

ON THE STABILITY OF A MONOTONICITY
CONSTRAINT FOR THE HUBBLE DIAGRAMTHOMAS SCHÜCKER[†]Centre de Physique Théorique, CNRS–Luminy
Case 907, 13288 Marseille Cedex 9, France
andUnité Mixte de Recherche (UMR 6207)
du CNRS et des Universités Aix–Marseille 1 et 2
et Sud Toulon–Var, Laboratoire affilié à la FRUMAM (FR 2291), France
`schucker@cpt.univ-mrs.fr`ILHEM ZOZZOU[‡]Laboratoire de Physique Théorique, Université Mentouri
25000 Constantine, Algeria
`izouzou1965@yahoo.fr`*(Received August 30, 2006)*

The Hubble diagram has a purely kinematical monotonicity constraint. We study its stability under small scalar perturbations in the Robertson–Walker metric and to small peculiar velocities of emitter and receiver. Our analysis is independent of any dynamical hypothesis.

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1. Introduction

The Hubble diagram of supernovae is one of the main pillars of modern cosmology. Indeed, its experimental accuracy has dramatically increased in recent years and we expect that this accuracy will continue to increase. Certainly the most surprising lesson from the Hubble diagram is the acceleration of the universe. It forces us to revise our assumptions on the dynamics. There is also the alternative proposal [1] to explain the acceleration by perturbations to the maximal spacelike symmetry.

[†] also at Université de Provence, France.[‡] also at Université de Skikda, Algeria.

Independently of this proposal, with increasing experimental accuracy deviations from maximal symmetry might become observable in the Hubble diagram in the not too distant future. The computations of these perturbations to the Hubble diagram [2] started well before they were discovered in the cosmic microwave background and are well understood, including in the framework of various assumptions on the dynamics.

On the other hand, the Hubble diagram allows for a purely kinematical test, *i.e.* a test that relies only on the kinematics of general relativity and that does not require any dynamical assumptions, like Einstein's equation, the matter content of the Universe, its equation of state or inflation. This test is crude but non-trivial. It expresses a monotonicity constraint on the Hubble diagram, whose qualitative origin is easy to understand: the apparent luminosity of a standard candle decreases with increasing distance from the observer except when the universe is spherical and the candle goes behind the observer's equator. The present-day supernova data are compatible with this monotonicity constraint at 95% confidence level [3] and this compatibility translates into a lower bound on the present radius of the universe: $a_0 > 1.2 \times 10^{26}$ m or $\Omega_{k0} > -1.29$. (Adding the dynamical assumptions of Λ CDM improves the bound derived from SN data considerably, $\Omega_{k0} > -1$ at 95% C.L.)

Suppose that with increasing accuracy, future SN data do exhibit a statistically significant non-monotonicity. At least since the meter has been abolished in 1983 we are unwilling to renounce the relativistic kinematics. What alternatives would remain? The first two that come to mind are fluctuations in the absolute luminosity of the standard candle and absorption by dust along the line of sight. We find it hard to believe that these two effects show a z dependence that will mimic a non-monotonicity in the Hubble diagram. We rather expect that they will produce a band in the redshift — apparent luminosity plane (single side band for dust). Two other excuses come to mind next, deviations from maximal symmetry and peculiar velocities of candle and observer. For small perturbations and small peculiar velocities one expects again that the Hubble diagram will widen from a line to a band.

The purpose of this work is to compute this widening to first order in scalar perturbations to the Robertson–Walker metric and to first order in peculiar velocities and without assuming any dynamics. We will also treat the perturbations and the peculiar velocities separately. Of course, anisotropies and peculiar velocities are intimately related, but this relation presupposes a gravitational dynamics. Naturally, the purely kinematical computations of the linear perturbations to the Hubble diagram are much simpler than the dynamical ones in the literature [2]. Therefore, and also because we want to keep the hypotheses minimal we propose a pedestrian approach.

