ROLE OF COLORED CROSS-CORRELATION IN ADDITIVE AND MULTIPLICATIVE WHITE NOISES ON UPPER BOUND OF TIME DERIVATIVE OF INFORMATION ENTROPY

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In this paper we have studied upper bound of time derivative of information entropy for colored cross-correlated noise driven open systems. The upper bound is calculated based on the Fokker–Planck equation and the Schwartz inequality principle. Our results consider the effect of the noise correlation strength and correlation time due to the correlation between additive and multiplicative white noises on the upper bound as well as relaxation time. The interplay of deterministic and random forces reveals extremal nature of the upper bound and its deviation from the time derivative of information entropy.

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The study of dynamical system subject to a noise perturbation has become a recurrent theme in physics, chemistry and biology, as well as in several other areas [1–11]. In this paper we have investigated relaxation behavior of the noise driven dynamical system. Although in traditional classical thermodynamics the specific nature of stochastic process is irrelevant, it plays an important role on the way to equilibration of a given non-equilibrium state of a noise driven system. An appropriate tool for the study of stationary and non-stationary states [9–15] in stochastic processes is Shanon's information measure [16–17]

$$S = -\int W(q,t)\ln W(q,t)dq \tag{1}$$

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which typically is not a conserved quantity. Here $W(q,t)$ is the continuous probability distribution function in phase space. S as defined in the above equation is called information entropy. If one considers Boltzmann's constant as the information unit and identifies Shannon's measure with the thermodynamic entropy, then the whole of statistical mechanics can be elegantly reformulated by extremization of S, subject to the constraints imposed by the *a priori* information one may possess concerning the system of interest [16–17]. The time evolution of S considers mainly the signature of the rate of phase space expansion and contraction in the stochastic process. This implies that the specific nature of the random process has a strong role to play with S. Generally we consider that the noise driven system is thermodynamically closed, which means that the noise of the medium is of internal origin so that the dissipation and fluctuation get related through the fluctuation–dissipation relation. However, in a number of situations the system is thermodynamically open, *i.e.*, the dissipation and the random force are *not* related through fluctuation–dissipation relation [18]. In general, origins of the noise in the open systems which exert two or more random forces are different.

The barrier crossing dynamics with multiplicative and additive white noises raised strong interest in the early eighties. In most of the works stated above, noise forces that are present simultaneously in the stochastic systems were usually treated as random variables uncorrelated with each other. However, cross-correlation between the random variables is possible in the noise driven dynamical systems as pointed out in Refs. [19–21]. Physically, the cross-correlation would mean that the noises are of the same origin (the same fluctuating quantity, either intrinsic or external, influencing two different kinetic parameters) [19]. But for the noises of different origin cross-correlation is also possible. This was pointed out in Refs. [20–21] where authors have assumed that external environmental fluctuation can influence internal fluctuation changing the internal structure [20–21]. If this happens, then the statistical properties of the noises should not be widely different, and can be correlated. The cross correlated noises were first considered by Fedchenia [22] in the context of hydrodynamics of vortex flows in ellipsoidal containments with regard to fluctuations. There the author introduced cross correlation among the noises of common origin which appear in the time evolution equation of dimensionless modes of flow rates. Fuliński and Telejko [19] also considered the interference of additive and multiplicative white noises in the bistable kinetic model, mentioning the physical possibility of cross correlated noises. However, very recently Madureira *et al.* [20] have pointed out the possibility of cross correlated noise in realistic model (ballast resistor) showing bistable behavior and have also discussed the influence of correlation of additive and multiplicative white noises on the activated rate processes. Transport of particles caused by cross-correlation between additive and multiplicative noises in the symmetric periodic potential has been investigated in Ref. [21]. The effect of correlation between additive and multiplicative noises is considered indispensable in explaining phenomena like stochastic resonance, phase transitions *etc.* [23–31]. Our aim in the present paper is to investigate the effect of interference of multiplicative and additive noises on time dependence of upper bound of time derivative of information entropy, when the coupling between two noise terms is colored with nonzero correlation time τ.

