

CHAOTIC AND REGULAR MOTION
IN DISSIPATIVE GRAVITATIONAL BILLIARDS

A.Z. GÓRSKI, T. SROKOWSKI

H. Niewodniczański Institute of Nuclear Physics
Polish Academy of Sciences
Radzikowskiego 152, 31-342 Kraków, Poland*(Received March 21, 2006)*

We consider the motion of a particle subjected to the constant gravitational field and scattered inelastically by oscillating boundaries which possess the shape of parabola, wedge, and hyperbola. The linear dependence of the restitution coefficient on the particle velocity is assumed. We demonstrate that this dynamical system can be either regular or chaotic, which depends on the billiard shape and the oscillation frequency. The trajectory calculations are compared with the experimental data; a good agreement has been achieved. Moreover, the properties of the system has been studied by means of the Liapunov exponents and the Kaplan–Yorke dimension. The period-doubling bifurcation route to chaos has been found. Chaotic and nonuniform patterns visible in the experimental data are interpreted as a result of large embedding dimension.

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From the mathematical point of view, billiards constitute an interesting class of dynamical systems because they exhibit — despite their simplicity — a variety of nonlinear phenomena, including both regular tori and completely chaotic, dense trajectories. Some of them are quite realistic and have a direct physical importance. An example of such a system is the gravitational billiard in which a point-mass particle bounces within a container of a given shape and its motion between the bounces is not free but ballistic. Obviously, the dynamics depends on the billiard shape. For the parabolic boundary the system is integrable and all orbits are regular and stable. Historically, the first study on the gravitational billiards has been performed for the two-dimensional wedge, defined as two intersecting straight lines [1, 2]. The motion in the wedge is fully chaotic if its vertex angle is larger than $\pi/2$ [3], otherwise there is a coexistence of regular and chaotic behaviour. Those findings were successfully tested experimentally [4]. The chaotic, as

well as regular, dynamics is present also for the hyperbolic shape of the gravitational billiard which involves both the wedge and the parabola as its asymptotic limits [5]. The chaotic component prevails for shapes close to the wedge.

All the above approaches assumed that the collisions between the particle and the billiard boundary were elastic. It is natural to require that, in order to make the problem more realistic, one should take into account the energy loss and allow for the energy exchange between the particle and the wall. Such handling of the dissipation is known in nuclear physics as the wall formula [6] and then billiard models are applied. They are derived in the framework of the liquid drop model. The atomic nucleus can then be regarded as a billiard possessing the oscillating boundary given by the Legendre polynomials of various kinds [7]. The system exhibits both regular and chaotic motion [8]. Similar investigations for the gravitational billiards were lacking [9]. Only recently they have been studied — experimentally — under the assumption that the energy loss during the collisions is to be compensated, on the average, by the motion of the container [10]. The authors of Ref. [10] constructed three aluminium containers with parabolic, wedge, and hyperbolic shape which exercised the horizontal oscillations. Inside, a ball of steel was scattered from the boundaries and a camera registered the position of each bounce and the ball velocity. The results clearly indicate the regular motion for the parabola and the chaotic one for the wedge; they also suggest some sort of regularity for the hyperbola at a small driving frequency.

In this paper we present a theoretical analysis of that experiment. To the best of our knowledge, it is the first theoretical approach to such — very realistic — systems. The parameters of our model are the same as in Ref. [10]. We assume the following boundaries: $f(x) = a(x - \bar{x})^2 + c$ (the parabola), $f(x) = b|x - \bar{x}| + c$ (the wedge), and $f(x) = \sqrt{\alpha(1 + \beta(x - \bar{x})^2)} - \delta$ (the hyperbola), where $a = 0.26 \text{ cm}^{-1}$, $b = 1.85$, $c = 0.63 \text{ cm}$, $\alpha = 40.3 \text{ cm}^2$, $\beta = 0.08 \text{ cm}^{-2}$, and $\delta = 4.45 \text{ cm}$. Those three shapes are depicted in Fig. 1; the relative dimensions correspond to the above parameters. The containers oscillate horizontally: $\bar{x}(t) = -A \sin \omega t$, where A is the amplitude and $\omega = 2\pi f$ is the oscillation frequency. Inside the container, the particle is subjected to the constant gravitational acceleration g . Collisions with the boundaries result in the energy loss, quantified by the restitution coefficient $r \in [0, 1]$ which is defined as a ratio of the absolute values of the velocity after and before the collision. The case $r = 1$ corresponds to the elastic collision. It is difficult to decide *a priori* which value for r should be assumed. An experiment with steel particles bounced on a steel block gives $r = 0.93$ [11]. However, taking into account effects connected with the sharing of energy between rotation and translation during the collision reduces this coefficient substantially and the effective r appears of about 0.7. Moreover, r can de-

