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A UNIVERSAL SHAPE OF EMPIRICAL MASS FORMULA FOR ALL LEPTONS AND QUARKS*

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A specific *universal* shape of empirical mass formula is proposed for all leptons ν_1, ν_2, ν_3 and e^-, μ^-, τ^- as well as all quarks u, c, t and d, s, b of three generations, parametrized by three free constants μ, ε, ξ assuming four different triplets of values. Four such triplets of parameter values are determined or estimated from the present data. Mass spectra in the four cases are related to each other by *shifting* the triplet of parameters μ, ε, ξ. For charged leptons $\xi \simeq 0$ (but probably $\xi \neq 0$). If for them ξ is put to be exactly 0, then $m_\tau = 1776.80$ MeV is *predicted* after the input of experimental m_e and m_μ (the central value of experimental $m_\tau = 1776.99^{+0.29}_{-0.26}$ MeV corresponds to $\xi = 1.8 \times 10^{-3} \neq 0$). For neutrinos $1/\xi \simeq 0$ (but $1/\xi \neq 0$ in the case of normal hierarchy $m^2_{\nu_1} \ll m^2_{\nu_2} \ll m^2_{\nu_3}$). If for neutrinos $1/\xi$ is conjectured to be exactly 0, then $(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) \sim (1.5, 1.2, 5.1) \times 10^{-2}$ eV are *predicted* after the input of experimental estimates $|m^2_{\nu_2} - m^2_{\nu_1}| \sim 8.0 \times 10^{-5}$ eV^2 and $|m^2_{\nu_3} - m^2_{\nu_2}| \sim 2.4 \times 10^{-3}$ eV^2. Thus, the mass ordering of neutrino states 1 and 2 is then *inverted*, while the position of state 3 is *normal*.

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1. Introduction

Some time ago we have found an efficient two-parameter mass formula for charged leptons, predicting reasonably the mass m_τ from the input of experimental masses m_e and m_μ [1]. Then, we have extended this formula to up and down quarks, introducing necessarily a third parameter [2]. Recently, we have considered a few versions of two-parameter mass formula for active neutrinos [3] (also some versions of one-parameter mass formulae have been

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taken into account). In the present paper, on the base of our previous experience (especially of that about the efficient mass formula for charged leptons), we propose a specific *universal* shape of empirical mass formula for all leptons ν_1, ν_2, ν_3 and e^-, μ^-, τ^- as well as all quarks u, c, t and d, s, b of three generations, parametrized by three free constants assuming four different sets of three values. This mass formula reads:

$$m_i = \mu\,\rho_i \left(N_i^2 + \frac{\varepsilon - 1}{N_i^2} - \xi \right) \quad (i = 1, 2, 3)\,, \tag{1}$$

where the numbers

$$N_1 = 1\,, \quad N_2 = 3\,, \quad N_3 = 5 \tag{2}$$

and

$$\rho_1 = \frac{1}{29}\,, \quad \rho_2 = \frac{4}{29}\,, \quad \rho_3 = \frac{24}{29}\,, \tag{3}$$

($\sum_i \rho_i = 1$) are *fixed* elements in all four cases, while μ, ε, ξ are three free parameters that take four different sets of three parameter values (the normalized fractions ρ_i may be called "generation-weighting factors"). Here,

$$(m_1, m_2, m_3) = \left\{ \begin{array}{ll} (m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) & \text{for active neutrinos,} \\ (m_e\,,\, m_\mu\,,\, m_\tau\,) & \text{for charged leptons,} \\ (m_u\,,\, m_c\,,\, m_t\,) & \text{for up quarks,} \\ (m_d\,,\, m_s\,,\, m_b\,) & \text{for down quarks} \end{array} \right. \tag{4}$$

are experimental masses (as we know them). Strictly speaking, m_{ν_i} in Eq. (1) are neutrino Dirac masses $m^{(\mathrm{D})}_{\nu_i}$, subject to recalculation into active-neutrino masses m_{ν_i}. The active mass neutrinos ν_i ($i = 1, 2, 3$) are related to the active weak-interaction neutrinos ν_α ($\alpha = e, \mu, \tau$) through the familiar unitary transformation $\nu_\alpha = \sum_i U_{\alpha i} \nu_i$.

From Eq. (1), rewritten in the explicit form

$$m_1 \;=\; \frac{\mu}{29} (\varepsilon - \xi)\,, \tag{5}$$

$$m_2 \;=\; \frac{\mu}{29} \frac{4}{9} (80 + \varepsilon - 9\,\xi)\,, \tag{6}$$

$$m_3 \;=\; \frac{\mu}{29} \frac{24}{25} (624 + \varepsilon - 25\,\xi)\,, \tag{7}$$

we can evaluate the parameters:

$$\mu \;=\; \frac{29 \times 25}{1536 \times 6} \left[m_3 - \frac{6}{25} (27 m_2 - 8 m_1) \right]\,, \tag{8}$$

$$\varepsilon \;=\; 10\frac{m_3 - \frac{6}{125}(351 m_2 - 904 m_1)}{m_3 - \frac{6}{25}(27 m_2 - 8 m_1)}\,, \tag{9}$$

$$\xi \;=\; 10\frac{m_3 - \frac{6}{125}(351 m_2 - 136 m_1)}{m_3 - \frac{6}{25}(27 m_2 - 8 m_1)}\,, \tag{10}$$

and find also the mass sum rule:

$$m_3 = \frac{1}{1 - \frac{\xi}{10}}\,\frac{6}{125}\left[351\left(1 - \frac{\xi}{26}\right)m_2 - 136\left(1 - \frac{\xi}{34}\right)m_1\right]. \tag{11}$$

