Vol. 38 (2007) *ACTA PHYSICA POLONICA B* No 1

AFFINE LIE ALGEBRAS WITH NON-COMPACT RANK ONE LEVI SUBALGEBRA AND THEIR INVARIANTS

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(Received August 28, 2006)

Using a maximal solvable subalgebra of the Lie algebras $\mathfrak{g}_n = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_n}(n+1)L_1$ we reduce the problem of obtaining the Casimir operators to the integration of only one linear partial differential equation. This reduction allows to prove various results on the admissible degrees of invariants of $\mathfrak{g}_n$, and to construct the quadratic Casimir invariant explicitly for even n. It is moreover shown, that only for 4 values $\mathfrak{g}_n$ arises as a non-trivial contraction of Lie algebras. We also point out that the order of a Casimir operator in a fundamental basis of invariants can exceed the dimension of the Lie algebra.

PACS numbers: 02.20.Sv, 02.20.Qs

1. Introduction

Generalized Casimir operators of Lie algebras have great significance for representation theory and many applications, as their eigenvalues provide labels to distinguish irreducible representations, and allow to infer various properties on the geometry of the co-adjoint orbits [1–3]. For Lie algebras with definite physical meaning, the invariants provide information on quantum numbers that allow to characterize the states of a system [4, 5]. Up to the semi-simple case, which was completely solved in the 60's, there is no general theory that allows to construct the generalized Casimir invariants of Lie algebras, although many results on their structure and number have been developed in the literature. Recently, large classes of Lie algebras have been analyzed for their invariants, such as triangular solvable and nilpotent Lie algebras or solvable algebras with various types of nilradicals, such as Abelian, Heisenberg or naturally graded [6–8]. The same problem has been considered for special classes of non-solvable Lie algebras, like inhomogeneous algebras [4, 9–12].

Physical applications like the classification of kinematical groups [13] motivated the problem of relating the invariants of a Lie algebra $\mathfrak{g}$ with those of a contraction $\mathfrak{g}'$. This procedure allows to compare the states of two systems related by some limiting process, and is also useful for establishing branching rules and the missing label problem [14–16]. However, contracting invariants constitutes a powerful method only when the contracted algebra has the same number of independent invariants as the starting Lie algebra. Otherwise we are led to determine the remaining invariants of the contraction by some other method. The contraction procedure has been employed for various types of inhomogeneous Lie algebras, as well as other contractions of simple Lie algebras [11, 14, 15, 17]. For Lie algebras that are not contractions of others, some special procedures have been developed, like the reduction to total differential equations [18] or some matrix methods related to specific representations [12]. Recently an original method based on moving frames has been proposed [19] and shown of considerable interest for solvable Lie algebras having determined nilradicals. We remark that the latter method is being used to recalculate and simplify the tables of invariants of low dimensional Lie algebras [20].

In this work, we analyze the invariant problem for the affine Lie algebras $\mathfrak{g}_n = \mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_n}(n+1)L_1$, where D_n is an irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. We first point out that for values of n different from $\{2,4,6,10\}$, these algebras cannot be obtained as a contraction of another Lie algebra. Thus the invariants have to be computed directly. By considering a maximal solvable subalgebra of $\mathfrak{g}_n$, we obtain a set of rational functions on the generators that allow to reduce the problem to the integration of only one linear equation. Considering this reduction we are able to make predictions on the degrees of the Casimir operators, which cannot be obtained by alternative methods or that involve lengthy proofs. We also construct explicitly the quadratic Casimir operator for even n, and show that for odd values it does not exist. Further, we indicate an invariant of minimal degree for arbitrary n.

Unless otherwise stated, any Lie algebra $\mathfrak{g}$ considered here is of finite dimension over the field $\mathbb{K}=\mathbb{R}$ and indecomposable, *i.e.* not splittable into a direct sum of ideals. Abelian Lie algebras of dimension n are denoted by nL_1.

