KASNER GENERALIZATION OF LEVI-CIVITA SPACE-TIME*

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We investigate some cylindrically symmetric nonstationary and nonstatic solutions of Einstein field equations. We first study some physical properties of a solution which can be considered as Kasner generalization of static Levi-Civita vacuum solution. Then we generalize this metric to include a solution where a space-time is filled with null dust or a stiff fluid.

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1. Introduction

One of the most important differences of the spherically and the cylindrically symmetric vacuum solutions of General Relativity is that, according to the Birkhoff theorem, there is a time-like Killing vector in the spherically symmetric vacuum solution. Thus, it can be said that the spherically symmetric vacuum is necessarily static. However, the situation drastically changes when we consider the cylindrically symmetric systems since there is no analogue of Birkhoff's theorem in cylindrical symmetry. During the gravitational collapse of a cylindrically symmetric system, gravitational waves can be emitted and the exterior region of a collapsing cylindrical body is not static [1]. This fact has important consequences in the studies of gravitational waves, cosmological models, quantum gravity and numerical relativity.

If ∂_z and ∂_ϕ are the axial and the angular Killing vectors describing cylindrical symmetry, then the most general cylindrically symmetric nonstationary metric can be written in the canonical form as [2]:

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{2U} dz^2 + e^{-2U} W^2 d\phi^2 \,, \tag{1}$$

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where K, U and W are the functions of r and t, in general. Here r is the radial, t is the time, z is the axial and ϕ is the angular coordinate with the ranges $0 \leq r < \infty$, $-\infty < z$, $t < \infty$, $0 \leq \phi \leq 2\pi$.

The static solution of the metric (1) represents the exterior gravitational field of a general static cylindrical line source and was found by Levi-Civita in 1919 [3]:

$$ds^2 = -\rho^{4s}dt^2 + \rho^{4s(2s-1)}(d\rho^2 + dz^2) + \alpha^2\rho^{2(1-2s)}d\phi^2, \tag{2}$$

where s and α are constant parameters. These parameters in general cannot be removed by a coordinate transformation if ϕ is an angular coordinate. Research on this solution was mainly focused on the understanding of these parameters and finding physically acceptable sources generating this metric since in order to understand the meaning and the behavior of the metric parameters s and α, one may need to match it with an interior solution. Static cylinders [4] and cylindrical shells [5] have been constructed as a source of this metric. Shells composed of various matter sources satisfying some energy conditions for certain ranges of s have also been studied [6]. The parameter s is related to the energy density of the source and α is an angular deficit parameter.

The nonstatic vacuum solutions of (1) have also been studied extensively. They have important consequences on cosmology, gravitational waves and also on quantum gravity. For a discussion of these solutions we refer to the book of Stephani *et al.* [2].

Furthermore, the time dependent cylindrically symmetric nonvacuum solutions of Einstein equations were studied for different cylindrical systems. An expanding cylindrical radiation filled Universe [7], a radiation Universe with heat and null radiation flow [8], nonstatic cosmic strings with a time dependent vacuum exterior [9–11], nonstatic global strings [12] are some examples of such solutions. Some of these solutions have an interesting property that their exterior vacuum solutions correspond to some particular values of the parameters the Levi-Civita metric having also a Kasner type time dependence. This fact motivates us to study the Kasner generalization of the Levi-Civita solution with the full range of its parameters. Thus in this paper, we will study the properties of cylindrically symmetric time dependent vacuum solutions in Kasner form. This solution can be considered as a Kasner generalization of the Levi-Civita (LC) solution since for every constant time slice it reduces to the LC solution. These kind of generalized Kasner solutions having more than one variable are well known and studied by different authors [13]. This solution is also equivalent to the Einstein–Rosen soliton wave solutions [14] by a coordinate transformation. Although this solution is well known, we will establish a direct relation between the parameters of the LC solution with the parameters of its Kasner generalization. We will also perform a detailed comparison of the LC solution and its nonstatic Kasner generalization by studying their singularity behavior, geodesic structure and radial acceleration of test particles in these space-times.

We also extend our discussion into some nonvacuum generalizations of this solution. Since the gravitational collapse of a physically reasonable source is one of the main topics in general relativity, the cylindrical collapse is studied extensively in the literature [1, 15]. The radiating Levi-Civita space-time [16], a space-time filled with a radially oriented null radiation in an otherwise empty static background is generally employed in these concerns and others [17], since this metric can represent the exterior region of a collapsing cylindrical body. However, since there is no analogue of Birkoff theorem in cylindrical symmetry, it might be reasonable to discuss a nonstatic generalization of this radiating Levi-Civita solution. We also present a nonstatic stiff fluid as an another example of this form.

The paper is organized as follows. In the next section we present the Kasner generalization of the LC solution. In Section 3 we discuss some physical properties of this solution. Section 4 discusses the radiating generalization of this solution and its some physical properties. In Section 5 we present a solution representing a Universe filled with a nonstatic isotropic stiff fluid. Lastly, we give a brief conclusion.