Note also the relations

$$\mu = \frac{29 m_1}{\varepsilon - \xi} = \frac{1}{1 - \frac{\xi}{10}}\;\frac{29(9 m_2 - 4 m_1)}{320} \tag{12}$$

and

$$\varepsilon - \xi = \left(1 - \frac{\xi}{10}\right)\,\frac{320 m_1}{9 m_2 - 4 m_1} \tag{13}$$

following from the formula (1).

Since the shape $m_i = F_i(\mu, \varepsilon, \xi)$ $(i = 1, 2, 3)$ of mass formula (1) is *the same* in four cases of fundamental fermions, the four mass spectra of them are related to each other by *shifting* the set of three parameters μ, ε, ξ. Three parameters μ, ε, ξ, assuming four different sets of three parameter values, determine four mass spectra of fundamental fermions. Then, the mass formula (1) gives $m_{f_i} = F_i(\mu^{(f)}, \varepsilon^{(f)}, \xi^{(f)})$ $(i = 1, 2, 3)$, where $f = \nu, l, u, d$ labels four kinds of fundamental fermions: neutrinos, charged leptons, up quarks and down quarks, respectively (the function $F_i(\mu, \varepsilon, \xi)$ is *universal i.e.* , independent of the label f). Strictly speaking, in the case of neutrinos, the mass formula (1) gives directly three neutrino Dirac masses $m^{(\mathrm{D})}_{\nu_i}$ $(i = 1, 2, 3)$ that generically ought to be recalculated afterwards into three physical active-neutrino masses m_{ν_i} $(i = 1, 2, 3)$ through the seesaw mechanism (or another analogical procedure).

Generically, the mass formula (1) does not predict the values of masses, when all parameters are free. However, it may lead to some specific *predictions* for $m_{f_1}, m_{f_2}, m_{f_3}$ (some specific relations for them), if *not* all three $\mu^{(f)}, \varepsilon^{(f)}, \xi^{(f)}$ for a particular f are really free parameters, for instance, if one of them happens to be fixed (*e.g.*, if $\xi^{(l)} = 0$ or if $1/\xi^{(\nu)} = 0$ as discussed below), while two others remain free parameters determined by the input of two of $m_{f_1}, m_{f_2}, m_{f_3}$ and so, predict the third of these masses through the formula (1). This opens a new field of phenomenological investigations.

2. Charged leptons

In this case, the experimental masses are [4]

$$m_e = 0.5109989 \text{ MeV}\,,\ m_\mu = 105.65837 \text{ MeV}\,,\ m_\tau = 1776.99^{+0.29}_{-0.26} \text{ MeV}\,. \tag{14}$$

Thus, from Eqs. (8) and (9) we find (with the central value of m_τ)

$$\mu = 86.0076 \text{ MeV}\,,\ \varepsilon = 0.174069\,, \tag{15}$$

and from Eq. (12)

$$\varepsilon - \xi = \frac{29 m_e}{\mu} = 0.172298\,. \tag{16}$$

Hence,

$$\xi \equiv \varepsilon - (\varepsilon - \xi) = 1.771 \times 10^{-3} = 1.8 \times 10^{-3}\,. \tag{17}$$

The same value of ξ follows from Eq. (10). Of course, we reproduce all three values (14) of masses m_e, m_μ and m_τ (its central value), when we make use of three values (15) and (17) of parameters μ, ε and ξ.

Notice that ξ for charged leptons is very small in comparison with the terms $N_i^2 + (\varepsilon - 1)/N_i^2$ in Eq. (1). If we put for charged leptons exactly $\xi = 0$, we would evaluate from Eqs. (12) and (13)

$$\mu = \frac{29(9m_\mu - 4m_e)}{320} = 85.9924 \text{ MeV} \tag{18}$$

and

$$\varepsilon = \frac{320 m_e}{9m_\mu - 4m_e} = 0.172329\,, \tag{19}$$

respectively, and would *predict* from Eq. (11) the simplified sum rule [1]

$$m_\tau = \frac{6}{125}(351 m_\mu - 136 m_e) = 1776.80 \text{ MeV} \tag{20}$$

in a very good agreement with experimental m_τ given in Eq. (14). In calculating the values (18) and (19) for μ and ε as well as the value (20) for m_τ we use as an input only the experimental m_e and m_μ.