2. Preliminaries

Given a basis $\{X_1,...,X_n\}$ of the Lie algebra $\mathfrak{g}$ and the structure tensor $\left\{C_{ij}^k\right\}$, then $\mathfrak{g}$ can be realized in the space $C^\infty(\mathfrak{g}^*)$ by means of the differential operators:

$$\widehat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}\,, \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k \quad (1 \leq i < j \leq n)$ and $\{x_1, ..., x_n\}$ is a dual basis of $\{X_1, ..., X_n\}$. An analytic function $F \in C^{\infty}(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system of partial differential equations:

$$\widehat{X}_i F = 0\,, \qquad 1 \leq i \leq n\,. \tag{2}$$

The cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions (in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis) is obtained from the classical criteria for differential equations, and equals

$$\mathcal{N}(\mathfrak{g}) := \dim\, \mathfrak{g} - \mathrm{rank}\left(C_{ij}^k x_k\right)_{1 \leq i < j \leq \dim \mathfrak{g}}\,, \tag{3}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix associated to the commutator table of $\mathfrak{g}$ over the given basis. We remark that formula (3) can be also deduced naturally using the Maurer–Cartan equations of the algebra [21].

We briefly recall the notions of contraction of Lie algebras for later use. Let $\Phi_t \in \mathrm{Aut}(\mathfrak{g})$ a family of automorphisms of $\mathfrak{g}$, where $t \in \mathbb{N}$. For any $X, Y \in \mathfrak{g}$ define

$$[X, Y]_{\Phi_t} := [\Phi_t(X), \Phi_t(Y)] = \Phi_t([X, Y])\,. \tag{4}$$

It follows that $[X, Y]_{\Phi_t}$ are the brackets over the transformed basis. If the limit

$$[X, Y]_{\infty} := \lim_{t \to \infty} \Phi_t^{-1}\left[\Phi_t(X), \Phi_t(Y)\right] \tag{5}$$

exists for any $X, Y \in \mathfrak{g}$, equation (5) defines a Lie algebra $\mathfrak{g}'$, called the contraction of $\mathfrak{g}$ [22]. If $\mathfrak{g}$ and $\mathfrak{g}'$ are nonisomorphic, we say that the contraction is non-trivial. Contractions also constitute a useful method for computing Casimir invariants of Lie algebras. For example, if $F(X_1, ..., X_n)$ is a Casimir operator of degree p of $\mathfrak{g}$, in the new basis $\{\Phi_t(X_1), ..., \Phi_t(X_n)\}$ it has the form

$$F'(X_1, ..., X_n) := \lim_{t \to \infty} t^p F(\Phi_t(X_1), ..., \Phi_t(X_n)). \tag{6}$$

Using (5) it can be easily verified that $F'(X_1, ..., X_n)$ is a Casimir operator of the contraction[1]. This argument has been applied by different authors to compute invariants of various types of Lie algebras [11, 14, 17, 23].

[1] Formally the method can be enlarged to cover also non-polynomial invariants.

3. General properties of the Lie algebras $\mathfrak{g}_n$

In this section we analyze some general properties of the affine algebras $\mathfrak{g}_n = \mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_n}(n+1)L_1$, where D_n is the irreducible representation of dimension $n+1$ and maximal weight n. Specifically we show that up to four values of n, these algebras cannot be obtained by contraction of Lie algebras. This will imply that the Casimir operators have to be derived directly. Further we recall a known reduction of the system (2) corresponding to these algebras, and prove that for odd values of n no quadratic Casimir operators exist.

In the following we will use for $\mathfrak{g}_n$ a basis $B = \{X_1, ..., X_{n+4}\}$ such that $\{X_1, X_2, X_3\}$ spans the Levi subalgebra with the usual structure tensor $C_{23} = 1, C_{12} = -C_{13} = 2$ and the action over the radical is given by:

$$\begin{aligned}
[X_1, X_{3+l}] &= (n+1-2l)X_l\,, \qquad 1 \leq l \leq n+1 \\
[X_2, X_{4+l}] &= (n+1-l)X_{3+l}\,, \qquad 1 \leq l \leq n \\
[X_3, X_{3+l}] &= lX_{4+l}\,, \quad 1 \leq l \leq n\,.
\end{aligned}$$

Observe in particular that X_2 can be identified with a lowering operator, while X_3 acts as a raising operator. In order to analyze which algebras contract onto some $\mathfrak{g}_n$, we need the following important result[2] concerning the topology of orbits of Lie algebras in the variety M of Lie algebra multiplications over a vector space V. We recall that over V, a Lie algebra is completely determined (up to isomorphism) by its structure tensor μ. Therefore, we can identify a Lie algebra with the pair (V, μ).