2. Our hypotheses

We assume the kinematics of general relativity:

- The gravitational field is coded in a time-space metric of signature $+ - - -$, we take the velocity of light to be one.
- Massive and massless, pointlike test particles, subject only to gravity, follow timelike and lightlike geodesics.
- Pointlike clocks, *e.g.* atomic clocks, are necessarily massive. They move on timelike curves and indicate proper time τ .

We add the following cosmological hypotheses:

- We assume that the metric is Robertson–Walker with small scalar perturbations [4],

$$d\tau^2 = (1 + 2\Psi)dt^2 - a^2(1 + 2\Phi) [d\chi^2 + s^2 d\theta^2 + s^2 \sin^2 \theta d\varphi^2] . \quad (1)$$

The scale factor $a(t)$ is a strictly positive function of time only, the perturbations Φ and Ψ are arbitrary functions on time-space. The separation of the perturbations into anisotropies and inhomogeneities makes no sense for closed universes and by abuse we call the perturbations collectively anisotropies. We define the function of one variable by $s(\chi) = \sin \chi$ for the sphere, $k = 1$, where $0 < \chi < \pi$ describes the northern hemisphere. We put $s(\chi) = \chi$ for the Euclidean space, $k = 0$, with $0 < \chi < \infty$ and $s(\chi) = \sinh \chi$ for the pseudo-sphere, $k = -1$, with $0 < \chi < \infty$. We take the coordinates χ, θ, φ dimensionless and call them “co-moving position”, while the scale factor is measured in meters. We suppose that the anisotropies Φ and Ψ together with their first dimensionless derivatives $a\partial\Phi/\partial t$, $\partial\Phi/\partial\chi, \dots$ are bounded in absolute value by $\epsilon \ll 1$.

- The test particles are (superclusters of) galaxies and photons. The former are at rest ($t = \tau$, $\chi, \theta, \varphi = \text{constant}$) plus a small “peculiar” velocity. Note that in absence of anisotropies and peculiar velocities, the proper time is universal for all these timelike geodesics and is taken as time coordinate. Under the same conditions, χ measures the dimensionless, co-moving, geodesic distance of a position from the origin at $\chi = 0$.

3. Christoffels

We list the non-vanishing Christoffel symbols in linear approximation in Φ and Ψ where the underlined terms refer to the symmetric case. We denote by ∂_t ordinary and partial derivative with respect to t and similarly for the other coordinates.