To begin with, we consider a stochastic process where both multiplicative and additive noises are present. The Langevin equation of motion for the present problem can be written as

$$\frac{dq}{dt} = -\frac{V'(q)}{\gamma} + \frac{q}{\gamma}\zeta(t) + \frac{1}{\gamma}\eta(t)\,, \tag{2}$$

where $V'(q)$ is the derivative of potential energy expressed as a function of the particle coordinate q. γ in Eq. (2) is the dissipation parameter. $\zeta(t)$ and $\eta(t)$ are white noises. The two noise terms are characterized by their means and variances as

$$\langle\zeta(t)\rangle = \langle\eta(t)\rangle = 0\,, \tag{3}$$

$$\langle\zeta(t)\zeta(t')\rangle = 2D\delta(t-t')\,, \tag{4}$$

and

$$\langle\eta(t)\eta(t')\rangle = 2D'\delta(t-t')\,. \tag{5}$$

Here D and D' are intensity of multiplicative and additive noises, respectively. In general, we express the influence of the internal fluctuation on the system as additive noise and the effect of the external environmental fluctuation on the system as multiplicative noise. Here we assume that the external environmental fluctuation can influence the internal fluctuation. Because of the influence of the external environmental fluctuation on the internal fluctuation, additive and multiplicative noise are not independent (there is correlation between them). We assume that the correlation time of the $\zeta(t)$ and $\eta(t)$ are nonzero [25,27–31]

$$\langle\zeta(t)\eta(t')\rangle = \langle\eta(t)\zeta(t')\rangle = \frac{\lambda\sqrt{DD'}}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right)\,, \tag{6}$$

where τ is the correlation time of the coupling between multiplicative and additive noises. λ in Eq. (6) corresponds to the coupling strength. In the limit $\tau \to 0$ the above equation becomes

$$\langle\zeta(t)\eta(t')\rangle = \langle\eta(t)\zeta(t')\rangle = 2\lambda\sqrt{DD'}\delta(t-t')\,. \tag{7}$$

A general equation satisfied by the probability distribution of equation (2) with (3)–(6) is given by [32]

$$\frac{\partial}{\partial t}\rho(q,t) = \frac{\partial}{\partial q}\frac{V'(q)}{\gamma}\rho(q,t) - \frac{\partial}{\partial q}\frac{q}{\gamma} \times \langle\zeta(t)\delta[q(t)-q]\rangle - \frac{\partial}{\partial q}\frac{1}{\gamma}\langle\eta(t)\delta[q(t)-q]\rangle\,, \tag{8}$$

where $\rho(q,t) = \langle\delta[q(t)-q]\rangle$; the average (8) can be calculated for Gaussian noise $\zeta(t)$ and $\eta(t)$ by the Novikov theorem [33]. The Fokker–Planck equation for (2) is obtained following Refs. [25,29–30].

$$\frac{\partial\rho}{\partial t} = \left[\frac{\partial}{\partial q}\frac{V'(q)}{\gamma} - \frac{\partial}{\partial q}\left(g(q)\frac{\partial g(q)}{\partial q}\right) + \frac{\partial^2 g(q)^2}{\partial q^2}\right]\rho\,, \tag{9}$$

where

$$g(q) = \frac{\left[D' + \frac{2\lambda\sqrt{DD'}}{1+2\tau_2}q + Dq^2\right]^{1/2}}{\gamma}\quad . \tag{10}$$

The above Fokker–Planck equation can be written in the form

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial q}\frac{V'(q)\rho}{\gamma} + \frac{\partial lq\rho}{\partial q} + \frac{\partial l_1\rho}{\partial q} + Q\frac{\partial^2\rho}{\partial q^2} - \frac{2D\rho}{\gamma^2} \tag{11}$$

with

$$l = \frac{3D}{\gamma^2}\,, \tag{12}$$

$$l_1 = \frac{3\lambda\sqrt{DD'}}{\gamma^2(1+2\tau_2)} \tag{13}$$

and

$$Q = \frac{D' + \frac{2\lambda\sqrt{DD'}}{1+2\tau_2}q + Dq^2}{\gamma^2}\,. \tag{14}$$

Now multiplying $\exp(2Dt/\gamma^2)$ on both sides of the Fokker–Planck equation, (11), followed by the transformation

$$W(q,t) = \rho(q,t)\exp\left(\frac{2Dt}{\gamma^2}\right)\,, \tag{15}$$

we get

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial q}\frac{V'(q)W}{\gamma} + \frac{\partial lqW}{\partial q} + \frac{\partial l_1 W}{\partial q} + Q\frac{\partial^2 W}{\partial q^2}\,. \tag{16}$$