2. Levi-Civita–Kasner solution

Let us consider the following ansatz for the functions of the metric (1):

$$\begin{aligned} W &= \alpha(c_1 r + c_2)(c_3 t + c_4)\,, && (3)\\ U &= k\ln(c_1 r + c_2) + q\ln(c_3 t + c_4)\,, && (4)\\ K &= k^2\ln(c_1 r + c_2) + q^2\ln(c_3 t + c_4)\,, && (5)\end{aligned}$$

where k, q, α and c_i's are constants. Here, when $c_3 = 0$, $c_4 \neq 0$, $c_1 \neq 0$ we get the Levi-Civita solution of the form:

$$ds^2 = r^{2(k^2-k)}(-dt^2 + dr^2) + r^{2k}dz^2 + \alpha^2 r^{2(1-k)}\, d\phi^2\,, \qquad (6)$$

where we have rescaled the coordinates r, t and z. One can recover the conventional form of the LC solution (2) by applying the following coordinate transformations:

$$R = \frac{r^\kappa}{\kappa}\,,\quad R = \frac{\rho^S}{S}\,,\quad \kappa = k^2 - k + 1\,,\quad S = 4s^2 - 2s + 1\,,\quad s = \frac{k}{2(k-1)}\,. \qquad (7)$$

When we choose $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$ the solution reduces to well known vacuum Kasner solution:

$$ds^2 = t^{2(q^2-q)}(-dt^2 + dr^2) + t^{2q}dz^2 + t^{2(1-q)}d\phi^2\,, \qquad (8)$$

where q is a real constant. For this case the coordinates can be thought of as the Cartesian coordinates. The coordinate transformation $t' = (Q\,t)Q^{-1}$ puts the Kasner solution in its familiar form as [19]:

$$ds^2 = -dt^2 + t^{2a}dr^2 + t^{2b}dz^2 + t^{2c}d\phi^2\,, \tag{9}$$

where we have again rescaled the metric, removed prime for clarity and $a = (q^2 - q)Q^{-1}$, $b = q\,Q^{-1}$, $c = (1-q)Q^{-1}$ with $Q = q^2 - q + 1$. Kasner solution corresponds to an anisotropic homogeneous cosmology. Here the constants a, b, c satisfy the Kasner constraints:

$$a + b + c = 1 = a^2 + b^2 + c^2\,. \tag{10}$$

Also, for $c_1 = c_3 = 0$ and others nonvanishing we get flat Minkowski space-time. Notice that c_1 and c_2 cannot vanish simultaneously in (3). The same is true also for c_3 and c_4.

If one calculates the Ricci tensor of the metric (3), the only nonvanishing term is

$$R_{01} = -c_1c_3(-1 + k^2 - 2kq + q^2)(c_1r + c_2)^{(-1+2k-2k^2)}(c_3t + c_4)^{(-1+2q-2q^2)}\,. \tag{11}$$

Here we see that when c_1 or c_3 vanish we have a vacuum solution as it should be. Assuming they do not vanish, equaling (11) to zero we get $q = k \pm 1$ which results (hereafter, we choose $c_2 = c_4 = 0$ and we absorb c_1 and c_3 in the coordinates r, t, z by redefining them):

$$\begin{aligned} ds^2 \;&=\; r^{2(k^2-k)}\,t^{2((k+\varepsilon)^2-k-\varepsilon)}(-dt^2 + dr^2) \\ &\quad + r^{2k}\,t^{2(k+\varepsilon)}dz^2 + P^2\,r^{2(1-k)}\,t^{2(1-k-\varepsilon)}\,d\phi^2\,, \end{aligned} \tag{12}$$

with $\varepsilon = \pm 1$. Thus, we have obtained the desired Kasner generalization of the LC solution, where we can call it as Levi-Civita–Kasner space-time (LCK). It is better to express them with the Levi-Civita parameter since we want to compare them with the static solution. The transformations:

$$\begin{aligned} R \;&=\; r^{\kappa}\,\kappa^{-1}\,, \quad \tau = Q^{-1}t^{Q}\,, \quad k = \frac{2s}{(2s-1)}\,, \\ Q \;&=\; (k+\varepsilon)^2 - (k+\varepsilon) + 1\,, \end{aligned} \tag{13}$$

leads to the following metric:

$$ds^2 = -R^{2D}d\tau^2 + \tau^{2A}dR^2 + R^{2E}\tau^{2B}dz^2 + \alpha^2\,R^{2F}\tau^{2C}d\phi^2, \tag{14}$$

where we again rescaled the coordinates τ, R, z, absorbed all constant into α and

$$H = \varepsilon(4s^2-1)+(1-2s)^2\,, \qquad A=\frac{2s+H}{S+H}, \tag{15}$$

$$B = \frac{(2s-1)(2s+\varepsilon(2s-1))}{S+H}\,, \quad C=\frac{(1-2s)(1+\varepsilon(2s-1))}{S+H}\,, \tag{16}$$

$$D = \frac{2s}{S}\,, \qquad E=\frac{2s(2s-1)}{S}\,, \qquad F=\frac{1-2s}{S}\,. \tag{17}$$