Theorem 1 [24] *Let $L = (V, \mu)$ be a Lie algebra and $\mathfrak{s}$ a semi-simple subalgebra of L. There exists a neighborhood $U^\mu \in M$ of μ such that if $\mu_1 \in U^\mu$, then the algebra $L_1 = (V, \mu_1)$ is isomorphic to a Lie algebra $L' = (V, \mu')$ that satisfies the conditions*

(i) $\mu(x, x') = \mu'(x, x'), \forall x, x' \in \mathfrak{s}\,,$

(ii) $\mu(x, y) = \mu'(x, y), \forall x \in \mathfrak{s}, y \in \mathfrak{r}\,.$

Proposition 1 *Let D_n be the irreducible representation of maximal weight $\lambda = n$. If the affine Lie algebra $\mathfrak{g}_n = \mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_n}(n+1)L_1$ is a non-trivial contraction of a Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is semi-simple of rank 2.*

Proof. Suppose that $\mathfrak{g} \rightsquigarrow \mathfrak{g}_n$ is a nontrivial contraction. By the preceding theorem, $\mathfrak{g} = (\mathbb{R}^{n+4}, \mu)$ must have a simple subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ such that

[2] This theorem is known as the stability theorem of Page and Richardson [24].

$$
\begin{aligned}
\mu(X_i, X_j) &= [X_i, X_j]\,, \quad 1 \le i < j \le 3\,, \\
\mu(X_i, X_{3+j}) &= [X_i, X_j]\,, \quad 1 \le i \le 3, 1 \le j \le n+1\,.
\end{aligned}
$$

Since the contraction is non-trivial, the radical $\mathfrak{r}$ of $\mathfrak{g}$ reduces to zero. Otherwise, since D_n is irreducible, it would be Abelian [25], which implies the isomorphism $\mathfrak{g} \simeq \mathfrak{g}_n$, contradicting the assumption. Therefore, $\mathfrak{r} = 0$ and $\mathfrak{g}$ must be semi-simple.

To prove the assertion on the rank, it suffices to consider the complexification of $\mathfrak{g}_n$. Let $\mathfrak{s}$ be a complex semi-simple Lie algebra of rank r and suppose we are given the non-trivial contraction $\mathfrak{s} \rightsquigarrow \mathfrak{g} \otimes \mathbb{C}$. This implies that the reduction of the adjoint representation Γ of $\mathfrak{s}$ with respect to the subalgebra $\mathfrak{sl}(2,\mathbb{C})$ must have the following form:

$$\Gamma = D_2 \oplus D_n\,, \tag{7}$$

i.e. the direct sum of the adjoint representation of $\mathfrak{sl}(2,\mathbb{R})$ and the defining representation. From the classical theory it is known that the number of irreducible components intervening in the decomposition of the adjoint representation for the chain $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{s}$ for a simple algebra $\mathfrak{s}$ equals at least its rank (see Theorem 5.2 in [26] for a proof), thus $r \le 2$, proving that $\mathfrak{s}$ is of rank 2. ∎

We remark that an alternative proof (although quite long and tedious) of this result can be obtained analyzing the representation indices for branching rules of simple Lie algebras [27]. By this result, only the Lie algebras $\mathfrak{g}_n$ for $n = 2, 4, 6, 10$ can arise as a contraction of a non-isomorphic Lie algebra. Table I specifies the real semi-simple Lie algebras that contract onto the $\mathfrak{g}_n$. However, it follows that the procedure of contracting Casimir operators provides a complete set of invariants of $\mathfrak{g}_n$ only for dimensions 6 and 8, *i.e.* $n = 2, 4$. For the remaining cases there are additional invariants that must be obtained directly.

TABLE I

Semi-simple algebras contracting onto $\mathfrak{g}_n$.

n	Dimension	Contracted algebra $\mathfrak{s}$	$\mathcal{N}(\mathfrak{s})$	$\mathcal{N}(\mathfrak{g}_n)$
2	6	$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	2	2
	6	$\mathfrak{so}(3,1)$	2	2
4	8	$\mathfrak{sl}(3,\mathbb{R})$	2	2
	8	$\mathfrak{su}(2,1)$	2	2
6	10	$\mathfrak{sp}(4,\mathbb{R})$	2	4
10	14	NG_2	2	8

