THE TUNNEL-EFFECT IN THE LOBACHEVSKY SPACE

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The problem of the tunnel-effect in the three-dimensional Lobachevsky space is formulated and solved. It is shown that the tunneling probability essentially decreases when radius of the space curvature is of the same order as linear sizes of the well in which a particle is locked.

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1. Introduction

It is well known, that processes using the mechanism of quantum tunneling are abundant in different areas of physics. The tunnel-effect can be met in various physical processes, such as chemical reactions at low temperatures, nuclear alpha and cluster decays, fission and fusion processes in thermonuclear reactors, shortly after the Big Bang and inside stellar matter that proceed through the Coulomb barrier.

However, as far as it is known to the authors, the problem of the tunnel-effect in spaces of constant curvature was not investigated.

Quantum-mechanical problems in the spaces of constant positive and negative curvature have been the object of interest of researchers since 1940, when Schrödinger [1] was first to solve the quantum-mechanical problem of the hydrogen atom on the three-dimensional sphere S_3 (Einstein's universe). The analogous problem in the three-dimensional Lobachevsky space $^1S_3$ was first solved by Infeld and Shild [2]. In recent years the quantum-mechanical problems in the spaces of constant curvature have attracted considerable attention due to their interesting mathematical features as well as the possibility of applications to physical problems. These problems are discussed in the monograph [3], reviews [4–6] and articles [7–12]. Thus, the quantum-mechanical models based on the geometry of spaces of constant curvature

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are used for the description of the phenomena in nuclear physics [13–15], particle physics [16] and nanostructure physics [18–21].

In this paper the tunnel-effect in the three-dimensional Lobachevsky space is considered. The dependence of tunneling on the space curvature is investigated. In conclusion the model of Lobachevsky's space inside the sphere of the three-dimensional space of Euclid is given. This model can be used as a model for description of the states in quantum dots with the discrete and continuous spectrum which are localized in the three-dimensional Euclidean space.

2. The quantum-mechanical tunneling in the Lobachevsky space

Let us consider a freely moving particle which, at a given time, meets a potential barrier higher than its energy. As it is known, quantum mechanics implies a non-vanishing probability for the particle to cross the barrier (*i.e.* the tunnel-effect).

We use embedding of the Lobachevsky space in four-dimensional pseudo-Euclidean space with Cartesian coordinates x_μ, $\mu = 1, 2, 3, 4$, given by the formula

$$\begin{aligned} x_\mu x_\mu &= \boldsymbol{x}^2 + x_4^2 = \boldsymbol{x}^2 - x_0^2 = -\rho^2 \,, \\ \boldsymbol{x} &= \{x_1, x_2, x_3\} \,, \qquad x_4 = i x_0 \,, \end{aligned} \tag{1}$$

where ρ denotes the radius of curvature.

We introduce the spherical coordinates in the Lobachevsky space as [2]

$$\begin{aligned} x_0 &= \rho\cosh\beta \,, \qquad\qquad x_1 = \rho\sinh\beta \sin\theta \cos\varphi, \\ x_2 &= \rho\sinh\beta \sin\theta \sin\varphi \,, \qquad x_3 = \rho\sinh\beta \cos\theta \,, \\ & 0 \le \beta < \infty \,, \quad 0 \le \theta \le \pi \,, \quad 0 \le \varphi < 2\pi \,. \end{aligned} \tag{2}$$

Metric of the Lobachevsky space in these coordinates is given by

$$dl^2 = \rho^2 \left(d\beta^2 + \sinh^2\beta \; d\theta^2 + \sinh^2\beta \; \sin^2\theta \; d\varphi^2 \right) . \tag{3}$$

The Laplace–Beltrami operator is

$$\varDelta_{\mathrm{LB}} = \frac{1}{\rho^2} \left(\frac{1}{\sinh^2\beta} \frac{\partial}{\partial\beta} \sinh^2\beta \frac{\partial}{\partial\beta} - \frac{1}{\sinh^2\beta} L^2 \right) , \tag{4}$$

where L^2 is the squared angular momentum operator in spherical coordinates

$$L^2 = -\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} \frac{\partial}{\partial\varphi^2} \right) . \tag{5}$$

Then the Schrödinger equation for motion of a particle in the centrally symmetric field $U(\beta)$ in the three-dimensional Lobachevsky space is

$$H\Psi = E\Psi\,, \tag{6}$$

where

$$H = -\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}} + U(\beta)\,. \tag{7}$$

Now we shall analyze the tunneling of a particle through a potential barrier in the three-dimensional Lobachevsky space. Let us assign the potential of the barrier as follows

$$U(\beta) = \begin{cases} U_{\mathrm{C}} = \frac{\alpha \coth\beta}{\rho}\,, & \text{for} \quad \beta \geq \beta_0\,, \\ 0\,, & \text{for} \quad \beta \leq \beta_0\,, \end{cases} \tag{8}$$

where U_{C} is the Coulomb potential in the Lobachevsky space.