3. Up and down quarks

In the case of up and down quarks, the medium experimental mass values are [4]

$$m_u \sim 2.8 \text{ MeV}\,, \quad m_c \sim 1.3 \text{ GeV}\,, \quad m_t \sim 174 \text{ GeV}\,, \tag{21}$$

and

$$m_d \sim 6 \text{ MeV}\,, \quad m_s \sim 110 \text{ MeV}\,, \quad m_b \sim 4.3 \text{ GeV}\,, \tag{22}$$

respectively. Thus, using Eqs. (8), (9) and (10) we obtain

$$\mu \sim 13 \text{ GeV}\,, \ \varepsilon \sim 9.2\,, \ \xi \sim 9.2 \tag{23}$$

and

$$\mu \sim 280 \text{ MeV}\,, \ \varepsilon \sim 7.5\,, \ \xi \sim 6.9\,, \tag{24}$$

respectively. More precisely,

$$\varepsilon - \xi = \frac{29 m_u}{\mu} \sim 0.0062\,, \ \frac{\varepsilon}{\xi} - 1 \sim 6.7 \times 10^{-4} \tag{25}$$

and

$$\varepsilon - \xi = \frac{29 m_d}{\mu} \sim 0.61\,, \ \frac{\varepsilon}{\xi} - 1 \sim 8.8 \times 10^{-2}\,, \tag{26}$$

respectively. Of course, we can reproduce all quark masses (21) and (22), when we use the values (23) and (24) of parameters μ, ε, ξ (and also Eqs. (25) and (26)).

We can see that for up and down quarks $\xi \simeq \varepsilon$ (especially for up quarks). If we put for them exactly $\xi = \varepsilon$, we would *predict* from Eq. (5) that $m_u = 0$ and $m_d = 0$, and then would evaluate from Eqs. (8) and (10) that $\mu \sim 13$ GeV and $\varepsilon = \xi \sim 9.2$ for up quarks and $\mu \sim 280$ MeV and $\varepsilon = \xi \sim 6.8$ for down quarks.

4. Neutrinos

The situation for neutrinos may be different than for three other kinds of fundamental fermions since, being electrically neutral, they may be Majorana fermions, in contrast to the others which are Dirac fermions. Denote by $\nu_i \equiv \nu_{i\mathrm{L}}$ and $N_i \equiv \nu_{i\mathrm{R}}$ $(i = 1, 2, 3)$ the three active (lefthanded) and three sterile (righthanded) mass neutrinos, and by m_{ν_i} and m_{N_i} their respective masses, being eigenstates of the corresponding 3×3 mass matrices $M^{(\nu)}$ and $M^{(N)}$ (in the flavor basis). Of course, the righthanded-neutrino mass states N_i must not be confused with the numbers $N_i = 1, 3, 5$.

Assume that the seesaw mechanism works and that $M^{(N)}$ commutes with the neutrino Dirac 3×3 mass matrix $M^{(\mathrm{D})}$ (in the flavor basis), giving the neutrino Dirac masses $m^{(\mathrm{D})}_{\nu_i}$ $(i = 1, 2, 3)$ as its eigenstates. Then, we can write

$$m_{\nu_i} = -\frac{m^{(\mathrm{D})\,2}_{\nu_i}}{m_{N_i}}\,, \tag{27}$$

as a consequence of the popular seesaw relation [5]

$$M^{(\nu)} = -M^{(\mathrm{D})\,\mathrm{T}} \frac{1}{M^{(N)}} M^{(\mathrm{D})} \tag{28}$$

with $M^{(\mathrm{D})\,\dagger} = M^{(\mathrm{D})}$ and $M^{(N)\mathrm{T}} = M^{(N)}$. Now, make (for simplicity) the conjecture that the assumed commutation $[M^{(N)}, M^{(\mathrm{D})}] = 0$ is realized (trivially) by the matrix proportionality $M^{(N)} = \zeta\, M^{(\mathrm{D})}$ with a very large parameter $\zeta > 0$. This implies the eigenvalue proportionality

$$m_{N_i} = \zeta m^{(\mathrm{D})}_{\nu_i}\,. \tag{29}$$

Then, from Eqs. (27) and (29) we obtain

$$m_{\nu_i} = -\left(\frac{1}{\zeta}\right) m^{(\mathrm{D})}_{\nu_i} = -\left(\frac{1}{\zeta}\right)^2 m_{N_i}\,. \tag{30}$$

Hence, the commutation $[M^{(\nu)}, M^{(N)}] = 0$. If $m_{\nu_i} > 0$, then it follows that $m_{N_i} < 0$ from Eq. (27) and $m^{(\mathrm{D})}_{\nu_i} < 0$ from Eq. (30) (masses for relativistic spin-1/2 fields may be negative, since only masses squared can be physical).