pend on the velocity and on the scattering angle [11]. The authors of Ref. [10] suggest $r = 0.9$. Since there is no unique empirical evidence about r , in the following we try to draw some conclusions about its value from the comparison of our results with the experimental data of Ref. [10]. In the experiment, the ball movement was kept planar by implementing two sheets of Plexiglas. One can expect that collisions with those sheets resulted in additional noise and in a reduction of the effective restitution coefficient.

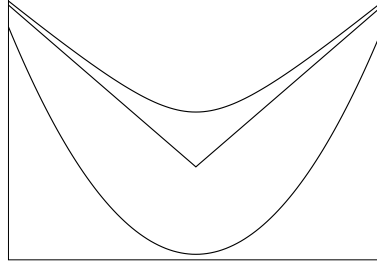


Fig. 1. The billiard shapes which are discussed in this paper: the parabola (lower), the wedge (middle) and the hiperbola (upper).

Let us assume that the particle hits the boundary at the time t_0 with the velocity $\mathbf{v}_0^C$, determined in respect to the frame connected with the billiard, and the collision point is $(x_0^C, y_0^C = f(x_0^C))$. The transformation of particle velocities $\mathbf{v}_0^C \rightarrow \mathbf{v}_1^C$, at this point, has the following form

$$\mathbf{v}_1^C = r (\mathbf{v}_0^C - 2\mathbf{u} (\mathbf{v}_0^C \cdot \mathbf{u})) , \quad (1)$$

where the components of the versor normal to the boundary, $\mathbf{u}$, depend on x_0^C and are given by: $u_x = -f'(x_0^C)/h(x_0^C)$ and $u_y = 1/h(x_0^C)$ with $h(x_0^C) = \sqrt{1 + f'^2(x_0^C)}$. The particle, after being reflected from the boundary, moves along the ballistic trajectory:

$$\begin{aligned} x^C(t) &= x_0^C + v_{x,1}^C(t - t_0) + A \sin \omega t - A \sin \omega t_0 , \\ y^C(t) &= y_0^C + v_{y,1}^C(t - t_0) - g(t - t_0)^2/2 , \end{aligned} \quad (2)$$

for $t > t_0$, up to the next section of this curve with the boundary:

$$(x^C(t_1), y^C(t_1)) = (x_1^C, y_1^C) .$$

The subsequent applying of Eqs. (1) and (2) produces a set of collision events which take place at times t_n . The time evolution can be characterised by the vector $\mathbf{X}_n = (x_n^C, y_n^C, v_{x,n}^C, v_{y,n}^C)$ in the four-dimensional phase space, where the velocities are taken just before the consecutive bounces. Therefore, we

can restrict the dynamics to the billiard boundary and describe it in terms of the following time-dependent mapping:

$$\mathbf{X}_{n+1} = \mathcal{P}_n(\mathbf{X}_n). \quad (3)$$

The above expressions have been formulated in the billiard coordinates because then the simple velocity transformation rule (1) holds. The transformation to the laboratory frame is straightforward: $x = x^C - A \sin \omega t$ and $v_x = v_x^C - A\omega \cos \omega t$; y -components remain the same. The equation for the boundary $f(x)$ reduces the 4-dimensional phase space, in which $\mathcal{P}_n$ is defined, to the 3-dimensional manifold.