$$\begin{aligned}
\Gamma^t_{tt} &= \partial_t \Psi, \\
\Gamma^t_{\chi\chi} &= \underline{a\partial_t a} + 2a\partial_t a(\Phi - \Psi) + a^2\partial_t \Phi, & \Gamma^t_{t\chi} &= \partial_\chi \Psi, \\
\Gamma^t_{\theta\theta} &= s^2 [\underline{a\partial_t a} + 2a\partial_t a(\Phi - \Psi) + a^2\partial_t \Phi], & \Gamma^t_{t\theta} &= \partial_\theta \Psi, \\
\Gamma^t_{\varphi\varphi} &= \sin^2 \theta s^2 [\underline{a\partial_t a} + 2a\partial_t a(\Phi - \Psi) + a^2\partial_t \Phi], & \Gamma^t_{t\varphi} &= \partial_\varphi \Psi, \\
\Gamma^\chi_{tt} &= a^{-2}\partial_\chi \Psi, & \Gamma^\chi_{t\chi} &= \underline{\partial_t a/a} + \partial_t \Phi, \\
\Gamma^\chi_{\chi\chi} &= \partial_\chi \Phi, & \Gamma^\chi_{\chi\theta} &= \partial_\theta \Phi, \\
\Gamma^\chi_{\theta\theta} &= \underline{-s\partial_\chi s} - s^2\partial_\chi \Phi, & \Gamma^\chi_{\chi\varphi} &= \partial_\varphi \Phi, \\
\Gamma^\chi_{\varphi\varphi} &= \sin^2 \theta [\underline{-s\partial_\chi s} - s^2\partial_\chi \Phi], \\
\Gamma^\theta_{tt} &= a^{-2}s^{-2}\partial_\theta \Psi, & \Gamma^\theta_{t\theta} &= \underline{\partial_t a/a} + \partial_t \Phi, \\
\Gamma^\theta_{\chi\chi} &= -s^{-2}\partial_\theta \Phi, & \Gamma^\theta_{\chi\theta} &= \underline{\partial_\chi s/s} + \partial_\chi \Phi, \\
\Gamma^\theta_{\theta\theta} &= \partial_\theta \Phi, & \Gamma^\theta_{\theta\varphi} &= \partial_\varphi \Phi, \\
\Gamma^\theta_{\varphi\varphi} &= \underline{-\sin \theta \cos \theta} - \sin^2 \theta \partial_\theta \Phi, \\
\Gamma^\varphi_{tt} &= a^{-2}s^{-2}\sin^{-2} \theta \partial_\varphi \Psi, & \Gamma^\varphi_{t\varphi} &= \underline{\partial_t a/a} + \partial_t \Phi, \\
\Gamma^\varphi_{\chi\chi} &= -s^{-2}\sin^{-2} \theta \partial_\varphi \Phi, & \Gamma^\varphi_{\chi\varphi} &= \underline{\partial_\chi s/s} + \partial_\chi \Phi, \\
\Gamma^\varphi_{\theta\theta} &= -\sin^{-2} \theta \partial_\varphi \Phi, & \Gamma^\varphi_{\theta\varphi} &= \underline{\cot \theta} + \partial_\theta \Phi, \\
\Gamma^\varphi_{\varphi\varphi} &= \partial_\varphi \Phi.
\end{aligned}$$

4. Anisotropies in the Hubble diagram

The Hubble diagram is a two-dimensional parametric plot. The parameter is the time of flight of the photon between the emitting galaxy and receiving one, us today. The two observables are the apparent luminosity ℓ and the spectral deformation z . According to our model they are functions of the unobserved time of flight, which is therefore treated as parameter and eliminated [5]. These calculations are feasible to first order in the perturbations of the Robertson-Walker metric.

4.1. Trajectories of emitter and receiver

Our first task is to compute how the trajectories of galaxies are perturbed by the anisotropies Φ and Ψ . In the symmetric case and without peculiar velocities these geodesics are at rest with respect to the co-moving coordinates and of course we take the affine parameter p to be proper time τ , $t = p = \tau$, $\chi = \chi_e$, $\theta = \theta_e$, $\varphi = \varphi_e$. We denote by an overdot the ordinary derivative of the trajectory with respect to its affine parameter. To first order, we have

$$\dot{t} =: 1 + \eta_t, \quad \dot{\chi} =: \eta_\chi, \quad \dot{\theta} =: \eta_\theta, \quad \dot{\varphi} =: \eta_\varphi. \quad (2)$$

With the Hubble rate $H := \partial_t a/a$, the deviations η . satisfy

$$\dot{\eta}_t + \partial_t \Psi = 0, \quad (3)$$

$$\dot{\eta}_\chi + 2H\eta_\chi + (a)^{-2} \partial_\chi \Psi = 0, \quad (4)$$

$$\dot{\eta}_\theta + 2H\eta_\theta + (sa)^{-2} \partial_\theta \Psi = 0, \quad (5)$$

$$\dot{\eta}_\varphi + 2H\eta_\varphi + (\sin \theta sa)^{-2} \partial_\varphi \Psi = 0. \quad (6)$$