Now, conjecture for neutrino Dirac masses $m^{(\mathrm{D})}_{\nu_i}$ our mass formula (1):

$$m^{(\mathrm{D})}_{\nu_i} = \mu\,\rho_i\left(N_i^2 + \frac{\varepsilon - 1}{N_i^2} - \xi\right) \quad (i = 1,2,3)\,. \tag{31}$$

Then, making use of the seesaw relation (30) (valid in the case of the proportionality (29)), we obtain for active neutrinos ν_i the following mass formula:

$$m_{\nu_i} = \left(\frac{\mu\,\xi}{\zeta}\right)\rho_i\left[1 - \left(\frac{1}{\xi}\right)\left(N_i^2 + \frac{\varepsilon - 1}{N_i^2}\right)\right] \quad (i = 1,2,3)\,. \tag{32}$$

Hence, rewriting this formula in the explicit form

$$m_{\nu_1} \;=\; \frac{\mu\,\xi/\zeta}{29}\left(1 - \frac{\varepsilon}{\xi}\right)\,, \tag{33}$$

$$m_{\nu_2} \;=\; \frac{\mu\,\xi/\zeta}{29} 4\left[1 - \frac{1}{9}\left(\frac{1}{\xi}\right)(80 + \varepsilon)\right]\,, \tag{34}$$

$$m_{\nu_3} \;=\; \frac{\mu\,\xi/\zeta}{29} 24\left[1 - \frac{1}{25}\left(\frac{1}{\xi}\right)(624 + \varepsilon)\right]\,, \tag{35}$$

we can evaluate the parameters $\mu\,\xi/\zeta$, ε , $1/\xi$ in terms of the active-neutrino masses $m_{\nu_1}, m_{\nu_2}, m_{\nu_3}$:

$$\frac{\mu\xi}{\zeta} \;=\; -\frac{29 \times 125}{1536 \times 3}\left[m_{\nu_3} - \frac{6}{125}(351 m_{\nu_2} - 136 m_{\nu_1})\right]\,, \tag{36}$$

$$\varepsilon \;=\; 10\,\frac{m_{\nu_3} - \frac{6}{125}(351 m_{\nu_2} - 904 m_{\nu_1})}{m_{\nu_3} - \frac{6}{25}(27 m_{\nu_2} - 8 m_{\nu_1})}\,, \tag{37}$$

$$\frac{1}{\xi} \;=\; \frac{1}{10}\;\frac{m_{\nu_3} - \frac{6}{25}(27 m_{\nu_2} - 8 m_{\nu_1})}{m_{\nu_3} - \frac{6}{125}(351 m_{\nu_2} - 136 m_{\nu_1})}\,, \tag{38}$$

and find also the mass sum rule:

$$m_{\nu_3} = \frac{1}{1-\frac{10}{\xi}}\,\frac{6}{25}\left[27\left(1-\frac{26}{\xi}\right)m_{\nu_2} - 8\left(1-\frac{34}{\xi}\right)m_{\nu_1}\right]. \tag{39}$$

Note also the relations

$$\frac{\mu\,\xi}{\zeta} = \frac{29 m_{\nu_1}}{1-\frac{\varepsilon}{\xi}} = \frac{1}{1-\frac{10}{\xi}}\,\frac{29}{32}(9m_{\nu_2} - 4m_{\nu_1}) \tag{40}$$

and

$$1-\frac{\varepsilon}{\xi} = \left(1-\frac{10}{\xi}\right)\,\frac{32 m_{\nu_1}}{9m_{\nu_2} - 4m_{\nu_1}} \tag{41}$$

following from the mass formula (32).

Unfortunately, in spite of the enormous progress in neutrino physics, data on neutrino masses are still unsatisfactory. In fact, only the mass-squared differences are reasonably well estimated [6]:

$$\begin{aligned} |\Delta m^2_{21}| &\equiv |m^2_{\nu_2} - m^2_{\nu_1}| \sim 8.0\times 10^{-5}\ \mathrm{eV}^2\,, \\ |\Delta m^2_{32}| &\equiv |m^2_{\nu_3} - m^2_{\nu_2}| \sim 2.4\times 10^{-3}\ \mathrm{eV}^2\,, \end{aligned} \tag{42}$$

giving in the case of normal hierarchy $m^2_{\nu_1} \ll m^2_{\nu_2} \ll m^2_{\nu_3}$ the estimations

$$\begin{aligned} m_{\nu_2} &\equiv \sqrt{\Delta m^2_{21} + m^2_{\nu_1}} \simeq \sqrt{\Delta m^2_{21}} \sim 8.9\times 10^{-3}\ \mathrm{eV}\,, \\ m_{\nu_3} &\equiv \sqrt{\Delta m^2_{32} + \Delta m^2_{21} + m^2_{\nu_1}} \simeq \sqrt{\Delta m^2_{32} + \Delta m^2_{21}} \sim 5.0\times 10^{-2}\ \mathrm{eV}\,, \end{aligned} \tag{43}$$

when $m^2_{\nu_1}$ can be neglected (it cannot be neglected *e.g.* for $m_{\nu_1} \sim 1\times 10^{-3}$ eV, where $m_{\nu_2} \sim 9.0\times 10^{-3}$ eV and $m_{\nu_3} \sim 5.0\times 10^{-2}$ eV). Their ratios are

$$\frac{\Delta m^2_{32}}{\Delta m^2_{21}} \sim 30\,, \qquad \frac{m_{\nu_3}}{m_{\nu_2}} \sim \sqrt{30} = 5.5\,. \tag{44}$$