For various shapes of the billiard and various driving forces, the mapping $\mathcal{P}_n$ can represent either a regular cycle or a strange attractor. The static, elastic billiard with the parabolic shape is always regular and possesses two stable orbits: the horizontal orbit, connected with the symmetric bouncing between the left and the right part of the boundary at the same height, and the vertical one which involves the top of the parabola [12]. In our case the limit cycle corresponds to the fixed point of the mapping $\mathcal{P}_n^2$ and can be obtained analytically by solving the equation $\mathbf{X}^* = \mathcal{P}^2(\mathbf{X}^*)$. The detailed equations are complicated and will not be presented here. As a result, we yield the stable horizontal orbit which moves to and fro together with the container. The vertical orbit of the elastic billiard shrinks now to a single point. The time interval between consecutive bounces of the horizontal orbit (2-point attractor on the boundary) $\tau^* = \pi/\omega$, moreover $v_y = \pi g/2\omega$. For the parabolic shape the agreement of τ^* with the data [10] is very good. For $r = 0.9$ we get the height of the orbit $y = 158$ cm which exceeds by far the experimental value $y \approx 7.5$ cm. The latter value can be obtained if we assume $r = 0.43$. Therefore, r seems to be well established by the experiment at the value $r = 0.43$ and we can try to apply it in numerical calculations for the other shapes. However, all trajectories calculated in this way, both for the wedge and for the hyperbola, do collapse to the bottom of the billiard instantly. Apparently, the assumption that r can be approximated by a constant cannot be maintained.

In the following we assume the dependence $r(v)$ in the simplest form:

$$r = 1 - \frac{|\mathbf{v}^C|}{v_{\text{cr}}}, \quad (4)$$

if $r > 0.01$, otherwise (for very large velocities) $r = 0.01$. The formula (4) takes into account that the dissipation should rise with the velocity and, since the small velocities are most important, in the first approximation we can neglect the terms higher than linear. The threshold value $r = 0.01$ has been introduced for numerical reasons; events which correspond to such r

are rare and the results are not sensitive to this quantity. The parameter v_{cr} can be determined by comparison with the data. It is especially easy for the parabola case because it is characterised by a simple cycle. The data [10] seem to confirm the presence of a cycle though it appears in a strongly diffused form which may result both from the low resolution in the experiment (*e.g.* due to the surface roughness) and from the rotational degree of freedom, not taken into account in the calculations. We get the position of the 2-point attractor in agreement with the data for $v_{\text{cr}} = 390$ cm/s. We apply this value in the calculations for the other shapes. The restitution coefficient determined in this way is effective, it incorporates *e.g.* the energy loss which results from the collisions with the sheets of Plexiglas which are not taken into account in our calculations.

Fig. 2 presents some properties of the map $\mathcal{P}_n$ — the height y and the time interval between consecutive bounces $\tau_n = t_n - t_{n-1}$ — for the wedge and the hyperbola which is driven by three different frequencies. Each figure represents a single trajectory, evolved up to $t = 2 \times 10^3$ s which corresponds to about 2×10^4 collisions with the boundary. The plot of y for the wedge is strongly nonuniform and indicates a high degree of chaoticity. The most of the points is concentrated just above the line $y_{n+1} = y_n$ and the fractal structure is hardly visible. The figure can be directly compared with the data (see Fig. 3 in Ref. [10]); Both figures are similar though the calculated quantities are extended to slightly larger values than the data. The nonuniformity is clearly visible also in the plot of the time intervals and the region close to the point (0.09, 0.09) is distinguished. The experimental data exhibit the same pattern. For the hyperbola, at the low driving frequency $f = 4.5$ Hz (Fig. 2, h1) the particle abides close to the bottom of the billiard which can be well approximated by the parabola. That results in apparently regular pattern. However, also a stochastic ingredient is present there and a magnification of the upper part of the figure reveals the strange attractor structure. In the experimental data the chaotic component seems to be absent completely. Instead, long-time tails in the plot of time intervals, possessing the intermittent structure, are observed, as well as very small values of the height y . Then the experimental results suggest a more regular motion than the calculations predict. For the larger frequency $f = 5.8$ Hz (Fig. 2, h2) the chaotic behaviour is overwhelming. The bands typical for the strange attractors are visible; it is not the case for the experimental data but such subtle structures may be smeared due to the low resolution. The bands vanish completely if we make the frequency still larger. The last row in Fig. 2 shows that it happens for $f = 8$ Hz (the case not studied experimentally) and the picture is similar to that for the wedge.

Some additional information about the phase space structure can be obtained by plotting the position y versus the tangential velocity after the collision, v_t , at collision points, normalized by y^{max} and v_t^{max} , respectively.