For any value of s we have in general two different solutions depending on $\varepsilon=\pm 1$. These solutions are in the form of the generalized Kasner space-times [13] and the metric functions A, B, C and E, F, G satisfy the Kasner constraints separately:

$$A+B+C=A^2+B^2+C^2 = 1\,, \tag{18}$$

$$D+E+F=D^2+E^2+F^2 = 1\,. \tag{19}$$

The LCK solution (12) is also equivalent to Einstein–Rosen soliton wave solutions [2, 14] by a transformation:

$$r = \sqrt{T-\sqrt{T^2-\varrho^2}}\,, \qquad t=\sqrt{T+\sqrt{T^2-\varrho^2}}\,, \tag{20}$$

which puts the metric functions into the form:

$$W = rt = \varrho\,, \tag{21}$$

$$U = k\ln\varrho + \frac{\varepsilon}{2}\ln\left(T+\sqrt{T^2-\varrho^2}\right)\,, \tag{22}$$

$$K = k^2\ln\varrho+\left(\varepsilon k+\frac{1}{2}\right)\ln\left(T+\sqrt{T^2-\varrho^2}\right)-\frac{1}{2}\ln\left(2\sqrt{T^2-\varrho^2}\right)\,. \tag{23}$$

This transformation is valid only for $t^2 > r^2$. For $r^2 > t^2$ we need the following transformation:

$$r = \sqrt{\varrho+\sqrt{\varrho^2-T^2}}\,, \qquad t=\sqrt{\varrho-\sqrt{\varrho^2-T^2}}\,, \tag{24}$$

which gives:

$$W = rt = T\,, \tag{25}$$

$$U = k\ln T + \frac{\varepsilon}{2}\ln\left(\varrho-\sqrt{\varrho^2-T^2}\right)\,, \tag{26}$$

$$K = k^2\ln T-\left(\varepsilon k+\frac{1}{2}\right)\ln\left(\varrho-\sqrt{\varrho^2-T^2}\right)-\frac{1}{2}\ln\left(2\sqrt{\varrho^2-T^2}\right)\,. \tag{27}$$

Here the first metric is not valid at $\varrho > T$ and the other is not valid at $\varrho < T$. Then we need to extend one to join with the other. After achieving this, the resulting space-time is the solution we consider in this article. Hence, the metric we discuss covers both regions.

3. Some physical properties of LCK solution

3.1. NP spin and Weyl coefficients

The static Levi-Civita metric is in general Petrov type I except it is flat for $s=0, 1/2$ and it is Petrov type D for $s=-1/2,\ 1/4, 1$ (see da Silva *et. al.* in [4]). Let us compare with LCK space-time.

Here, using a NP tetrad, we will present the nonvanishing spin and Weyl scalars of this space-time since in this formalism some of the curvature components have direct physical meaning [20]. The canonical form of the metric (12) is more appropriate for our purposes. We chose the NP tetrad as follows:

$$\begin{aligned} ds^2 &= \boldsymbol{l}\otimes\boldsymbol{n}-\boldsymbol{m}\otimes\bar{\boldsymbol{m}}\,, && (28)\\ \sqrt{2}\,\boldsymbol{l} &= \boldsymbol{e}^0+\boldsymbol{e}^1\,, \qquad \sqrt{2}\,\boldsymbol{n}=\boldsymbol{e}^0-\boldsymbol{e}^1\,, \qquad \sqrt{2}\,\boldsymbol{m}=\boldsymbol{e}^2+i\boldsymbol{e}^3\,, && (29)\end{aligned}$$

$$\begin{aligned} \boldsymbol{e}^0 &= r^{k^2-k}t^{(k+\varepsilon)^2-k-\varepsilon}dt\,, \quad \boldsymbol{e}^1=r^{k^2-k}t^{(k+\varepsilon)^2-k-\varepsilon}dr\,, && (30)\\ \boldsymbol{e}^2 &= r^k t^{k+\varepsilon}dz, \qquad\qquad\quad \boldsymbol{e}^3=\alpha r^{1-k}t^{1-k-\varepsilon}d\phi\,. && (31)\end{aligned}$$

For $\varepsilon=1$ the nonvanishing components of spin coefficients and Weyl scalars are:

$$\begin{aligned}
\sigma &= -\frac{(1+2k)\,r+(1-2k)\,t}{2\sqrt{2}r^{k^2-k+1}t^{k^2+k+1}}\,, \quad \lambda=\frac{(1+2k)\,r-(1-2k)\,t}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2+k+1}}\,, && (32)\\
\rho &= \frac{t-r}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2+k+1}}\,, \qquad \mu=\frac{t+r}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2+k+1}}\,, && (33)\\
\varepsilon &= \frac{k((1+k)\,r+(1-k)\,t)}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2+k+1}}\upsilon, \quad \gamma=\frac{k((1-k)\,t-(1+k)\,r)}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2+k+1}}\,, && (34)\\
\kappa &= \nu=\tau=\pi=\alpha=\beta=0\,, && (35)\\
\Psi_0 &= \frac{k\left((1+k)(1+2k)\,r^2+4(1-k^2)\,r\,t+(1-k)(1-2k)\,t^2\right)}{2\,r^{2k^2-2k+2}\,t^{2k^2+2k+2}}\,, && (36)\\
\Psi_2 &= \frac{k((1+k)\,r^2+(1-k)\,t^2)}{2\,r^{2k^2-2k+2}\,t^{2k^2+2k+2}}\,, && (37)\\
\Psi_4 &= \frac{k\left((1+k)(1+2k)\,r^2-4(1-k^2)\,r\,t+(1-k)(1-2k)\,t^2\right)}{2\,r^{2k^2-2k+2}\,t^{2k^2+2k+2}}\,. && (38)
\end{aligned}$$

This shows us that the LCK space-time with $\varepsilon=1$ is again Petrov type I in general except it is flat for $k=0\,(s=0)$ and $k\to\infty$ $(s=1/2)$ and Petrov type D for $k=1$ $(s\to\infty)$ and $k=-1$ $(s=-1/4)$. Also, since $\kappa=0$, $\boldsymbol{l}$ is geodesics but it is not affinely parameterized since $\varepsilon\neq 0$ except $k=0\,(s=0)$. It also has expansion $(-\rho\neq 0)$ and shear $(|\sigma|\neq 0)$ but it is not twisting.