Eq. (15) implies that $W(q,t)$ is not normalized if $\rho(q,t)$ is a normalized probability distribution function. It also implies that a factor $\exp(-2Dt/\gamma^2$

has to be multiplied with $\rho(q,t)\exp(-2Dt/\gamma^2)$ for $W(q,t)$ to be normalized. Hence normalized $W(q,t)$ and $\rho(q,t)$ are the same. For mathematical convenience we will use Eq. (16) and $W(q,t)$ for further calculations. Eq. (16) can be rearranged as

$$\frac{\partial W}{\partial t} = -\frac{\partial FW}{\partial q} + Q\frac{\partial^2 W}{\partial q^2}\,, \tag{17}$$

where

$$F = -\Gamma q - l_1 \tag{18}$$

and

$$\Gamma = \frac{V'(q)}{\gamma} + l\,. \tag{19}$$

The Fokker–Planck equation (17) can be rearranged into the general form of continuity equation

$$\frac{\partial W(q,t)}{\partial t} = -\frac{\partial j}{\partial q}\,, \tag{20}$$

where the current j is defined as

$$j = FW - Q\frac{\partial W}{\partial q}\,. \tag{21}$$

We shall now define the upper bound for the time derivative of information entropy using Eqs.(1) and (20). The time evolution equation for S can be written as

$$\frac{dS}{dt} = \int dq\frac{\partial j}{\partial q}\ln W\,. \tag{22}$$

Performing partial integration on the right hand side of the above equation and then putting natural boundary conditions [10], $j|_{\mathrm{boundary}} = 0$, and $j\ln W|_{\mathrm{boundary}} = 0$, one obtains

$$\frac{dS}{dt} = -\int dq\frac{1}{W}j\frac{\partial W}{\partial q}\,. \tag{23}$$

In the next step an application of the Schwartz inequality $|\int dqAB|^2 \leq \int dq|A|^2\int dq|B|^2$ to the integral(23), where A and B can be appropriately identified, yields an upper bound for the rate of entropy change

$$\begin{aligned} \frac{ds}{dt} &\leq U_{\mathrm{B}}\,, \\ U_{\mathrm{B}} &= \left(\int dq\frac{j^2}{W}\right)^{1/2}\left(\int dq\frac{1}{W}(\frac{\partial W}{\partial q})^2\right)^{1/2}\,. \end{aligned} \tag{24}$$

It is to be noted here that the second integral is the same as the trace of Fisher information matrix [10]. Thus the maximum rate of increase of S for an isolated system, is limited by the Fisher information level.

To find the explicit time dependence of the above quantity we consider a simple external force field. In this context we choose $V(q)$ in Eq. (2) as potential energy of a simple harmonic oscillator, having frequency ω. For this linear stochastic process we then search for the Green's function or conditional probability solution [34–36] for the system at q, at time t given that it had the value of U' at $t=0$. This initial condition may be represented by the δ-function

$$\delta(q-q') = \lim_{\epsilon\to\infty}\sqrt{\frac{\epsilon}{\pi}}\exp[-\epsilon(q-q')^2]\,. \tag{25}$$

$\sqrt{\epsilon/\pi}$ is the normalization constant. We now look for a solution of Eq. (17) of the form

$$W(q,t|q',0) = \exp[G(t)]\,, \tag{26}$$

where $G(t) = -\frac{1}{\sigma(t)}(q-\alpha(t))^2 + \ln\nu(t)\,.$

We will see that by suitable choice of $\alpha(t), \sigma(t), \nu(t)$ one can solve Eq. (17) subject to the initial condition

$$W(q,0|q',0) = \lim_{\epsilon\to\infty}\sqrt{\frac{\epsilon}{\pi}}\exp[-\epsilon(q-q')^2]\,. \tag{27}$$

Comparing Eq. (26) with (27) and $G(0)$ we have $\sigma(0)=\frac{1}{\epsilon}$, $\alpha(0)=q'$, $\nu(0)=\sqrt{\epsilon/\pi}$.