To first order, the first equation decouples and we get

$$\frac{dt}{d\tau} = 1 - \Psi(\tau, \chi_e, \theta_e, \varphi_e). \quad (7)$$

The other three equations produce peculiar velocities,

$$\begin{aligned} \frac{d\chi}{d\tau} = & - \left\{ \exp \left[- \int_{t_e}^{\tau} 2H(t(\tilde{\tau})) d\tilde{\tau} \right] \right\} \\ & \times \left\{ \int_{t_e}^{\tau} a(\hat{\tau})^{-2} \partial_\chi \overset{\circ}{\Psi}(\hat{\tau}) \exp \left[\int_{t_e}^{\hat{\tau}} 2H(t(\tilde{\tau})) d\tilde{\tau} \right] d\hat{\tau} \right\}, \end{aligned} \quad (8)$$

and similarly for the perpendicular components. In this section we will ignore peculiar velocities, the emitter is held at rest, only its proper time is affected by the perturbations.

The perturbed proper time of the receiver is given by a similar formula.

4.2. Trajectories of photons

We solve (in first order) the geodesic equation of a photon emitted from a galaxy at time t_e and position χ_e , θ_e , φ_e and received at time t_0 , today, at our position, which of course we take in the center of the universe, $\chi = 0$.

Fortunately, the singularity of the metric tensor, equation (1), in the center is only a coordinate singularity. We need the link between the time of flight $t_0 - t_e$ and the geodesic distance χ . To zeroth order in the anisotropies Φ and Ψ , the trajectory of the photon is given by $\dot{t} = a_e/a$, $\dot{\chi} = -a_e/a^2$, $\dot{\theta} = \dot{\varphi} = 0$ with $a_e := a(t_e)$. To first order we write

$$\dot{t} =: a_e/a + \epsilon_t, \quad \dot{\chi} =: -a_e/a^2 + \epsilon_\chi, \quad \dot{\theta} =: \epsilon_\theta, \quad \dot{\varphi} =: \epsilon_\varphi. \quad (9)$$

The geodesic equation becomes:

$$\begin{aligned} \dot{\epsilon}_t - \frac{a_e}{a} H \epsilon_t - 2a_e H \epsilon_\chi + \left(\frac{a_e}{a}\right)^2 \partial_t(\Phi + \Psi) - 2\frac{a_e^2}{a^3} \partial_\chi \Psi \\ + 2\left(\frac{a_e}{a}\right)^2 H(\Phi - \Psi) = 0, \end{aligned} \quad (10)$$

$$\dot{\epsilon}_\chi + 2\frac{a_e}{a} H \epsilon_\chi - 2\frac{a_e^2}{a^3} \partial_t \Phi + \frac{a_e^2}{a^4} \partial_\chi(\Phi + \Psi) = 0, \quad (11)$$

$$\dot{\epsilon}_\theta + 2\frac{a_e}{a} \left[H - \frac{\partial_\chi s}{as} \right] \epsilon_\theta - s^{-2} \frac{a_e^2}{a^4} \partial_\theta(\Phi - \Psi) = 0, \quad (12)$$

$$\dot{\epsilon}_\varphi + 2\frac{a_e}{a} \left[H - \frac{\partial_\chi s}{as} \right] \epsilon_\varphi - \sin^{-2} \theta s^{-2} \frac{a_e^2}{a^4} \partial_\varphi(\Phi - \Psi) = 0. \quad (13)$$

To first order, the first two equations decouple and we get the solution

$$\epsilon_t + a\epsilon_\chi = \frac{a_e}{a}(\overset{\circ}{\Phi} - \overset{\circ}{\Psi}), \quad (14)$$

where $\overset{\circ}{\Phi}$ is the function Φ evaluated along the zeroth order geodesic:

$$\overset{\circ}{\Phi}(p) := \Phi(\overset{\circ}{t}(p), \overset{\circ}{\chi}(p), \theta_e, \varphi_e). \quad (15)$$

The desired link between the time of flight of the photon and its geodesic distance covered is given to first order by:

$$\frac{d\chi}{dt} = -\frac{1}{a} + \frac{\overset{\circ}{\Phi} - \overset{\circ}{\Psi}}{a} =: -\frac{1}{\alpha}. \quad (16)$$