We note that this potential differs from the potential originally used by Infeld and Schild [2] by a constant term α/ρ and does not vanish as $\beta \longrightarrow \infty$. This choice was motivated by the wish to retain similarity of expressions between the cases of positive and negative curvature.

The Coulomb barrier height can be estimated as

$$B = \frac{\alpha \coth\beta_0}{\rho}\,. \tag{9}$$

Separating in the solution of the Schrödinger equation the dependence on the angles θ and φ by using spherical harmonics, namely $\Psi = S_l(\beta)Y_l^m(\theta,\varphi)$ we obtain the radial equation as follows

$$\left[\frac{\hbar^2}{2m\rho^2}\left(-\frac{1}{\sinh^2\beta}\frac{d}{d\beta}\sinh^2\beta\frac{d}{d\beta} + \frac{l(l+1)}{\sinh^2\beta}\right) - E + U(\beta)\right] S_l(\beta) = 0\,. \tag{10}$$

Using the substitution $S(\beta) = f(\beta)/\sinh\beta$ we have

$$\frac{d^2 f(\beta)}{d\beta^2} + \left[\frac{2m\rho^2}{\hbar^2}\{E - U(\beta)\} - \frac{l(l+1)}{\sinh^2\beta} - 1\right] f(\beta) = 0\,. \tag{11}$$

Then introducing the effective potential

$$V(\beta) = U(\beta) + \frac{\hbar^2}{2m}\frac{l(l+1)}{\rho^2\sinh^2\beta} + \frac{\hbar^2}{2m\rho^2}\,, \tag{12}$$

equation (11) can be rewritten in the form

$$\frac{d^2 f(\beta)}{d\beta^2} + \left[\frac{2m\rho^2}{\hbar^2}\{E - V(\beta)\}\right] f(\beta) = 0\,. \tag{13}$$

As a result the centrally symmetric problem was reduced to one-dimensional one. Thus, the barrier penetration factor in the WKB semiclassical approximation is

$$G = \left\{ -2 \int_{\beta_0}^{\beta_1} \sqrt{\frac{2m\rho^2}{\hbar^2}[V(\beta) - E]}\, d\beta \right\}, \tag{14}$$

where β_0 and β_1 are the two turning points.

The condition of applicability of the semiclassical approximation in our case has the form

$$\frac{(\sinh\beta_1 - \sinh\beta_0)\rho}{\lambda} \gg 1\,, \tag{15}$$

i.e. the barrier width is significantly larger than the de Broglie wavelength.

As a result in the WKB approximation for the tunneling probability in the Lobachevsky space we have

$$D \sim \exp\left\{ -2 \int_{\beta_0}^{\beta_1} \rho \sqrt{\frac{2m}{\hbar^2}\left[\frac{\alpha\coth\beta}{\rho} + \frac{\hbar^2}{2m}\frac{l(l+1)}{\rho^2\sinh^2\beta} + \frac{\hbar^2}{2m\rho^2} - E\right]}\, d\beta \right\}. \tag{16}$$

We note that in the limit of the flat space, *i.e.* when $\rho \to \infty$, the expression (16) transforms to the barrier penetration factor in the flat space

$$D \sim \exp\left\{ -2 \int_{r_0}^{r_1} \sqrt{\frac{2m}{\hbar^2}\left[\frac{\alpha}{r} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - E\right]}\, dr \right\}. \tag{17}$$

From (16) we can see that with the growth of angular momentum l the probability of tunnel-effect in the Lobachevsky space decreases as in the flat space.

In the case when the particle has nonzero orbital angular momentum, it possesses the centrifugal energy

$$U_l = \frac{\hbar^2}{2m\rho^2}\,\frac{l(l+1)}{\sinh^2\beta}\,. \tag{18}$$

This energy is added to the Coulomb energy U_{C} and thus it increases the potential barrier. However, as can be seen from comparison (16) and (17) there is a difference from analogous problem in the flat space. In the case of the Lobachevsky space that it appears additional constant is proportional to the curvature of the space,

$$U_\rho = \frac{\hbar^2}{2m\rho^2}\,, \tag{19}$$

which is not present in the Hamiltonian in the flat space. This constant appears when we get rid of first-order derivative using the replacement $S(\beta) = f(\beta)/\sinh\beta$ in equation (10). We note that this constant can be removed after redefinition of the Hamiltonian as follows

$$H = -\frac{\hbar^2}{2m}\varDelta_{\rm LB} - \frac{\hbar^2}{2m\rho^2}\,. \tag{20}$$

3. The α-decay in the three-dimensional Lobachevsky space

Let us consider the phenomenon of α-decay in the Lobachevsky space. In the phenomenon of α-decay there should be distinguished two stages:

1. The formation of α-particle inside the nucleus.
2. The nucleus-decay via α emission.

We will examine idealization when parent nucleus consists of the daughter nucleus with charge Ze and α-particle. To emerge from the nucleus α-particle should penetrate through the potential barrier (8), where $\alpha = 2Ze^2$.