If we put (26) in (17) and equate the coefficients of equal powers of q we obtain after some algebra

$$\dot\sigma(t) = -2\Gamma\sigma(t) + 4Q\,, \tag{28}$$

$$\dot\alpha(t) = -\Gamma\alpha(t) - l_1 \tag{29}$$

and

$$\frac{1}{\nu(t)}\dot\nu(t) = -\frac{1}{2\sigma(t)}\dot\sigma(t)\,. \tag{30}$$

Here it is to be noted that in the above calculation we have used approximate values of q and q^2 in the diffusion coefficient Q as $< q >_{\mathrm{eq}}$ and $< q^2 >_{\mathrm{eq}}$ respectively. $< q >_{\mathrm{eq}}$ and $< q^2 >_{\mathrm{eq}}$ are the average values of q and q^2 at equilibrium. Now we consider the relevant solutions of $\sigma(t)$ and $\alpha(t)$ for the present problem which satisfy the above initial conditions and are given by

$$\sigma(t) = \frac{2Q}{\Gamma}[1-\exp(-2\Gamma t)] + \sigma(0)\exp(-2\Gamma t) \tag{31}$$

and

$$\alpha(t) = \left(\alpha(0) + \frac{l_1}{\Gamma}\right) \exp(-\Gamma t) - \frac{l_1}{\Gamma} \ . \tag{32}$$

Now making use of Eqs. (26), (31) and (32) in Eq. (24) we finally obtain the explicit time dependence of the upper bound $U_{\mathrm{B}}(t)$ for the rate of entropy change as

$$U_{\mathrm{B}} = \frac{(2\alpha^2\Gamma^2\sigma + 4\alpha\Gamma l_1\sigma + 2l_1^2\sigma + \Gamma^2\sigma^2 + 4Q^2 - 4Q\Gamma\sigma)^{1/2}}{\sigma} \, . \tag{33}$$

Similarly, one can calculate the rate of change of information entropy with time using Eq. (26) in Eq. (22)

$$\frac{dS}{dt} = -\Gamma + \frac{2Q}{\sigma} \, . \tag{34}$$

Since the information entropy is the negative of the Shannon information, the rate of change of entropy can be interpreted as the rate of information transmission. So the upper bound for (33) is interesting in the sense that the amount of information transmitted per unit time cannot exceed this quantity. Deviation(dU_{B}) of the bound from $\frac{dS}{dt}$ can be calculated from Eqs.(33)–(34) as

$$dU_{\mathrm{B}} = U_{\mathrm{B}} - \frac{dS}{dt} = \frac{2(\alpha\Gamma + l_1)^2}{(2\sigma(\alpha\Gamma + l_1)^2 + (\Gamma\sigma - 2Q)^2)^{1/2} - \Gamma + \frac{2Q}{\sigma}} \, . \tag{35}$$

Now we explore how these quantities vary with time and other system parameters. First, we calculate both U_{B} and dU_{B} at different time and plot in Fig. 1. It shows that the upper bound and its deviation from the rate of change of entropy with time decrease monotonically as system approaches the stationary state. This is due to the fact that at very short time the motion of the particle is mainly governed by the deterministic force and gradually random force becomes effective *i.e.*, the random force has maximum tendency to expand the phase space against the deterministic one at $t \to 0$ and it reduces with progress of time. Finally they balance each other at equilibrium. Thus the rate of change of width of distribution function and entropy decreases regularly. Since possibility of the deviation(dU_{B}) at large value of $\frac{dS}{dt}$ is greater, the upper bound and its deviation are maximum at $t \to 0$ and superpose at long time.

Now we consider the relaxation time. Eqs. (31)–(32) show that the relaxation time increases with increase of damping constant(γ), because Γ decreases here. Γ, on the other hand, rises for increase of system frequency or strength of multiplicative noise and as a result of that, the non-equilibrium system relaxes more rapidly with increase of ω or D.