Since in the case of normal hierarchy $m^2_{\nu_1} \ll m^2_{\nu_2} \ll m^2_{\nu_3}$ the mass m_{ν_1} is very small, consider the following range of its possible value

$$m_{\nu_1} \sim (0 \text{ to } 1)\times 10^{-3}\ \mathrm{eV}\,. \tag{45}$$

Then, $m_{\nu_2} \sim (8.9 \text{ to } 9.0)\times 10^{-3}$ eV and $m_{\nu_3} \sim 5.0\times 10^{-2}$ eV (*cf.* Eqs. (42)). In this situation, we can evaluate from Eq. (36)

$$\frac{\mu\xi}{\zeta} \sim (7.9 \text{ to } 7.5)\times 10^{-2}\ \mathrm{eV}\,, \tag{46}$$

from Eq. (37)

$$\varepsilon \sim (120 \text{ to } 89) \tag{47}$$

and from Eq. (38)

$$\frac{1}{\xi} \sim (8.1 \text{ to } 6.9) \times 10^{-3}\,, \qquad \frac{\varepsilon}{\xi} \sim (1 \text{ to } 0.61)\,. \tag{48}$$

Of course, we can reproduce all active-neutrino masses $m_{\nu_1}, m_{\nu_2}, m_{\nu_3}$, if we know adequate values of the parameters μ, ε, ξ.

It is interesting to notice that, in the case of normal hierarchy $m_{\nu_1}^2 \ll m_{\nu_2}^2 \ll m_{\nu_3}^2$, the parameter $1/\xi$ for neutrinos is very small, though not 0 (*cf.* Eqs. (48)). In contrast, for charged leptons, the parameter ξ is very small, but not 0 for the central value of experimental m_τ (*cf.* Eq. (17)). This suggests the existence of a kind of *complementarity* between neutrinos and charged leptons.

It seems worthwhile to investigate for neutrinos the strict limit of $1/\xi \to 0$ that implies a new mass sum rule (49) below. Similarly, in the case of charged leptons, the strict limit of $\xi \to 0$ leads to the mass sum rule (20) which is consistent with their experimental masses within the uncertainty limits of m_τ. To this end observe that, in the limit of $1/\xi \to 0$, Eq. (39) gives the simplified sum rule:

$$m_{\nu_3} = \frac{6}{25}\left(27 m_{\nu_2} - 8 m_{\nu_1}\right)\,, \tag{49}$$

implying the equality

$$\left[\frac{6}{25}(27 m_{\nu_2} - 8 m_{\nu_1})\right]^2 - m_{\nu_2}^2 = \Delta m_{32}^2 \equiv \frac{\Delta m_{32}^2}{\Delta m_{21}^2}\left(m_{\nu_2}^2 - m_{\nu_1}^2\right). \tag{50}$$

Denoting

$$r \equiv \frac{m_{\nu_2}}{m_{\nu_1}}\,,\ \lambda \equiv \frac{\Delta m_{32}^2}{\Delta m_{21}^2} \sim \pm 30 \tag{51}$$

and dividing Eq. (50) by $m_{\nu_1}^2$, we find the following quadratic equation for r:

$$\left(\frac{6}{25}\right)^2 (27r - 8)^2 - (1+\lambda) r^2 + \lambda = 0\,. \tag{52}$$

With $\lambda \sim 30 > 0$, this equation gets two complex solutions for r that cannot be physical. Thus, in the strict limit of $1/\xi \to 0$ the neutrino mass orderings $m_{\nu_1}^2 < m_{\nu_2}^2 < m_{\nu_3}^2$ and $m_{\nu_3}^2 < m_{\nu_2}^2 < m_{\nu_1}^2$ leading to $\lambda > 0$ are both excluded. On the contrary, with $\lambda \sim -30 < 0$, there appear two real solutions for r:

$$r \sim \left\{ \begin{array}{ll} -0.46 & < 0 \\ 0.81 & > 0 \end{array} \right.\,, \tag{53}$$

corresponding to

$$m_{\nu_2} \equiv r\, m_{\nu_1} \sim \begin{cases} -0.46\, m_{\nu_1} & < 0 \\ 0.81\, m_{\nu_1} & > 0 \end{cases} , \tag{54}$$

where we choose $m_{\nu_1} > 0$ (here and below, only two decimals are significant as only two are such in λ). The requirement that all three masses m_{ν_i} of active-neutrino triplet ν_1, ν_2, ν_3 should have the same sign, excludes the first solution (54). Then, only the second solution (54) remains as physical.