As a result from (16) in case when $l = 0$ for the probability of α-decay in the three-dimensional Lobachevsky space we have

$$D \sim \exp\left\{-2\int\limits_{\beta_0}^{\beta_1}\rho\sqrt{\frac{2m}{\hbar^2}\left[\frac{2Ze^2\coth\beta}{\rho}+\frac{\hbar^2}{2m\rho^2}-E\right]}d\beta\right\}. \tag{21}$$

Let us illustrate α-decay with the concrete example. Namely, we estimate the tunneling probability for ^{238}U. Let $Z = 92$ and $\beta_0 = \operatorname{arcsinh}(r_0/\rho)$, where $r_0 = 0.10^{-11}$cm is the radius of the daughter nucleus in the flat space and turning point β_1 is

$$\beta_1 = \operatorname{arccth}\left(\frac{E\rho - \hbar^2/2m\rho}{2Ze^2}\right).$$

Now for s-states we find the value of G. In the case of the flat space when $l = 0$ the fist energy level for our example is $E_1 = 4.008\,\mathrm{MeV}$. This energy corresponds to the value $G = -94.478$. In case of the Lobachevsky space we have the following equation for the energy levels of the system [22]

$$\sqrt{k^2-1}\cot(\beta_0\sqrt{k^2-1}) = -\sqrt{\lambda^2+1}\,, \tag{22}$$

where

$$\lambda = \sqrt{\frac{2m\rho^2}{\hbar^2}E}\,, \qquad k = \sqrt{\frac{2m\rho^2}{\hbar^2}(B-E)}\,. \tag{23}$$

We use equations (21), (22) to find the values of the first energy level E_1 and the tunneling probability D for various values of the curvature radius.

These data are shown in Table I. We see from Fig. 1 and Table I that the tunneling probability slump when radius of the space curvature is of the same order as linear sizes of the system.

TABLE I

The values of the first energy level of the system in the case of $l = 0$ and tunneling probability depending on the radius of curvature.

ρ [cm]	E [MeV]	G
$\rho = 0.105 \times 10^{-11}$	$E = 1.617$	$G = -737.692$
$\rho = 0.11 \times 10^{-11}$	$E = 1.948$	$G = -585.275$
$\rho = 0.12 \times 10^{-11}$	$E = 2.340$	$G = -450.361$
$\rho = 0.13 \times 10^{-11}$	$E = 2.595$	$G = -379.435$
$\rho = 0.14 \times 10^{-11}$	$E = 2.784$	$G = -332.963$
$\rho = 0.15 \times 10^{-11}$	$E = 2.933$	$G = -299.398$
$\rho = 0.16 \times 10^{-11}$	$E = 3.055$	$G = -273.767$
$\rho = 0.17 \times 10^{-11}$	$E = 3.15$	$G = -253.465$
$\rho = 0.18 \times 10^{-11}$	$E = 3.241$	$G = -236.953$
$\rho = 0.19 \times 10^{-11}$	$E = 3.314$	$G = -223.252$
$\rho = 0.2 \times 10^{-11}$	$E = 3.377$	$G = -211.700$
$\rho = 0.25 \times 10^{-11}$	$E = 3.593$	$G = -173.550$
$\rho = 0.5 \times 10^{-11}$	$E = 3.899$	$G = -119.086$
$\rho = 0.1 \times 10^{-10}$	$E = 3.980$	$G = -101.962$
$\rho = 0.15 \times 10^{-10}$	$E = 3.995$	$G = -98.104$
$\rho = 0.5 \times 10^{-10}$	$E = 4.007$	$G = -94.855$
$\rho = 0.1 \times 10^{-5}$	$E = 4.008$	$G = -94.478$

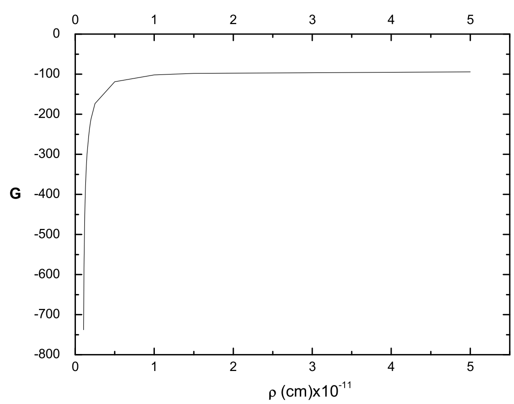

Fig. 1. The graph of the dependence G on ρ.