Lemma 1 *For any irreducible representation D_n of dimension $n+1 \geq 4$ the Lie algebra $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_n}(n+1)L_1$ has exactly $n-2$ Casimir operators depending only on the variables $\{x_4, \cdots, x_{n+4}\}$ of the radical.*

A proof of this result can be found in [29], Theorem 2. Using the Maurer–Cartan equations of $\mathfrak{g}_n$ the assertion follows at once [21]. This lemma shows that the problem is reduced to integrate the system:

$$\begin{aligned}
\widehat{X}_1'(F) &= \sum_{k=4}^{n+4}(n+8-2k)x_k\frac{\partial F}{\partial x_k} = 0\,, \\
\widehat{X}_2'(F) &= \sum_{k=4}^{n+3}(n+4-k)x_k\frac{\partial F}{\partial x_{k+1}} = 0\,, \\
\widehat{X}_3'(F) &= \sum_{k=5}^{n+4}(k-4)x_k\frac{\partial F}{\partial x_{k-1}} = 0\,.
\end{aligned} \tag{8}$$

However, for values $n \geq 5$ a direct integration of (8) is far from being trivial. For $n = 2, 4$ there is no need to integrate, since the invariants follow by contracting the Casimir operators of the semi-simple algebras of Table I, while for $n = 3$ the only invariant can be computed with determinants [28].

Lemma 2 *For odd n the Lie algebras $\mathfrak{g}_n$ have no quadratic Casimir operator.*

Proof. Let $n = 2m+1$. It follows at once from the first equation of (8) that a quadratic invariant must have the form:

$$C = \sum_{l=1}^{m+1} \lambda_l x_{3+l}x_{2m+6-l}, \lambda_l \in \mathbb{R}\,. \tag{9}$$

In fact, since the action of the Cartan subalgebra of $\mathfrak{g}_n$ on the radical is diagonal, the sum of the weights must cancel, which implies that only generators with opposite weights can appear in (9). Evaluating C in the second equation of the system, we obtain

$$\widehat{X}_2'(C)=\sum_{k=1}^{m}(k\lambda_k+(2m+2-k)\lambda_{k+1})x_{3+k}x_{2m+5-k}+\lambda_{m+1}(m+1)x_{m+4}^2\,. \tag{10}$$

If $\widehat{X}_1'(C) = 0$ holds, then the linear system

$$\begin{aligned}
k\lambda_k + (2m+2-k)\lambda_{k+1} &= 0\,, \quad 1 \leq k \leq m \\
\lambda_{m+1} &= 0\,,
\end{aligned}$$

is satisfied, which only has the trivial solution $\lambda_1 = ... = \lambda_{m+1} = 0$. Therefore, no quadratic Casimir operator exists. ■

We will prove at a later stage that for odd n there are no Casimir operators of odd order. To this extent, we will need a further reduction of system (8) that will simplify the proofs.

4. The reduced system

In this section we reduce system (8) by first obtaining a fundamental set of invariants of a maximal solvable subalgebra of $\mathfrak{g}_n$. Introducing new variables associated to these invariants, we are able to reduce the system to only one equation in $n-1$ variables. Although the complete integration of the reduced equation is still a difficult problem, it allows considerable simplification in the explicit expressions of invariants, and allows us to make precise prediction on the degrees of Casimir operators.

It follows from the brackets of $\mathfrak{g}_n$ that the subalgebra $\mathfrak{b}_n$ generated by $\{X_1, X_2, X_4, ..., X_{n+4}\}$ is solvable of co-dimension one, therefore, maximal in $\mathfrak{g}_n$. This algebra is easily seen to have $n-1$ independent invariants that only depend on the variables $\{X_4, ..., X_{n+4}\}$.

Theorem 2 *For any $n \geq 4$ the system of PDEs*

$$\sum_{k=4}^{n+4} (n+8-2k) x_k \frac{\partial F}{\partial x_k} = 0\,, \tag{11}$$

$$\sum_{k=4}^{n+3} (n+4-k) x_k \frac{\partial F}{\partial x_{k+1}} = 0 \tag{12}$$

associated to $\mathfrak{b}_n$ admits a fundamental system of solutions formed by the $n-1$ rational functions

$$P_1 = x_4^{\frac{4}{n}-2} \left(n x_4 x_6 - \frac{n-1}{2} x_5^2 \right)\,, \tag{13}$$

