsymbol{p})$. This feature is quite inconvenient, because it suggests that one should take into account the difference between momentum distribution of protons and neutrons, if appropriate prediction of $d\sigma/dE_\mu$ for high E_μ is required.

6. Combination of the two effects

As we have seen in Sec. 4, the enhanced treatment of the mean field SF affected the shape of the cross section's peak, but it was a bit too high, whereas the change of the momentum distribution in Sec. 5 reduced the hump's height, however its shape disagreed. When we combine the two improvements, using both "the Benhar's momentum distribution" and the Gaussian mean field SF, discrepancy vanishes almost completely, see Fig. 6.

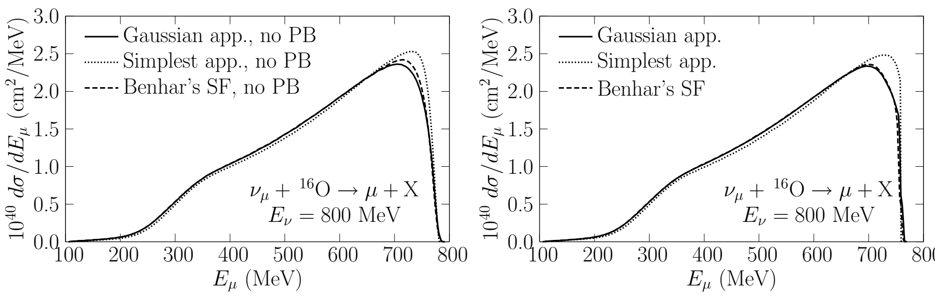

Fig. 6. Gaussian approximation with "the Benhar's distribution" (solid line) is significantly better then the simplest approximation with the original distribution (dotted line) and nicely reproduces results of the Benhar's spectral function (dashed line). Left: Without Pauli blocking. Right: With Pauli blocking.

We conclude that the remarks made in [9] were right: the cross section of the Benhar's SF can be satisfactory reproduced when one uses the same momentum distribution and the refined treatment of the mean field SF.

Our approximation remains simple enough and can be applied to other nuclei. The next goal will be ${}^{40}_{20}$Ca where the needed nuclear data is available.

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