Note that in this computing the condition (15) is fulfilled.

In conclusion, the model of Lobachevsky's space in the sphere of the three-dimensional space of Euclid is given. This model can be used as a model for description of the quantum-mechanical system with the discrete and continuous spectrum which is localized in the three-dimensional Euclidean space, such as, quantum dots, nuclei, molecules and so on.

4. The model of the three-dimensional Lobachevsky space inside the sphere of the three-dimensional Euclidean space

We realize the model of the three-dimensional space of Lobachevsky inside the sphere in three-dimensional Euclidean space as follows. Let us assume that in the three-dimensional Euclidean space there is a sphere with the center at coordinate origin and $r < \rho$, where $\boldsymbol{r} = \{x, y, z\}$ is the radius-vector of points inside the sphere (see Fig. 2).

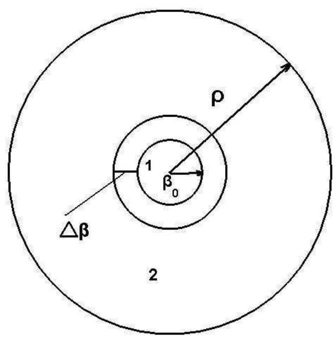

Fig. 2. The model of the three-dimensional Lobachevsky space inside the sphere of the three-dimensional Euclidean space. 1 — is the region of discrete spectrum, 2 — is the region of continuous spectrum, $\Delta\beta$ — is the potential barrier width.

Let us carry out the transformation

$$\begin{aligned} \boldsymbol{r} &= \{x, y, z\} \to \left\{ \frac{\boldsymbol{r}}{\sqrt{1 - \frac{r^2}{\rho^2}}}, \frac{\rho}{\sqrt{1 - \frac{r^2}{\rho^2}}} \right\} = \{\boldsymbol{x}, x_0\}\,, \\ \boldsymbol{r} &= \rho\,\frac{\boldsymbol{x}}{x_0}\,, \end{aligned} \tag{24}$$

where it is obvious that the introduced four-dimensional coordinates coincide with initial Cartesian coordinates x_μ and satisfy the condition (1), since these four-dimensional coordinates are independent. Therefore, we can use coordinates of the three-dimensional dean subspace $\boldsymbol{x} = \{x1, x2, x3\}$.

Then Laplace–Beltrami operator for three-dimensional Lobachevsky space is rewritten as

$$\Delta_{\mathrm{LB}} = \rho^2\nabla^2 - (\boldsymbol{x}\nabla)^2 - 2\,(\boldsymbol{x}\nabla)\ . \tag{25}$$

So the Hamiltonian of the free particle is

$$\begin{aligned} H &= -\frac{\hbar^2}{2m\rho^2}\Delta_{\mathrm{LB}} = -\frac{\hbar^2}{2m\rho^2}\left[\rho^2\nabla^2 - (\boldsymbol{x}\nabla)^2 - 2\,(\boldsymbol{x}\nabla)\right] \\ &= H_0 + V(x,\rho)\,, \end{aligned} \tag{26}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 \tag{27}$$

is the Hamiltonian of the free particle in the three-dimensional Euclidean space and the term

$$V(x,\rho) = \frac{\hbar^2}{2m\rho^2}\left[(\boldsymbol{x}\nabla)^2 + 2\,(\boldsymbol{x}\nabla)\right] \tag{28}$$

may be considered as certain quasi-potential in the three-dimensional flat space.

Thus, the exactly-solvable quantum-mechanical models in the spaces of constant curvature expand the range of the exactly-solvable problems in nonrelativistic quantum mechanics which can be used for describing the physical systems in the flat space.

5. Conclusion

In this paper the quantum-mechanical problem of the tunnel-effect in the three-dimensional Lobachevsky space is formulated and solved. It is demonstrated that the effect of the curvature gives rise to the first energy level shift. So, this effect can reduce to particles blocking (see Table I). The model of Lobachevsky's space, presented in Sec. 4, can be used as a model for description of the quantum-mechanical system with the discrete and continuous spectrum which is localized in the three-dimensional Euclidean space, such as for example, quantum dots, nuclei, molecules.

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