Let us rewrite this equation in terms of the emission time t_e

$$\frac{d\chi}{dt_e} = -\frac{1 - (\Phi - \Psi)(t_e, \overset{\circ}{\chi}(t_e), \theta_e, \varphi_e)}{a(t_e)}, \quad \overset{\circ}{\chi}(t_e) := \int_{t_e}^{t_0} \frac{dt}{a(t)} \quad (17)$$

and integrate

$$\chi(t_e) = \overset{\circ}{\chi}(t_e) - \int_{t_e}^{t_0} \frac{(\Phi - \Psi)(t, \overset{\circ}{\chi}(t), \theta_e, \varphi_e)}{a(t)} dt. \quad (18)$$

To first order and for a fixed direction (θ_e, φ_e) we still have a one-to-one correspondence between emission time and geodesic distance. This correspondence is, of course, direction dependent.

4.3. Spectral deformation

Now we are ready to compute the spectral deformation of the photon emitted at $(t_e, \chi_e, \theta_e, \varphi_e)$ with period T_e measured by the proper time of the emitter τ_e and received at $(t_0, 0, \cdot, \cdot)$. Let us denote by T_0 the Doppler-shifted period as measured by the proper time of the receiver τ_0 . As the period of the photon is infinitesimal with respect to its time of flight we have

$$\chi_e = \int_{t_e}^{t_0} \frac{dt}{\alpha(t)} = \int_{t_e + T_e dt/d\tau_e}^{t_0 + T_0 dt/d\tau_0} \frac{dt}{\alpha(t)}. \quad (19)$$

Taylor expanding we obtain

$$\frac{T_e dt/d\tau_e}{\alpha_e} = \frac{T_0 dt/d\tau_0}{\alpha_0} \quad (20)$$

and the spectral deformation,

$$z := \frac{T_0 - T_e}{T_e} = \frac{a_0}{a_e} [1 + \Phi_0 - \Phi_e] - 1, \quad \Phi_0 := \Phi(t_0, 0, \cdot, \cdot). \quad (21)$$

Note that to first order the spectral deformation is independent of the perturbation Ψ . Note also that the spectral deformation now depends on the direction via $\Phi_e := \Phi(t_e, \chi_e, \theta_e, \varphi_e)$.

4.4. Apparent luminosity

We suppose known the absolute luminosity L of the standard candle in Joule per second. Our hypothesis about photons flying on geodesics implies that the number of photons is constant. The energy E of each photon changes as its frequency $1/T$. A unit time interval $\tilde{T}_e$ during which a certain number of photons are emitted is measured by the proper time τ_e . The apparent luminosity ℓ is measured in Joule per second and per square meter.

Now the unit time interval $\tilde{T}_0$ during which we count the received photons is measured by the proper time τ_0 . The relation between the unit time intervals is computed by a formula similar to equation (19):

$$\frac{\tilde{T}_0}{\tilde{T}_e} = \frac{a_0}{a_e} [1 + \Phi_0 - \Phi_e]. \quad (22)$$

We also need the (orthogonal) detector area determined by a given (infinitesimal) solid angle $d\Omega$ in the direction θ_e, φ_e at time t_0 and at (co-moving) geodesic distance χ_e . This area is measured by the velocity of light times the proper time τ_0 all squared and is given by

$$dA = a_0^2 s^2(\chi_e) [1 + 2\Phi_0] d\Omega. \quad (23)$$

Note that to first order in Φ and Ψ we may still speak about photons propagating in a given solid angle. Finally the apparent luminosity is:

$$\ell = \frac{L}{4\pi} \frac{d\Omega}{dA} \frac{E_0}{E_e} \frac{\tilde{T}_e}{\tilde{T}_0} = \frac{L}{4\pi a_0^2 s^2(\chi_e)} \left(\frac{a_e}{a_0} \right)^2 [1 - 4\Phi_0 + 2\Phi_e]. \quad (24)$$

Note that as for the spectral deformation, the apparent luminosity to first order does not depend on the perturbation Ψ but does depend on the direction.