$$P_k = \Bigg(n^k x_4^k x_{5+k} + (-1)^k \frac{\prod_{s=1}^{k}(n-s)}{(k+1)(k-1)!}\, x_5^{k+1} + \sum_{l=1}^{k-1} (-1)^{k+1} \frac{n^l \prod_{s=l+1}^{k}}{(k-l)!} x_5^{k-l} x_4^l x_{5+l} \Bigg) x_4^{(k+1)(\frac{2}{n}-1)}, \quad 2 \leq k \leq n-2, \tag{14}$$

$$P_{n-1} = x_4^{2-n} \Bigg(n^n x_4^{n-1} x_{4+n} + (-1)^{n-1}(n-1) x_5^n + \sum_{l=1}^{n-2} (-1)^{n-1+l} n^{1+l} x_5^{n-1-l} x_4^l x_{5+l} \Bigg). \tag{15}$$

The proof follows by induction on n. If we now define the degree of P_k as the difference of degrees of numerator Q_k and denominator $R_k = x_4^{(k+1)(\frac{2}{n}-1)}$, we obtain that

$$\deg P_k = \deg Q_k - \deg R_k = \frac{2k+2}{n}, \qquad 1 \leq k \leq n-1. \tag{16}$$

In order to obtain Casimir operators of $\mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_n} (n+1)L_1$ we still have to find $n-2$ independent polynomials $\Phi(P_1, ..., P_{n-1})$ that satisfy the equation

$$\widehat{X}(F) := \sum_{k=5}^{n+4} (k-4) x_k \frac{\partial F}{\partial x_{k-1}} = 0. \tag{17}$$

We observe that if $\Phi = \sum_{i_1 ..., i_r} a^{i_1 \cdots i_r} P_1^{\alpha_{i_1}} ... P_{n-1}^{\alpha_{i_r}}$ is such a (homogeneous) polynomial in the variables $\{x_4, ..., x_{n+4}\}$, then it satisfies the constraints

$$\sum_{j=1}^{r} \alpha_{i_j} \deg P_{i_j} = \sum_j (2k_{i_j} + 2)\alpha_{i_j} = n\zeta, \qquad \text{for some } \zeta \in \mathbb{N}, \tag{18}$$

$$x_4^{\zeta} \quad \text{divides} \quad \sum_{i_1 ..., i_r} a^{i_1 \cdots i_r} Q_1^{\alpha_{i_1}} ... Q_{n-1}^{\alpha_{i_r}}. \tag{19}$$

To illustrate how these rational functions will allow us to reduce system (8) further, we consider the nine dimensional Lie algebra $\mathfrak{g}_5 = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_5} 6L_1$, which has 3 Casimir operators. In this case, system (12) has the structure

$$\begin{aligned} 5x_4 \frac{\partial F}{\partial x_4} + 3x_5 \frac{\partial F}{\partial x_5} + x_6 \frac{\partial F}{\partial x_6} - x_7 \frac{\partial F}{\partial x_7} - 3x_8 \frac{\partial F}{\partial x_8} - 5x_9 \frac{\partial F}{\partial x_9} &= 0, \\ 5x_4 \frac{\partial F}{\partial x_5} + 4x_5 \frac{\partial F}{\partial x_6} + 3x_6 \frac{\partial F}{\partial x_7} + 2x_7 \frac{\partial F}{\partial x_8} + x_8 \frac{\partial F}{\partial x_9} &= 0. \end{aligned}$$

According to Theorem 2, this system has a fundamental system formed by the four rational functions:

$$\begin{aligned} P_1 &= x_4^{-\frac{6}{5}} \left(5x_4x_6 - 2x_5^2\right); \; P_2 = x_4^{-\frac{9}{5}} \left(25x_4^2x_7 + 4x_5^4 - 15x_4x_5x_6\right), \\ P_3 &= x_4^{-\frac{12}{5}} \left(125x_4^3x_8 - 3x_5^4 + 15x_4x_5^2x_6 - 50x_4^2x_5x_7\right), \\ P_4 &= x_4^{-3} \left(3125x_4^4x_9 + 4x_5^5 - 25x_4x_5^3x_6 + 125x_4^2x_5^2x_7 - 625x_4^3x_5x_8\right). \end{aligned}$$