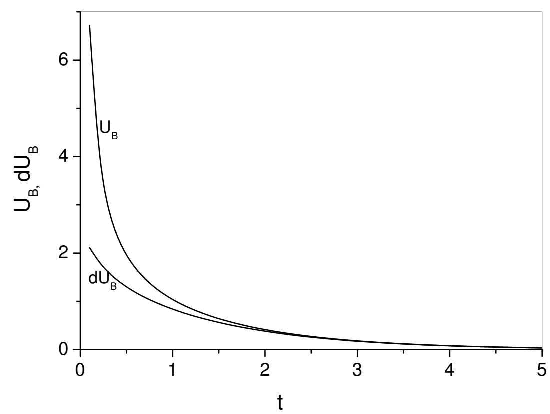

Fig. 1. Plot of $U_{\rm B}$ and $dU_{\rm B}$ *vs* time using Eqs. (33),(35) for the parameter set $\gamma = 1.0$, $\sigma(0) = 0$, $\alpha(0) = 1.0$, $D = D' = \omega = 0.25$, $\tau = 0.0$ and $\lambda = 0.5$ (units are arbitrary).

We examine now how the upper bound is affected by noise correlation strength λ. In Fig. 2 we have plotted $U_{\rm B}$ and $dU_{\rm B}$ *vs* λ. It shows that at large λ the deviation increases at faster rate than the $U_{\rm B}$. This happens because

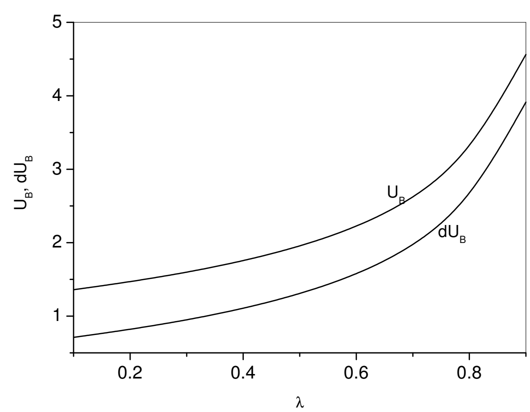

Fig. 2. Plot of $U_{\rm B}$ and $dU_{\rm B}$ *vs* strength of cross-correlation(λ) using Eqs. (33),(35) for the parameter set $\gamma = 1.0$, $\sigma(0) = 0$, $\alpha(0) = 1.0$, $D = D' = \omega = 0.25$, $\tau = 0.0$ and $t = 2.0$ (units are arbitrary).

the effective deterministic force is dominating over the random force in the dynamics with increase of λ as the constant force (l_1) increases and the effective noise strength (Q) decreases. Thus the greater value of λ makes the distribution function narrower and leads to higher value of both the $U_{\rm B}$ and $dU_{\rm B}$. Whereas the deviation and the bound are found to decrease for increase of correlation time (τ) of colored cross correlation since the l_1 decreases and the Q increases as τ becomes larger. It is shown in Fig. 3.

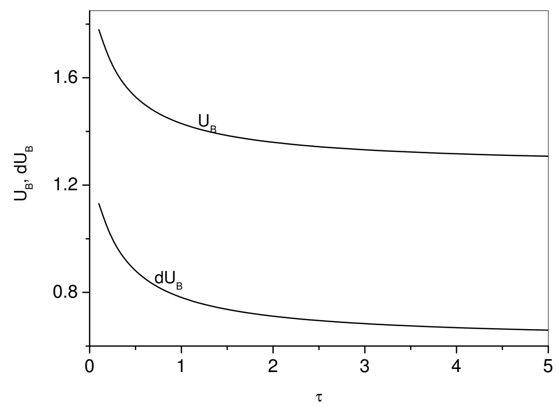

Fig. 3. Plot of $U_{\rm B}$, $dU_{\rm B}$ *vs* τ using Eqs.(33),(35) for the parameter set $\gamma = 1.0$, $\sigma(0) = 0$, $\alpha(0) = 1.0$, $D = D' = \omega = 0.25$, $\lambda = 0.5$ and $t = 2.0$ (units are arbitrary).

In the next step we explore the role of damping constant (γ) on the upper bound. The γ affects both the deterministic force and diffusion constant through Γ, l_1 and Q respectively. Since the role of former is opposite to latter on the upper bound and the deviation, in the interplay of damping constant, noise strength and system frequency variation of the upper bound and the deviation with γ show extremal nature what is shown in Fig. 4. Similarly the variation of $U_{\rm B}$ and $dU_{\rm B}$ with the strength of multiplicative noise also exhibits extremal nature. This is shown in Fig. 5. Thus in the persistence of non-equilibrium situation γ and D have important role. The extremal nature disappears if we increase the strength of additive noise keeping fixed other parameters. The upper bound and the deviation decrease with increase of D' as the effective noise strength becomes greater.