4.5. Eliminating the time of flight

Our last task is the elimination of the unobserved parameter, the time of flight. To this end we differentiate the relation between time of flight and spectral deformation, equation (21),

$$z(t_e) + 1 = \frac{a_0}{a_e} \left[1 + \Phi_0 - \Phi(t_e, \overset{\circ}{\chi}(t_e), \theta_e, \varphi_e) \right], \quad (25)$$

with respect to t_e :

$$\begin{aligned} \frac{dz}{dt_e} &= -\frac{a_0}{a(t_e)^2} \dot{a}(t_e) \left[1 + \Phi_0 - \Phi(t_e, \overset{\circ}{\chi}(t_e), \theta_e, \varphi_e) \right] + \frac{a_0}{a(t_e)} \\ &\quad \times \left[-\partial_t \Phi(t_e, \overset{\circ}{\chi}(t_e), \theta_e, \varphi_e) + \partial_{\chi} \Phi(t_e, \overset{\circ}{\chi}(t_e), \theta_e, \varphi_e) / a(t_e) \right]. \end{aligned} \quad (26)$$

From this and equation (17) we get

$$\begin{aligned} \frac{d\chi}{dz} &= \frac{d\chi}{dt_e} / \frac{dz}{dt_e} = \frac{1}{a_0 H(z)} \left[1 - \Phi_0 + \Psi(t_e(z), \overset{\circ}{\chi}(z), \theta_e, \varphi_e) \right. \\ &\quad \left. - H(z)^{-1} (\partial_t - a(z)^{-1} \partial_{\chi}) \Phi(t_e(z), \overset{\circ}{\chi}(z), \theta_e, \varphi_e) \right], \end{aligned} \quad (27)$$

with

$$\overset{\circ}{\chi}(z) = \frac{1}{a_0} \int_0^z \frac{d\tilde{z}}{H(\tilde{z})}, \quad (28)$$

and integrating

$$\chi(z) = \overset{\circ}{\chi}(z) + \delta, \quad (29)$$

$$\begin{aligned} \delta(z) := & -\Phi_0 \overset{\circ}{\chi}(z) + \int_0^z \frac{\Psi(t_e(\tilde{z}), \overset{\circ}{\chi}(\tilde{z}), \theta_e, \varphi_e)}{a_0 H(\tilde{z})} d\tilde{z} \\ & - \int_0^z \frac{(\partial_t - a(\tilde{z})^{-1} \partial_\chi) \Phi(t_e(\tilde{z}), \overset{\circ}{\chi}(\tilde{z}), \theta_e, \varphi_e)}{a_0 H(\tilde{z})^2} d\tilde{z}. \end{aligned} \quad (30)$$

4.6. Hubble diagram

Finally the Hubble diagram is to first order in the scalar perturbations Φ and Ψ :

$$\ell(z) = \frac{L}{4\pi a_0^2 (z+1)^2 s^2 (\overset{\circ}{\chi}(z))} \left[1 - 2\Phi_0 - 2 \frac{s'}{s} (\overset{\circ}{\chi}(z)) \delta(z) \right]. \quad (31)$$

Our unit of time is chosen today and here on earth. Therefore we set $\Psi_0 = 0$. Likewise our unit of length or more precisely the numerical value of the speed of light is chosen here and now and we set $\Phi_0 = 0$. This shows that the apparently strongest z -dependence of the linear correction to the Hubble diagram stemming from the term $-\Phi_0 \overset{\circ}{\chi}$ is a coordinate artifact. The remaining terms are weighted averages of Ψ and a derivative of Φ along the zeroth order path of the photon between the standard candle and us today. We have the following bound, which is far from optimal:

$$|\delta(z)| \leq \epsilon \left[\int_0^z \frac{d\tilde{z}}{a_0 H(\tilde{z})} + 2 \int_0^z \frac{d\tilde{z}}{a_0 a(\tilde{z}) H(\tilde{z})^2} \right]. \quad (32)$$

A recent fit to the Hubble diagram [6] up to $z=1.8$ gives $H(z) = H_0(z+1)^{0.69}$, $k=0$, yielding $|\delta| \leq \epsilon[1.21/(a_0 H_0) + 2.88/(a_0 H_0)^2]$.