For $\varepsilon = -1$ we have

$$\sigma = \frac{(3-2k)\,r + (2k-1)\,t}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}\,, \quad \lambda = \frac{(2k-3)\,r + (2k-1)\,t}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}\,, \tag{39}$$

$$\rho = \frac{t-r}{2\sqrt{2}r^{k^2-k+1}\,t^{k^2-3k+3}}\,, \quad \mu = \frac{t+r}{2\sqrt{2}r^{k^2-k+1}\,t^{k^2-3k+3}}\,, \tag{40}$$

$$\varepsilon = \frac{(k-1)((k-2)r - k\,t)}{2\sqrt{2}r^{k^2-k+1}\,t^{k^2-3k+3}}\,, \quad \gamma = \frac{(k-1)((2-k)r - k\,t)}{2\sqrt{2}r^{k^2-k+1}\,t^{k^2-3k+3}}\,, \tag{41}$$

$$\kappa = \nu = \tau = \pi = \alpha = \beta = 0\,, \tag{42}$$

$$\Psi_0 = \frac{(k-1)\left((6-7k+2k^2)\,r^2 - 4k(k-2)\,r\,t + k(2k-1)\,t^2\right)}{2r^{2k^2-2k+2}\,t^{2k^2-6k+6}}\,, \tag{43}$$

$$\Psi_2 = \frac{(k-1)\left((k-2)\,r^2 - kt\right)}{2r^{2k^2-2k+2}\,t^{2k^2-6k+6}}\,, \tag{44}$$

$$\Psi_4 = \frac{(k-1)\left((6-7k+2k^2)\,r^2 + 4k(k-2)\,r\,t + k(2k-1)\,t^2\right)}{2r^{2k^2-2k+2}\,t^{2k^2-6k+6}}\,. \tag{45}$$

Thus, LCK with $\varepsilon = -1$ is also Petrov type I in general except $k = 1$ $(s \to \infty)$ and $k \to \infty\,(s = 1/2)$ where it is flat and Petrov type D for $k = 0$ $(s = 0)$ and $k = 2$ $(s = 1)$. Again, the vector $\boldsymbol{l}$ is geodesics but not affinely parameterized except $k = 1$. Also, it has nonvanishing expansion and shear but it is not twisting. Thus, the LC and LCK solutions have common Petrov types in general, but they differ for some particular values of the parameter s.

3.2. Singularity behavior

The Kretschmann scalar $\mathcal{K} = R_{abcd}R^{abcd}$ of the metric (14) are

$$\begin{aligned}\mathcal{K} &= 64s^2\,(1-2s)^2\left(\frac{(1-4s)^2 r^{-8s/(1-2s+4s^2)}}{(1-6s+12s^2)^3\,t^4} + \frac{t^{8s(1-4s)/(1-6s+12s^2)}}{(1-2s+4s^2)^3\,r^4}\right.\\ &\quad\left. - \frac{2\,(1-4s)^2\,r^{-4s/(1-2s+4s^2)}\,t^{4s(1-4s)/(1-6s+12s^2)}}{(1-2s+4s^2)^2(1-6s+12s^2)^2\,r^2\,t^2}\right), \quad (\varepsilon = 1)\,,\end{aligned} \tag{46}$$

$$\begin{aligned}\mathcal{K} &= 64(1-2s)^2\left(\frac{(s-1)^2\ r^{-8s/(1-2s+4s^2)}}{(3-6s+4s^2)^3 t^4} + \frac{s^2\,t^{4(1-2s)^2/(3-6s+4s^2)}}{(1-2s+4s^2)\,r^4\,t^4}\right.\\ &\quad\left. - \frac{8s^2(s-1)^2 r^{4s/(1-2s+4s^2)} t^{2(1-2s)^2/(3-6s+4s^2)}}{(3-6s+4s^2)(1-2s+4s^2) r^2 t^4}\right), \quad (\varepsilon = -1)\,.\end{aligned} \tag{47}$$

We see that, this space-time has singularities as in the static case. It is well known that the static Levi-Civita space-time is singular at $r = 0$, except for $s = 0$, $s = \pm 1/2$ and $s \to \infty$. For these values of s, the solution is regular and flat. If one compares the static solution with the nonstatic solutions, one realizes that there are similarities and differences. For $\varepsilon = 1$ only when $s = 0$ or $s = 1/2$, the solution is locally flat. For other cases, there are singularities. For $s = 1/4$ we have a singularity at $r = 0$ whereas for $s \to \infty$ we have singularity at $t = 0$. For all other values of s we have singularities at both $r = 0$ and $t = 0$. When $\varepsilon = -1$ the situation is also different. For this case when $s = 0$ the solution is not locally flat but contains a singularity at $t = 0$. There are locally flat solutions when $s = 1/2$ or $s \to \infty$. For $s = 1$ we have a singularity at $r = 0$. For other values of s we have both line and Big Bang singularities.