For such a unique solution it follows that

$$\Delta m_{21}^2 \equiv (r^2-1) m_{\nu_1}^2 \sim -0.35\, m_{\nu_1}^2 < 0 \tag{55}$$

and so, $\Delta m_{21}^2 \sim -8.0 \times 10^{-5}\ \mathrm{eV}^2 < 0$, while $\Delta m_{32}^2 \sim 2.4 \times 10^{-3}\ \mathrm{eV}^2 > 0$ as $\lambda < 0$ in this case. Thus, from Eq. (55)

$$m_{\nu_1}^2 \equiv \frac{\Delta m_{21}^2}{r^2-1} \sim 2.3 \times 10^{-4}\ \mathrm{eV}^2 , \tag{56}$$

$$m_{\nu_2}^2 \equiv \Delta m_{21}^2 + m_{\nu_1}^2 \sim 1.5 \times 10^{-4}\ \mathrm{eV}^2 \tag{57}$$

and

$$m_{\nu_3}^2 \equiv \Delta m_{32}^2 + m_{\nu_2}^2 \sim 2.6 \times 10^{-3}\ \mathrm{eV}^2 . \tag{58}$$

This gives

$$\Delta m_{31}^2 \equiv m_{\nu_3}^2 - m_{\nu_1}^2 \sim 2.3 \times 10^{-3}\ \mathrm{eV}^2$$

and

$$m_{\nu_2}^2 : m_{\nu_1}^2 : m_{\nu_3}^2 \sim 0.65 : 1 : 11 . \tag{59}$$

We can see that

$$m_{\nu_2}^2 < m_{\nu_1}^2 < m_{\nu_3}^2 \tag{60}$$

i.e., the mass ordering of neutrino states 1 and 2 is *inverted*, while the position of neutrino state 3 is *normal*. The hierarchy is here more moderate than in the case of very small m_{ν_1} (Eq. (45)). From Eqs. (56), (57) and (58) we *predict* that

$$m_{\nu_1} \sim 1.5 \times 10^{-2}\ \mathrm{eV}\, ,\ m_{\nu_2} \sim 1.2 \times 10^{-2}\ \mathrm{eV}\, ,\ m_{\nu_3} \sim 5.1 \times 10^{-2}\ \mathrm{eV}\, , \tag{61}$$

implying the proportion

$$m_{\nu_2} : m_{\nu_1} : m_{\nu_3} \sim 0.81 : 1 : 3.3 . \tag{62}$$

It is easy to check that these masses of active neutrinos really satisfy the mass sum rule (49) valid in the strict limit of $1/\xi \to 0$. Recall that the ordering of indices $i = 1, 2, 3$ is established by the form of neutrino mixing matrix $U = (U_{\alpha i})$ transforming active weak-interaction neutrinos ν_α $(\alpha = e, \mu, \tau)$ into active mass neutrinos ν_i $(i = 1, 2, 3)$.

With the values (61) of m_{ν_1} and m_{ν_2} we can evaluate $\mu\xi/\zeta$ and ε/ξ in the strict limit of $1/\xi \to 0$, applying Eqs. (40) and (41) considered in this limit:

$$\frac{\mu\xi}{\zeta} \;\to\; \frac{29}{32}(9m_{\nu_2} - 4m_{\nu_1}) \sim 4.5 \times 10^{-2}\ \mathrm{eV} \tag{63}$$

and

$$\frac{\varepsilon}{\xi} \;\to\; 1 - \frac{32m_{\nu_1}}{9m_{\nu_2} - 4m_{\nu_1}} \sim -8.8 < 0\,. \tag{64}$$

Thus, $\mu \to +0$ and $1/\varepsilon \to -0$ when $1/\xi \to +0$, but only the constants $\mu\xi/\zeta$ and ε/ξ as well as $1/\xi$ appear in the mass formula (32) which, in the limit of $1/\xi \to 0$, takes the form

$$m_{\nu_i} = \mu' \rho_i \left(1 + \frac{\varepsilon'}{N_i^2}\right)\,, \tag{65}$$

where $\mu\xi/\zeta \to \mu' \sim 4.5 \times 10^{-2}$ eV and $-\varepsilon/\xi \to \varepsilon' \sim 8.8 > 0$.

Finally, not passing with $1/\xi$ strictly to 0, consider $1/\xi$ smaller than the values in Eq. (48) corresponding to the range $m_{\nu_1} \sim (0 \text{ to } 1) \times 10^{-3}$ eV (Eq. (45)), where still $m^2_{\nu_1} \ll m^2_{\nu_2} \ll m^2_{\nu_3}$. For instance, put $1/\xi \sim 1.8 \times 10^{-3}$ eV which is the value of neutrino $1/\xi \equiv 1/\xi^{(\nu)}$ related to the charged-lepton $\xi^{(l)}$ (Eq. (17)) through the simplest form of complementarity: $\xi^{(\nu)}\xi^{(l)} \sim 1$ or $1/\xi^{(\nu)} \sim \xi^{(l)} = 1.8 \times 10^{-3}$.