We conclude that the scalar perturbations produce a Hubble diagram which is a band in the $z\ell$ plane with more or less constant relative vertical width. This relative width is of the same order of magnitude as the perturbations, ϵ .

5. Peculiar velocities in the Hubble diagram

In this section our metric is Robertson–Walker without perturbations, $\Phi = \Psi = 0$. However we admit peculiar velocities of emitter and receiver with respect to the co-moving coordinates χ , θ , φ , or, put more physically, with respect to the cosmic microwave background. We compute the changes in the Hubble diagram to first order in the peculiar velocities divided by the speed of light, which we have set to one.

5.1. Trajectories

We take the line of sight in the direction $\theta = \pi/2$, $\varphi = 0$, and decompose the peculiar velocities into parallel and perpendicular components with respect to this direction: $\vec{v}_e = \vec{v}_{e\parallel} + \vec{v}_{e\perp}$, $\vec{v}_0 = \vec{v}_{0\parallel} + \vec{v}_{0\perp}$. Then we get the initial conditions of the emitter at $t = t_e$

$$\dot{t} = \frac{dt}{d\tau_e} = \sqrt{1 + v_e^2}, \quad \dot{\chi} = \frac{v_{e\parallel}}{a_e}, \quad \dot{\theta} = 0, \quad \dot{\varphi} = \frac{v_{e\perp}}{a_e s_e}, \quad (33)$$

the initial conditions of the receiver at $t = t_0$

$$\dot{t} = \frac{dt}{d\tau_0} = \sqrt{1 + v_0^2}, \quad \dot{\chi} = \frac{v_{0\parallel}}{a_0}, \quad \dot{\theta} = 0, \quad \dot{\varphi} = \frac{v_{0\perp}}{a_0 s_0}, \quad (34)$$

and the initial conditions of the go-between at $t = t_e$

$$\dot{t} = \frac{dt}{dp} = 1, \quad \dot{\chi} = -\frac{1}{a_e}, \quad \dot{\theta} = 0, \quad \dot{\varphi} = 0. \quad (35)$$

The connection between geodesic distance covered by the photon and its time of flight is

$$\chi_e = \int_{t_e}^{t_0} \frac{dt}{a(t)}. \quad (36)$$

5.2. Spectral deformation

To compute the spectral deformation we have a second photon emitted a period T_e later with respect to the proper time τ_e of the emitter. Therefore this photon will be emitted at $t = t_e + \sqrt{1 + v_e^2} T_e$ and at position $\chi = \chi_e + v_{e\parallel} T_e / a_e$. It will be received at $t = t_0 + \sqrt{1 + v_0^2} T_0$ and at position $\chi = v_{0\parallel} T_0 / a_0$. We therefore have

$$\chi_e + \frac{v_{e\parallel} T_e}{a_e} - \frac{v_{0\parallel} T_0}{a_0} = \int_{t_e + \sqrt{1 + v_e^2} T_e}^{t_0 + \sqrt{1 + v_0^2} T_0} \frac{dt}{a(t)}. \quad (37)$$

Taylor expanding as before yields

$$\frac{T_e}{T_0} = \frac{a_e}{a_0} \frac{\sqrt{1+v_0^2} + v_{0\parallel}}{\sqrt{1+v_e^2} + v_{e\parallel}}. \quad (38)$$

To first order, the spectral deformation,

$$z = \frac{a_0}{a_e} (1 - (v_{0\parallel} - v_{e\parallel})) - 1, \quad (39)$$

depends only on the difference of the parallel components of the peculiar velocities.