As we have mentioned, for $s = 0$ and $s = 1/2$ static Levi-Civita solution (2) is flat. Corresponding solutions for $\varepsilon = \pm 1$ are

$$ds^2 = -d\tau^2 + dr^2 + \tau^2 dz^2 + \alpha^2 r^2 d\phi^2 \,, \qquad (s = 0\,,\ \varepsilon = 1)\,, \tag{48}$$

$$ds^2 = -d\tau^2 + \tau^{4/3} dR^2 + \tau^{-2/3} dz^2 + \alpha^2 R^2 \tau^{4/3} d\phi^2 \,, \ (s = 0\,,\, \varepsilon = -1), \tag{49}$$

$$ds^2 = -R^2 d\tau^2 + \tau^2 dR^2 + dz^2 + \alpha^2 d\phi^2 \,, \qquad (s = 1/2\,,\ \varepsilon = \pm 1)\,. \tag{50}$$

Here the first and third metrics are flat whereas the second one is curved. The first and third metrics can be put into standard Minkowski form with a suitable coordinate transformation. The first solution is presented in [9] as a possible nonstatic exterior solution corresponding to a nonstatic string. Also, the second solution (49) was introduced in [8, 11, 12] as an exterior vacuum solution to their interior nonstatic string-like cylindrical source. It is not singular at $r = 0$ but has a Big Bang singularity at $t = 0$ since its Kretschmann scalar is $\mathcal{K} \sim t^{-4}$. Thus, for some specific values of parameters, the LCK solution reduces to some previously known solutions.

3.3. Radial acceleration of test particles

The radial acceleration of a free test particle at rest in the coordinate system of (14) is given by:

$$\frac{d^2 R}{d\tau^2} = -\frac{D}{R\,\tau^{2A}}\,. \tag{51}$$

The radial acceleration in the static Levi-Civita space-time can be found by taking $A = 0$. For the Levi-Civita space-time, when the parameter s is positive, the axis is attractive and when s is negative, the axis is repulsive. For a particle in a constant radius, the magnitude of the acceleration is increasing with increasing s when $0 < s < 1/2$, and decreases with increasing s when $s > 1/2$. For $s = 0$, no radial force is exerted on a particle at rest. For nonzero s, when the radial distance increases, radial acceleration decreases.

As in the Levi-Civita space-time, for the Levi-Civita–Kasner space-times (14) when s positive the axis is attractive and when s negative the axis is repulsive. And also when $s = 0$, no radial acceleration is felt by a particle at rest. However, since the solution is time dependent, the behavior of acceleration is changing with time. A typical behavior for $t = 3$ can be seen in Fig. 1. Here, for $\varepsilon = 1$ the magnitude of acceleration increases with increasing s up to $s \sim 0.17$, then it starts to decrease sharply up to $s \sim 0.3$, and then it decreases monotonically with increasing s. For $\varepsilon = -1$ situation is different. It increases monotonically up to $s = 1/2$ then increases more sharply up to $s = 3/2$ then starts to decrease with increasing s. When the time evolves, the radial acceleration is getting stronger for certain ranges of s and out of this range, particle feels very tiny force. For $\varepsilon = 1$ this range is in between 0 and 0.2. For $\varepsilon = -1$ the situation is reverse. For small s particle feels very small force. The region where acceleration is very strong is near $s \sim 1$. For other values of s a test particle feels very tiny radial force on it.

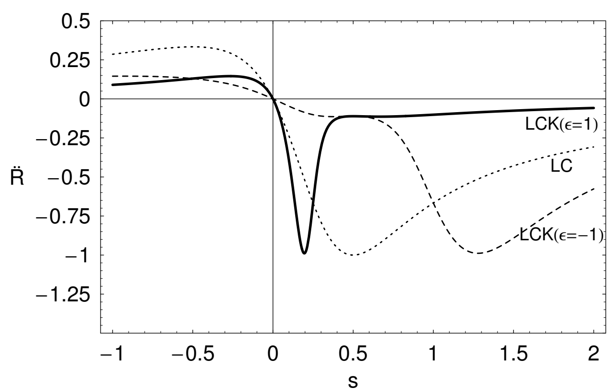

Fig. 1. The radial acceleration of a particle at $r = 1$, $t = 3$ for the static Levi-Civita and Levi-Civita–Kasner space-times with $\varepsilon = \pm 1$. Dotted line represents static Levi-Civita space-time, the solid line represents the solution with $\varepsilon = 1$ and dashed line represents $\varepsilon = -1$ case.

An important difference between LC and LCK space-times is that for $s = 1/2$ the radial acceleration of test particles becomes maximum for LC metric. This fact has been discussed in previous studies of LC metric since when the parameter s increases the energy density increases for $0 < s < 1/2$ but decreases for $s > 1/2$. This fact suggested that the parameter s is somehow related with the energy density of the source but not proportional to it. For the LCK space-time the maximum value of the radial acceleration is different than $1/2$ (Fig. (1)). Thus they have different gravitational fields.

3.4. Circular geodesics

Here we study the equations of a test particle following a circular geodesics in the space-time (14). The circular geodesics in the LC space-time is discussed in detail by da Silva *et. al.* [4].