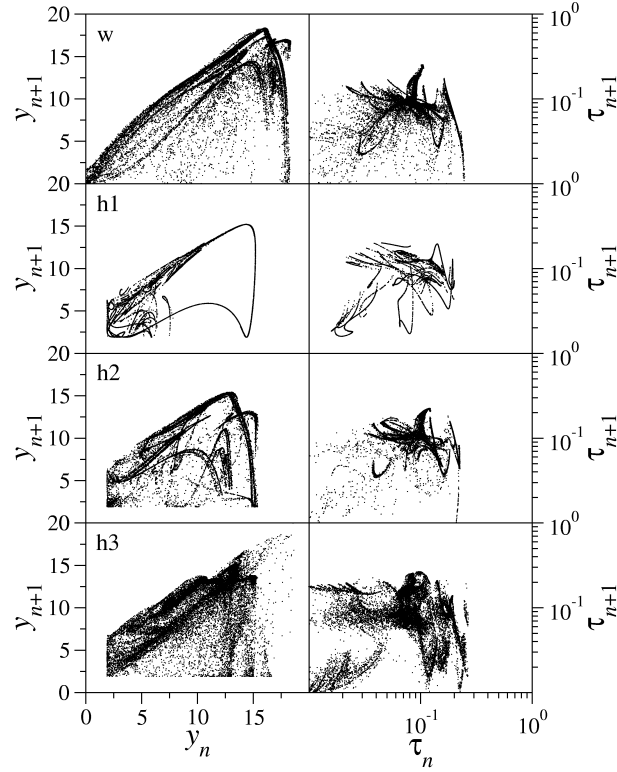


Fig. 2. The coordinate y of a collision point *versus* y for the previous collision (left column) and the time of flight between two subsequent collisions (right column). The rows correspond to the following cases: the wedge with $f = 6.6$ Hz (w) and the hyperbola with: $f = 4.5$ Hz (h1), $f = 5.8$ Hz (h2), and $f = 8$ Hz (h3). The units are: [cm] for y and [s] for the time.

The quantities $y^{\max}$ and $v_t^{\max}$ mean the largest possible values of y and v_t at each collision, obtained under the assumption that the particle is either at rest or bounces almost horizontally at the bottom of the billiard. Therefore, the region around $v_t = 0$ corresponds to the head-on collisions which may result in instability of the motion and the onset of the chaotic behavior. For the parabolic shape we have obtained 2-point limiting cycle, corresponding to very small values of the tangential velocity: $(v_t/v_t^{\max}, y/y^{\max}) = (\pm 0.044, 0.39)$. The experimental result presents itself as a narrow band, due to the noise in the system. Results of the calculations for the other shapes are presented in Fig. 3. The case of the hyperbola for the largest frequency $f = 8$ Hz shows the greatest disorder (Fig. 3, h3) whereas the trajectory for the wedge (Fig. 3, w) is able to fill the entire

region which is allowed by the energy partition rule [10]. The figure for the hyperbola at the intermediate frequency $f = 5.8$ Hz (Fig. 3, h2) indicates, in turn, a pronounced fractal structure. On the other hand, the pattern for $f = 4.5$ Hz is predominantly regular (Fig. 3, h1). However, a vertical strip at small $|v_t|$ bears apparent signs of chaos with a fractal structure, visible after a magnification of this region. Also the experiment [10] distinguishes the region of small tangential velocities for this case but the lack of any fine structure in the data prevents detailed comparisons.

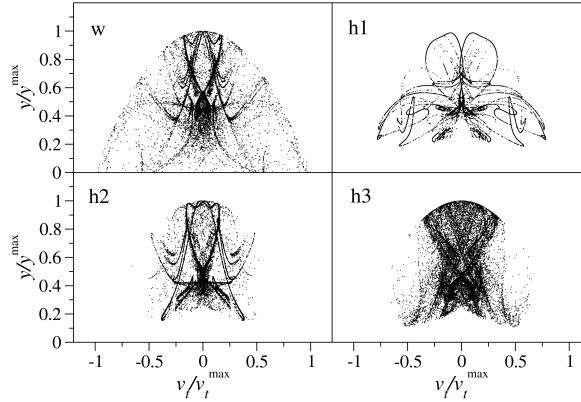


Fig. 3. The normalized coordinate y of a collision point *versus* the normalized tangential velocity v_t at the same point for the cases described in Fig. 2.