We now examine the long time limit of the above result (33). At $t \to \infty$ Eqs.(31) and (32) reduce to

$$\sigma(\infty) = \frac{2Q}{\Gamma} \tag{36}$$

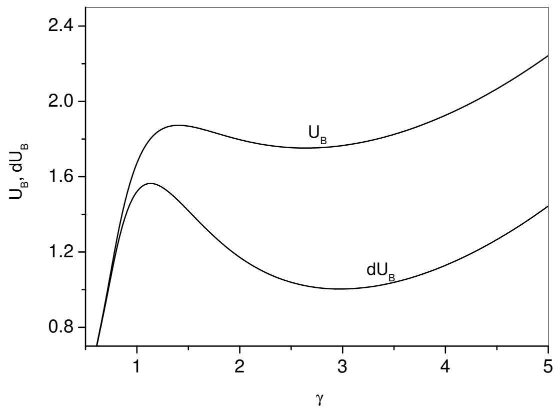

Fig. 4. Plot of $U_{\rm B}$, $dU_{\rm B}$ *vs* damping constant (γ) using Eqs.(33),(35) for the parameter set $\tau = 0.0$, $\sigma(0) = 0$, $\alpha(0) = 1.0$, $D = D' = \omega = 0.25$, $\lambda = 0.5$ and $t = 2.0$ (units are arbitrary).

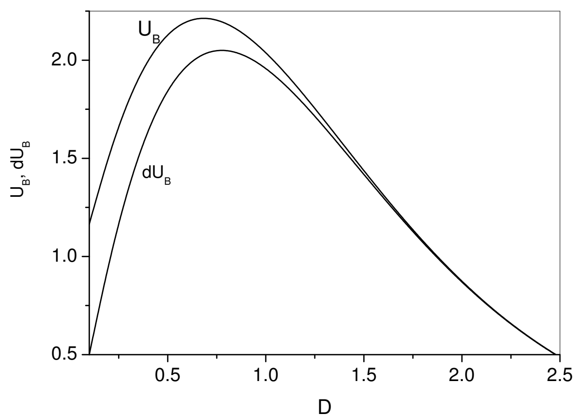

Fig. 5. Plot of $U_{\rm B}$, $dU_{\rm B}$ *vs* strength of multiplicative noise (D) using Eqs.(33),(35) for the parameter set $\tau = 0.0$, $\sigma(0) = 0$, $\alpha(0) = 1.0$, $\gamma = 1.0$, $D' = \omega = 0.25$, $\lambda = 0.5$ and $t = 2.0$ (units are arbitrary).

and

$$\alpha(\infty) = -l_1/\Gamma\,. \tag{37}$$

Eqs.(36) and (37) imply that at $t \to \infty$ the numerator of the right hand side of Eq. (33) vanishes. Therefore we obtain the equation

$$\frac{dS}{dt} = 0\,. \tag{38}$$

Thus the above result satisfies our natural demand.

In conclusions, we have studied here the non-stationary states of a noise driven dynamical system in terms of information entropy, when the couplings between additive and multiplicative noises are colored with noise correlation time τ, based on the Fokker–Planck description of stochastic processes and Schwartz inequality. We consider time evolution of the upper bound of time derivative of information entropy and its deviation from the rate of change of information entropy. Our main observations include the following points.

(1) The deviation and the upper bound monotonically decrease to zero with increase of time. The relaxation time increases with increase of damping constant and it decreases for the increase in system frequency as well as the strength of multiplicative noise. The cross-correlation time and the strength of cross correlation have no effect on the relaxation time.

(2) Rate of increase of the deviation is greater than the upper bound with increase of λ.

(3) The U_{B} and the dU_{B} both decrease with increase of correlation time of cross-correlation.

(4) The interplay of deterministic and random forces reveals extremal nature of the upper bound and the deviation.

These observations are, of course, restricted to the harmonic oscillator (HO). However, this is such an important system that HO insights usually have a wide impact.

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