Let us denote the angular velocity of a particle moving along a geodesics as $\omega = d\phi/d\tau$ and its tangential velocity as $W^\mu = (0,0,0,W^\phi)$ with $W^\phi = \omega/\sqrt{-g_{tt}}$, then we have (here τ is the time coordinate, not the proper time and dot represents derivation with respect to an affine parameter η):

$$\left(\frac{ds}{d\tau}\right)^2 = -R^{2D} + R^{2F}\tau^{2C}\left(\frac{d\phi}{d\tau}\right)^2 , \tag{52}$$

$$\omega^2 = \left(\frac{\dot\phi}{\dot\tau}\right)^2 = \frac{D}{\alpha^2 F}R^{2(D-F)}\tau^{-2C}, \tag{53}$$

$$\ddot\tau = -C\alpha^2 R^{2(F-D)}\tau^{2C-1}\dot\phi^2 , \tag{54}$$

$$\dot r = 0 , \qquad \dot z = 0 . \tag{55}$$

Then,

$$W^2 = \frac{D}{F} . \tag{56}$$

Replacing this into the first and the third equations, we get

$$\left(\frac{ds}{d\tau}\right)^2 = \left(W^2 - 1\right)R^{2D} , \tag{57}$$

$$\dot\tau = \frac{d\tau}{d\eta} = \frac{\tau_0}{\tau^C\sqrt{W}} . \tag{58}$$

Thus, the circular geodesics are time-like for $W < 1$ ($s < 1/4$), space-like for $W > 1$ ($s > 1/4$) and null for $W = 1$ ($s = 1/4$). We have the same conditions with the static Levi-Civita space-time. Thus the time dependence does not affect the circular geodesics. Also, as in the static case, for a given s the tangential velocity of a particle is constant. The only difference between LC and LCK space-times that is ∂_τ is not a Killing vector for LCK space-times. This does not affect the dependence of the character of the circular geodesics to the parameter s, although they have different gravitational fields, since $\dot\tau$ is not constant for this metric and also since the previous section suggests.

4. Radiating Levi-Civita–Kasner space-time

It is well known [2, 18] that for any Einstein–Rosen wave solution with ($K = K_0$, $U = U_0$, $W = W_0$) solving vacuum Einstein equations for this metric, there is a corresponding radiative solution ($K = F(r-t) + K_0$, $U = U_0$, $W = W_0$) satisfying:

$$T_{\mu\nu} = \eta\, \boldsymbol{k}_\mu \boldsymbol{k}_\nu\,, \tag{59}$$

where $\boldsymbol{k}_\mu$ is a null vector satisfying $\boldsymbol{k}_\mu \boldsymbol{k}^\mu = 0$ and η is energy density of the pure radiation (null dust). Using this property we can easily construct the Kasner generalization of radiating Levi-Civita solution. For the functions K_0, U_0, W_0 we will use the functions K, U, W of LCK solutions, namely have the metric

$$\begin{aligned} K &= F(r-t) + k^2 \ln(c_1 r + c_2) + q^2 \ln(c_3 t + c_4)\,, \\ U &= k \ln(c_1 r + c_2) + q \ln(c_3 t + c_4)\,, \\ W &= \alpha (c_1 r + c_2)(c_3 t + c_4)\,, \\ q &= k + \varepsilon\,, \qquad \varepsilon = \pm 1\,, \end{aligned} \tag{60}$$

which are solutions of (59) with the energy density:

$$\eta = \frac{(c_2 c_3 - c_1(c_4 + c_3(t-r)))\,\dot{F}}{(c_1 r + c_2)(c_3 t + c_4)}\,. \tag{61}$$

Notice that both c_1 and c_2 cannot vanish simultaneously. This is also true for c_3 and c_4. When $F = \text{const.}$ this solution reduces to the LCK space-time. Also, when we take $c_3 = 0$, $c_4 \neq 0$ we get the radiating Levi-Civita solution of the form:

$$ds^2 = e^{2F} r^{2(k^2-k)}(-dt^2 + dr^2) + r^{2k} dz^2 + \alpha^2 r^{2(1-k)} d\phi^2\,, \tag{62}$$

The energy density is

$$\eta = -\frac{\dot{F}}{r}\,. \tag{63}$$

To have a positive energy density, here we need $\dot{F} < 0$. Also for $c_1 = 0$ $c_2 \neq 0$ we get the radiating Kasner solution with the metric:

$$ds^2 = e^{2F(t-r)} t^{2(k^2-k)}(-dt^2 + dr^2) + t^{2k} dz^2 + t^{2(1-k)} d\phi^2\,. \tag{64}$$

For this radiating Kasner solution it is better to think the coordinates as the Cartesian coordinates. This metric describes a pure radiation moving in the r direction in the Kasner space-time. The energy density is the negative of the Levi-Civita case and $\dot{F}$ must be positive in order to have positive energy density since for this case

$$\eta = \frac{\dot{F}}{t}\,. \tag{65}$$

At $t=0$ this metric has a Kasner type cosmological singularity except $k=0$ and $k=1$.

If we have

$$c_2c_3 - c_1c_4 \geq 0\,, \quad c_1c_3(r-t)\dot{F} > 0\,, \quad \dot{F} > 0\,, \tag{66}$$

or

$$c_2c_3 - c_1c_4 \leq 0\,, \quad c_1c_3(r-t)\dot{F} < 0 \quad \dot{F} < 0\,, \tag{67}$$

in (61) then the energy density of the solution (60) is positive.