Now we calculate the largest Liapunov exponent λ_1 for the mapping $\mathcal{P}_n$ by applying a standard method for billiards [13, 14]. We utilize the fact that the distance between two close trajectories is governed by the linearized equations. Then we perform the time evolution of two trajectories, whose initial conditions differ by δr , by the time interval δt which must be small enough to keep the trajectories close. In the next step we renormalize the relative distance to δr and continue the procedure for a long time. The dependence $\lambda_1(f)$ for the hyperbola is presented in Fig. 4. For $f < 3.5$ Hz trajectories collapse, then finite cycles emerge ($\lambda_1 < 0$) with an exception of the quasiperiodic window at $f \in (3.97, 4.13)$ (at $f \approx 3.97$ Hz the Hopf bifurcation occurs). The transition to chaos proceeds via a period-doubling sequence. By plotting subsequent attractors, we obtained the following bifurcation points: $f_1 = 4.2242721$, $f_2 = 4.22429535$, $f_3 = 4.2243071$, $f_4 = 4.22430985$, and $f_5 = 4.224310525$, which corresponds to the 24-cycle up to 384-cycle, respectively, and we estimated $f_\infty = 4.224310715$. The above points satisfy the scaling relation $f_\infty - f_n \sim \delta^{-n}$ where the scaling parameter $\delta = 4.55$, evaluated from the last two points, agrees reasonable well with the Feigenbaum constant $4.669 \dots$ [15].

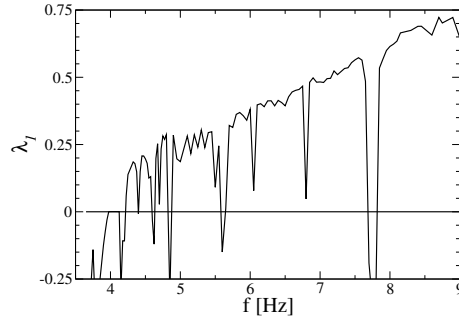


Fig. 4. The largest Liapunov exponent as a function of the frequency for the hyperbolic shape.

The spectrum of four Liapunov exponents λ_i can be obtained by the orthogonalization scheme [16]. More precisely, we evolve trajectories connected with all directions in the phase space and apply the Gramm–Schmidt method at each renormalization step to sort manifolds related to subsequent unstable and stable directions. Finally, we multiply the exponents by the mean time between successive bounces, different for each case. The results are summarized in Table I. All presented cases are characterized by one positive exponent, except the parabola for which all exponents are negative. We can now determine the Kaplan–Yorke dimension which in many cases may be identified with the Hausdorff dimension (the Kaplan–Yorke conjecture [17]). The definition is the following: $D_{KY} = j - \sum_i^j \lambda_i / \lambda_{j+1}$, where j is the largest integer such that $\lambda_1 + \lambda_2 + \dots + \lambda_j > 0$. The results, presented in Table I, indicate that the embedding dimension of the attractor for the cases w and h3 equals 3, *i.e.* it is equal to the entire available manifold. This conclusion explains the nature of chaotic, but also nonuniform and diffused, pattern observed in the figures for those cases, in respect both to the theoretical predictions and to the experimental data: for so high dimensionality of the

TABLE I

The Liapunov exponents for the mapping $\mathcal{P}_n$ and the Kaplan–Yorke dimension for the parabolic shape (p) and the cases described in Fig. 2.

	λ_1	λ_2	λ_3	λ_4	D_{KY}
p	−0.408	−0.579	−0.579	−2.17	0.00
w	0.447	−0.288	−1.17	−1.98	2.14
h1	0.208	−0.579	−1.05	−1.42	1.36
h2	0.367	−0.394	−1.21	−1.78	1.93
h3	0.611	−0.0637	−1.05	−2.08	2.52

attractor its structure, projected to the plane, cannot be clearly visible. For the wedge, that diffused, non-fractal pattern is restricted to the area just above the line $y_{n+1} = y_n$ but in the case h3 it extends to the almost whole picture. On the other hand, the fractal structure is apparent for the case h2.

We have studied the inelastic gravitational billiards with the time-dependent driving which exhibit both regular and chaotic behavior. A simple dependence of the restitution coefficient on the velocity has been proposed. It appears to be essential to get results consistent with the experimental data. The nonuniform patterns, revealed by the experiment, can be explained by the attractor dimensionality. Generally, the model predictions agree quite well with the data. However, the values of the height y are slightly too large, which results in the more irregular motion than the experiment shows, for the hyperbolic shape with small driving frequency. This discrepancy may be a suggestion that the damping at small velocities, as predicted by the simple formula (4), is slightly too weak. To refine this formula, some experimental effort is necessary: the resolution of the data should be improved and, first of all, the restitution coefficient, which is the essential quantity in the model, should be determined in the wide range of the ball velocity.

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