5.3. Apparent luminosity

We suppose that our candle emits its absolute luminosity L isotropically in its rest frame. When moving with velocity $\vec{v}_e - \vec{v}_0$, its emission profile with respect to the receiver is

$$\frac{dN}{d\Omega} = \frac{L}{4\pi} \frac{1 - |\vec{v}_e - \vec{v}_0|^2}{(1 - |\vec{v}_e - \vec{v}_0| \cos \varphi)^2} \sim \frac{L}{4\pi} \frac{1}{(1 - (v_{0\parallel} - v_{e\parallel}))^2}, \quad (40)$$

where φ is the angle between the line of sight and $\vec{v}_e - \vec{v}_0$. Our convention of orientation is such that $\varphi = 0$ and $v_{0\parallel} - v_{e\parallel} := |\vec{v}_e - \vec{v}_0| \cos \varphi$ positive when the emitter moves towards the receiver in which case the forward emission is enhanced. The deformed emission profile, the first of equations (40), is a special relativistic formula and contains a difference of velocities at different points. Its first order approximation, the second part of (40), only contains projections of velocities onto a geodesics and makes sense also in general relativity.

Neglecting Lorentz contractions, which are quadratic in velocity, the detector area seen by the first photon is $dA = a_0^2 s^2(\chi_e) d\Omega$, while the second photon sees

$$\begin{aligned} dA &= a_0^2 s^2 \left(\chi_e + \frac{v_{e\parallel} T_e}{a_e} - \frac{v_{0\parallel} T_0}{a_0} \right) d\Omega \\ &\sim a_0^2 s^2(\chi_e) \left[1 - 2(v_{0\parallel} - v_{e\parallel}) \frac{T_0 s'(\chi_e)}{a_0 s(\chi_e)} \right] d\Omega. \end{aligned} \quad (41)$$

The term $T_0/(a_0 s_e)$ is an atomic period divided by the time of flight and can safely be dropped. Note also that we do not have to worry about the angle between the detection area and the line of sight which for a moving observer optimizing her efficiency deviates from 90° by an amount quadratic in her velocity. Therefore to first order the apparent luminosity is:

$$\ell = \frac{dN}{d\Omega} \frac{d\Omega}{dA} \frac{E_0}{E_e} \frac{\tilde{T}_e}{\tilde{T}_0} = \frac{L}{4\pi a_0^2 s^2(\chi_e)} \left(\frac{a_e}{a_0} \right)^2 [1 + 4(v_{0\parallel} - v_{e\parallel})]. \quad (42)$$

5.4. Hubble diagram

We eliminate the time of flight as in the preceding section and get the Hubble diagram with its linear perturbations coming from the peculiar velocities of emitter and receiver:

$$\ell(z) = \frac{L}{4\pi a_0^2(z+1)^2 s^2(\overset{\circ}{\chi}(z))} \left[1 + 2\sigma(\overset{\circ}{\chi}(z)) (v_{0\parallel} - v_{e\parallel}) \right]. \quad (43)$$

The function $\sigma(\chi) := 1 - \chi s'(\chi)/s(\chi)$ vanishes identically for flat universes, $k = 0$. For curved universes, $k = \pm 1$, $\sigma(\chi)$ is small,

$$\sigma(\chi) = \frac{k}{3}\chi^2 + \frac{1}{45}\chi^4 + \frac{2k}{945}\chi^6 \pm \dots \quad (44)$$

We conclude that peculiar velocities do not perturb the Hubble diagram to first order if the universe is flat. For curved universes the linear perturbation is small for small redshift and grows with z .

6. Conclusions

After the above calculations, we find it hard to believe, that anisotropies and peculiar velocities can account for violations of the monotonicity constraint in the Hubble diagram.

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