In the case of arbitrary $1/\xi$, we derive for the ratio $r \equiv m_{\nu_2}/m_{\nu_1}$ the following generalized form of Eq. (52) (the latter being valid in the limit $1/\xi \to 0$):

$$\frac{1}{\left(1 - \frac{10}{\xi}\right)^2} \left(\frac{6}{25}\right)^2 \left[27\left(1 - \frac{26}{\xi}\right) r - 8\left(1 - \frac{34}{\xi}\right)\right]^2 - (1+\lambda)r^2 + \lambda = 0\,, \tag{66}$$

where $|\lambda| \sim 30$. With $\lambda \sim 30$ the discriminant of the quadratic algebraic equation (66) for r is negative when $1/\xi$ is smaller than $1/\xi \sim 6.1 \times 10^{-3}$, giving two complex solutions, while with $\lambda \sim -30$ it is always positive, providing two real solutions. If $\lambda \sim 30$, the parameter $1/\xi$ gets the upper bound $1/\xi \sim 8.1 \times 10^{-3}$ corresponding to $m_{\nu_1} \sim 0$ eV, $m_{\nu_2} \sim \sqrt{80}$ eV and $m_{\nu_3} \sim \sqrt{2480}$ eV. If $\lambda \sim -30$, the parameter $1/\xi$ is upper-bound by $1/\xi \sim 3.8 \times 10^{-2}$, what is realized when $m_3 \gtrsim m_1 \gtrsim m_2 \gg \sqrt{\Delta m^2_{32}} \sim \sqrt{2.4} \times 10^{-3}$ eV *i.e.* , when m_3, m_1 and m_2 are practically degenerate.

Now, consider $1/\xi \sim 1.8 \times 10^{-3}$. For $\lambda \sim 30$, both solutions to Eq. (66) turn out to be complex. For $\lambda \sim -30$, both solutions are real, but only one positive:

$$r \sim 0.81 > 0\,, \tag{67}$$

corresponding to $m_{\nu_2} \sim 0.81 m_{\nu_1}$ (a difference with the limiting value (53) appears at the level of further decimals). Then, with $|\Delta m^2_{21}| \sim 8.0 \times 10^{-5}$ eV^2 and $|\Delta m^2_{32}| \sim 2.4 \times 10^{-3}$ eV^2 we obtain

$$m^2_{\nu_1} \sim 2.4 \times 10^{-4}\ \mathrm{eV}^2\,, \quad m^2_{\nu_2} \sim 1.6 \times 10^{-4}\ \mathrm{eV}^2\,, \quad m^2_{\nu_3} \sim 2.6 \times 10^{-4}\ \mathrm{eV}^2 \tag{68}$$

and

$$m_{\nu_1} \sim 1.5\times 10^{-2}\ \mathrm{eV}\,, \quad m_{\nu_2} \sim 1.3\times 10^{-2}\ \mathrm{eV}\,, \quad m_{\nu_3} \sim 5.1\times 10^{-2}\ \mathrm{eV}\,. \tag{69}$$

Thus, the mass ordering of 1 and 2 neutrino states is *inverted* ($\Delta m^2_{21} \sim -8.0 \times 10^{-5}$ eV^2), but the position of 3 neutrino state is *normal* ($\Delta m^2_{32} \sim 2.4 \times 10^{-3}$ eV^2 as $\lambda < 0$).

In the case of $1/\xi \sim 1.8\times 10^{-3}$, using the values (69), we can also evaluate

$$\frac{\mu\xi}{\zeta} = \frac{29}{32}(9m_{\nu_2} - 4m_{\nu_1}) \sim 4.6 \times 10^{-2}\ \mathrm{eV}\,, \quad \frac{\mu}{\zeta} \sim 8.3 \times 10^{-5}\ \mathrm{eV} \tag{70}$$

and

$$\frac{\varepsilon}{\xi} = 1 - \frac{29 m_{\nu_1}}{\frac{\mu\xi}{\zeta}} \sim -8.7\,, \quad \varepsilon \sim -4800\,. \tag{71}$$

So, we can see that, in this case, the results (67)–(71) at the level of two decimals are practically identical to those in the strict limit of $1/\xi \to 0$ (except for μ/ζ and ε which could not be obtained with $\xi \to \infty$).

5. Conclusions

In this paper, we have proposed a specific *universal* shape (1) of empirical mass formula for all fundamental fermions: leptons ν_1, ν_2, ν_3 and e^-, μ^-, τ^- as well as quarks u, c, t and d, s, b of three generations, parametrized by four different sets of three free constants μ, ε, ξ. Mass spectra in the four cases are related to each other by *shifting* the set of three parameters μ, ε, ξ. The parameter μ plays the role of "radius" in the three-dimensional mass space of m_1, m_2, m_3, while ε and ξ are connected with "spherical angles" in this space. Strictly speaking, for active neutrinos the mass formula holds in the form (32) related (in the seesaw mechanism) to the primary mass formula (1), when the latter is valid for neutrino Dirac masses and when the matrix proportionality $M^{(N)} = \zeta\, M^{(\mathrm{D})}$ with a very large parameter $\zeta > 0$ is assumed (for simplicity). In the mass formula (32) the primary parameters μ and ε are replaced by $\mu\xi/\zeta$ and ε/ξ. Note that the seesaw ζ parameter is equal to