For example the following choice

$$c_2 = c_4 = 0\,, \quad c_1 = c_3 = 1\,, \quad F = -a(t-r)^n\,, \quad n = 1,2,3\ldots\,, \tag{68}$$

where $a > 0$ is a constant leads to positive energy solutions when n is even.

This space-time (60) contains in general a Kasner type cosmological singularity at $t=0$ and also it is singular at the axis (we take $c_2 = c_4 = 0$ in (60)). The space-time is not singular for the particular values of the parameters $\varepsilon = 1$, $k=0$ and $\varepsilon = -1$, $k=1$. The cosmological singularity seems to be unavoidable but if one is able to find a regular interior radiating solution containing the symmetry axis, then we can avoid having a line singularity at $r=0$ since our solution could be an exterior solution of a radiating nonstatic cylindrical source. The space-time is well behaved for $t>0$ and $r>0$.

4.1. Some properties of the solution

4.1.1. NP coefficients

Here we analyze Ricci and Weyl scalars of the metric (60) using a null tetrad. For $\varepsilon = 1$ we have the spin coefficients:

$$\begin{aligned}
\Phi_{00} &= \frac{(t-r)F'}{e^{2F}\, r^{2k^2-2k+1}\, t^{2k^2+2k+1}}\,, && (69)\\
\Psi_0 &= \Big(k((1+k)(1+2k)r^2 - 4(k^2-1)rt + (k-1)(2k-1)t^2) && (70)\\
&\quad -((1+2k)r+(1-2k)t)2rtF'\Big)\Big/\Big(2e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}\Big)\,, && (71)\\
\Psi_2 &= \frac{k((1+k)r^2+(1-k)t^2)}{2e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}}\,, && (72)\\
\Psi_4 &= \frac{k((1+k)(1+2k)r^2 + 4(k^2-1)rt + (k-1)(2k-1)t^2}{2e^{2F}\, r^{2k^2-2k+2} t^{2k^2+2k+2}}\,. && (73)
\end{aligned}$$

This shows that only for $\varepsilon = 1$ case, for $k=0$, Ψ_2 and Ψ_4 vanish and the space-time is Petrov type N. For other values of k, Ψ_0, Ψ_2 and Ψ_4 are nonvanishing and Petrov type is I. For $\varepsilon = -1$ we have:

$$\Phi_{00} = \frac{(t-r)F'}{e^{2F}\, r^{k^2-2k+1}\, t^{2k^2-6k+5}}\,, \tag{74}$$

$$\Psi_0 = \Big((k-1)((k-2)(2k-3)r^2 - 4k(k-2)rt + k(2k-1)t^2) \tag{75}$$

$$- ((2k-3)r + (1-2k)t)2rtF'\Big)\Big/\Big(2e^{2F} r^{2k^2-2k+2} t^{2k^2-6k+6}\Big), \tag{76}$$

$$\Psi_2 = \frac{(k-1)(k-2)r^2 - kt^2}{2\,e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2-6k+6}}\,, \tag{77}$$

$$\Psi_4 = \frac{(k-1)(k-2)(2k-3)r^2 + 4k(k-2)rt + k(2k-1)t^2}{2\,e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2-6k+6}}\,. \tag{78}$$

For the $\varepsilon = -1$ case, Ψ_0, Ψ_2 and Ψ_4 are nonvanishing and the space-time is Petrov type I except for $k = 1$ where Ψ_2 and Ψ_4 are vanishing and the space-time is Petrov type N.

4.1.2. Radial acceleration of test particles

The radial acceleration of a test particle initially at rest in a constant radius in the space-time (60) is given by:

$$\ddot{r} = \frac{(k-k^2)r^{-1} - F'}{e^F r^{k^2-k}\, t^{q^2-q}}\,. \tag{79}$$

If we compare (79) with the LCK metric, we see that the main difference is the term $\sim F'$ which characterizes the null radiation. When the F' is positive, the axis is more attractive whereas when it is negative, the axis is less attractive. Thus, the presence of null dust may alter the particle motion.

4.1.3. Circular geodesics

Let us study the equations of a test particle following a circular geodesics in the space-time (60). Let us denote the angular velocity of a particle moving along a geodesics as w, then we have

$$\omega^2 = \frac{(k^2-k)r^{-1} + F'}{(1-k)e^{2F} r^{2k^2-1} t^{2(q^2-1)}}\,, \tag{80}$$

which results

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{k^2-k+rF'}{1-k} - 1\right) e^{2F}\, r^{2(k^2-2)}\, t^{2(q^2-q)}\,. \tag{81}$$

Thus, the circular geodesics are time-like if the expression inside the parentheses is negative, null if it is zero and space-like if it is positive. For the

Levi-Civita and LCK metrics the ranges of k where the geodesics are time-like, space-like or null are the same. However, here we have extra terms proportional to $r\,F'$ and they, in general, depend on time and the radial coordinate. This might have some consequences on particle motion. For example, when time passes, a particle following a circular geodesics may not continue to its motion since such geodesics become space-like. Also for a given k, the circular geodesics might be restricted to a certain radius. Hence, the presence of the null radiation clearly affects the dependence of these ranges to the parameter k.