$$\zeta \equiv \frac{\mu^{(l)}}{\mu^{(\nu)}/\zeta}\,\frac{\mu^{(\nu)}}{\mu^{(l)}} \sim \frac{86\times 10^{6}}{(6.4 \text{ to } 5.2)\times 10^{-4}}\,\frac{\mu^{(\nu)}}{\mu^{(l)}} = (1.3 \text{ to } 1.7)\times 10^{11}\,\frac{\mu^{(\nu)}}{\mu^{(l)}}\,, \quad (72)$$

where $\mu^{(l)} = 86.0076$ MeV and $\mu^{(\nu)}/\zeta \sim (6.4 \text{ to } 5.2)\times 10^{-4}$ eV for charged leptons and neutrinos, respectively, as can be seen from our listing below. If $\mu^{(\nu)} \simeq \mu^{(l)}$, then $\zeta = O(10^{11})$. If rather $\mu^{(\nu)} : \mu^{(l)} \simeq \mu^{(u)} : \mu^{(d)} \sim 46$ (Eqs. (23) and (24)), then $\zeta = O(10^{13})$.

From the mass formula (1) we have evaluated or estimated the following parameter values:

for e^-, μ^-, τ^- :	$\mu = 86.0076$ MeV ,	$\varepsilon = 0.174069$,	$\xi = 1.8\times 10^{-3}$,
when	$m_e = 0.510999$ MeV,	$m_\mu = 105.658$ MeV,	$m_\tau = 1776.99$ MeV ,
for $u\,,\, c\,,\, t$:	$\mu \sim 13$ GeV ,	$\varepsilon \sim 9.2$,	$\xi \sim 9.2$,
when	$m_u \sim 2.8$ MeV ,	$m_c \sim 1.3$ GeV,	$m_t \sim 174$ GeV ,
for $d\,,\, s\,,\, b$:	$\mu \sim 280$ MeV ,	$\varepsilon \sim 7.5$,	$\xi \sim 6.9$,
when	$m_d \sim 6$ MeV ,	$m_s \sim 110$ MeV ,	$m_b \sim 4.3$ GeV ,
for ν_1, ν_2, ν_3 :	$\frac{\mu\xi}{\zeta} \sim (7.9 \text{ to } 7.5)\times 10^{-2}$ eV,	$\frac{\varepsilon}{\xi} \sim (1 \text{ to } 0.61)$,	$\frac{1}{\xi} \sim (8.1 \text{ to } 6.9)\times 10^{-3}$,
[and	$\frac{\mu}{\zeta} \sim (6.4 \text{ to } 5.2)\times 10^{-4}$ eV,	$\varepsilon \sim (120 \text{ to } 89)$,	$\xi \sim (120 \text{ to } 140)]$,
when	$m_{\nu_1} \sim (0 \text{ to } 1)\times 10^{-3}$ eV,	$m_{\nu_2} \sim (8.9 \text{ to } 9.0)\times 10^{-3}$ eV,	$m_{\nu_3} \sim 5.0\times 10^{-2}$ eV .

We can see that for charged leptons $\xi \simeq 0$ and for neutrinos $1/\xi \simeq 0$ (but here they are not exactly 0 in both cases) while for up and for down quarks $\xi \simeq \varepsilon$. In a way, the value of ξ characterizes four kinds of fundamental fermions.

For charged leptons, in the strict limit of $\xi \to 0$, we have *predicted* one linear relation between m_e, m_μ and m_τ (Eq. (20)), giving $m_\tau = 1776.80$ MeV *versus* the experimental value $m_\tau = 1776.99^{+0.29}_{-0.26}$ MeV, when the experimental values of m_e and m_μ are used. For neutrinos, in the strict limit of $1/\xi \to 0$, we have *predicted* one linear relation between m_{ν_1}, m_{ν_2} and m_{ν_3} (Eq. (49)), leading to their values (61), when the experimental estimates $|m^2_{\nu_2} - m^2_{\nu_1}| \sim 8.0\times 10^{-5}$ eV^2 and $|m^2_{\nu_3} - m^2_{\nu_2}| \sim 2.4\times 10^{-3}$ eV^2 are applied. Then $m^2_{\nu_2} < m^2_{\nu_1} < m^2_{\nu_3}$ *i.e.* , the mass ordering of neutrino states 1 and 2 is *inverted*, while the position of neutrino state 3 is *normal*.

If the simplest form of *complementarity* relation works between the neutrino and charged-lepton ξ's: $1/\xi^{(\nu)} \sim \xi^{(l)} = 1.8\times 10^{-3}$, then the predictions for neutrinos at the level of two decimals are practically identical to those in the strict limit of $1/\xi^{(\nu)} \to 0$.

Finally, we would like to stress that the numbers N_i and ρ_i $(i = 1, 2, 3)$, being fixed structural elements of the mass formula (1), can be constructed and interpreted in the formalism described in Ref. [1]. We hope that, eventually, the three terms appearing in the empirical mass formula (1) will be understood as three kinds of internal interactions within intrinsically composite leptons and quarks (for a proposal see Ref. [7]).

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