4.2. A radiating nonstatic string-like object

Using the property of the Einstein–Rosen type solutions, we can construct examples of interior solutions having a nonstatic radiating object with a cosmic string like equation of state and generating outer radiating space-time for particular values of the parameters k and q. The interior and exterior metrics are given by:

$$ds_-^2 = t^4\left(e^{2F(r-t)}(-dt^2+dr^2)+A(r)^2\,d\phi^2\right)+t^{-2}dz^2, \tag{82}$$

$$ds_+^2 = t^4\left(e^{2F(r-t)}(-dt^2+dr^2)+\alpha^2 r^2\,d\phi^2\right)+t^{-2}dz^2\,, \tag{83}$$

with the energy momentum tensor:

$$T_{\mu\nu-} = T^{(R)}_{\mu\nu-}+T^{(S)}_{\mu\nu-}\,, \qquad T_{\mu\nu+}=\eta_+\boldsymbol{k}_\mu\boldsymbol{k}_\nu\,, \tag{84}$$

$$T^{(R)}_{\mu\nu-} = \kappa\,\eta_-\boldsymbol{k}_\mu\boldsymbol{k}_\nu\,, \qquad k_\mu=(1,1,0,0)\,, \tag{85}$$

$$T^{0\ (S)}_{\ 0-} = T^{z\ (S)}_{\ z-} = -\kappa\,\mu\,, \tag{86}$$

$$\eta_- = \frac{(t\,A'-A)\,F'}{t\,A}\,, \qquad \mu=\frac{-A'}{A\,e^{2F}t^{-4}}\,, \qquad \eta_+=\frac{(t-r)\,F'}{t\,r}\,. \tag{87}$$

In these solutions, the interior and exterior metrics can be smoothly matched if the metrics and their first derivatives are continuous on the boundary of the string-like object. Since we have chosen same inner and outer coordinates, this can be fulfilled if $A(r_0)=\alpha r_0$ and $A'(r_0)=\alpha$. These are called Lichnerowicz boundary conditions [21] and can be satisfied for the present case easily. For example, if we choose $A(r)=\sin(b\,r)$ then the junction conditions yield:

$$\alpha=\sin(br_0)\,, \qquad r_0=\frac{\tan(b\,r_0)}{b}\,, \tag{88}$$

which can be easily satisfied since we have more parameters than equations.

Here the problem of these solutions is that, unlike μ, it seems that it may be impossible for η_- to be positive for all ranges of r and t. However, we can avoid negative energy density if we limit the r and t ranges with limited values where $\eta_- > 0$. Then, the solution can represent a radiating nonstationary cosmic string like object emitting null radiation.

5. A stiff fluid of generalized Kasner form

Let us consider the following metric:

$$ds^2 = r^{2(k^2-k)} t^{2(q^2-q+a)} (-dt^2 + dr^2) + r^{2k} t^{2q} dz^2 + P^2 r^{2(1-k)} t^{2(1-q)} d\phi^2 \,, \quad (89)$$

which deviates from LCK solution by a parameter a. For $a = -k^2 - q^2 + 2kq + 1$ we have a nonstatic stiff fluid with the equation of state

$$-G^0{}_0 = G^r{}_r = G^z{}_z = G^\phi{}_\phi = a\ r^{2(k-k^2)} t^{2(q-q^2-a-1)} \,. \quad (90)$$

When $a \to 0$ we recover the LCK solution. This solution has the similar singularity behavior as the LCK metric and it is singular in general at $r = 0$ and $t = 0$. For a special values of k, q we can avoid the singularity at the axis. For example for $k = q = 0$ we have $a = 1$ and the metric becomes

$$ds^2 = t^2(-dt^2 + dr^2 + r^2 d\phi^2) + dz^2, \quad (91)$$

with

$$-G^0{}_0 = G^r{}_r = G^z{}_z = G^\phi{}_\phi = t^{-4}. \quad (92)$$

This metric describes a cosmological solution where at $t = 0$ we have a Big Bang singularity, the Kretchman scalar is $K \sim t^{-8}$, then we have an Universe filled with an isotropic stiff fluid with the equation of state $\rho = p$. Since the energy density goes with t^{-4}, for large t it becomes negligible at late times and practically at $(t \to \infty)$ we get vacuum Universe.

6. Conclusions

In this paper we have first investigated some physical properties of the nonstatic vacuum solutions in cylindrical coordinates with Kasner type time dependence. They can describe the exterior regions of nonstatic line sources and nonstatic straight strings [9, 12] having nonvanishing gravitational potential. For each constant time slice they reduce to the Levi-Civita metric. For each Levi-Civita parameter, s, there are in general two corresponding nonstatic vacuum solutions of this form depending on $\varepsilon = \pm 1$. We have studied some physical properties of this space-time and compared them with the static Levi-Civita space-time. This metric is in the form of generalized Kasner solutions studied before [13]. Also, by a coordinate transformation, it reduces to Einstein–Rosen soliton waves [14]. We have discovered some differences and similarities between LC and LCK space-times. We believe that the form of the metric (14) is suitable for future applications.

Next, we generalized the discussion to a cylindrical nonstatic metric corresponding to an exterior atmosphere of a cylindrical radiating nonstatic source having generalized Kasner type metric. The atmosphere has an outgoing radial pure radiation as well as incoming and outgoing gravitational radiation. For some special cases of our parameter k, these solutions reduce to the exterior field of the radiating nonstatic cosmic string-like objects.

Finally, we have presented a stiff fluid solution by a small deviation of g_{tt} component of the metric from LCK space-time. This solution is also nonstatic and nonstationary and it is another sign off richness of cylindrically symmetric sources of general relativity together with the previous solutions we have discussed.

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