We are left to find solutions of the form $\Phi(P_1, ..., P_4)$ of the equation

$$\widehat{X}(F) = x_5 \frac{\partial F}{\partial x_4} + 2x_6 \frac{\partial F}{\partial x_5} + 3x_7 \frac{\partial F}{\partial x_6} + 4x_8 \frac{\partial F}{\partial x_7} + 5x_9 \frac{\partial F}{\partial x_8} = 0.$$

To this extent, we evaluate how the functions P_i transform under the differential operator $\widehat{X}$. We obtain

$$\widehat{X}\left(P_1\right) = \tfrac{3}{5}x_4^{-\frac{2}{5}}P_2\,, \qquad \widehat{X}\left(P_2\right) = x_4^{-\frac{2}{5}}\left(\tfrac{4}{5}P_3 - \tfrac{6}{5}P_1^2\right)\,,$$

$$\widehat{X}\left(P_3\right) = x_4^{-\frac{2}{5}}\left(\tfrac{1}{5}P_4 - \tfrac{4}{5}P_1P_2\right)\,, \qquad \widehat{X}\left(P_4\right) = -2x_4^{-\frac{2}{5}}P_1P_3\,.$$

Introducing the new variables $u_i = P_i$, we obtain the linear differential operator

$$\widehat{Y} := \left(\frac{3}{5}u_2\frac{\partial}{\partial u_1} + \left(\frac{4}{5}u_3 - \frac{6}{5}u_1^2\right)\frac{\partial}{\partial u_2} + \left(\frac{1}{5}u_4 - \frac{4}{5}u_1u_2\right)\frac{\partial}{\partial u_3} - 2u_1u_3\frac{\partial}{\partial u_4}\right)x_4^{-\frac{2}{5}}\,.$$

If now F satisfies the equality $\widehat{Y}\left(F\right) = 0$, then F is a solution of the equation

$$\frac{3}{5}u_2\frac{\partial F}{\partial u_1} + \left(\frac{4}{5}u_3 - \frac{6}{5}u_1^2\right)\frac{\partial F}{\partial u_2} + \left(\frac{1}{5}u_4 - \frac{4}{5}u_1u_2\right)\frac{\partial F}{\partial u_3} - 2u_1u_3\frac{\partial F}{\partial u_4} = 0\,. \quad (20)$$

As a consequence, a fundamental system of solutions of (20) provides the invariants of $\mathfrak{g}_5$ after replacing u_i by P_i. It is straightforward to verify that the function

$$F_1 = u_4^2 + u_1u_2u_4 + 8u_1u_3^2 - 3u_2^2u_3 + 6u_1^3u_3 - 2u_1^2u_2^2$$

is a solution of (20). Observe that F_1 is not homogeneous in the u_i, but that all monomials have the same degree (16), namely $d = 4$. Replacing u_i by the corresponding rational function P_i we obtain

$$\begin{aligned} F\left(x_1, ..., x_9\right) \;=\; & \frac{1}{x_4^6}\Big(15625x_4^6\Big(-19x_5x_6x_7x_8 - 15x_6^2x_5x_9 + 25x_4x_6x_7x_9 \\ & + 40x_9x_5^2x_7 + 625x_4^2x_9^2 + 9x_5^2x_8^2 - 250x_4x_5x_8x_9 - 2x_6^2x_7^2 \\ & + 6x_6^3x_8 - 15x_4x_7^2x_8 + 6x_7^3x_5 + 40x_4x_6x_8^2\Big)\Big)\,, \end{aligned}$$

which is a Casimir operator of fourth order since x_4^6 is a common factor, according to conditions (18)–(19). In analogous way, the solution

$$\begin{aligned} F_2 \;=\; & {u_4}^4 - 100\,{u_1}^3\,{u_2}^2\,{u_3}^2 - 135\,{u_2}^4\,{u_3}^2 + 400\,{u_1}^4\,{u_3}^3 + 720\,u_1\,{u_2}^2\,{u_3}^3 \\ & - 640\,{u_1}^2\,{u_3}^4 + 256\,{u_3}^5 + 80\,{u_1}^3\,{u_2}^3\,u_4 + 108\,{u_2}^5\,u_4 - 360u_1^4u_2u_3u_4 \\ & - 630u_1u_2^3u_3u_4 + 560u_1^2u_2u_3^2u_4 320u_2u_3^3u_4 + 108u_1^5u_4^2 + 165u_1^2u_2^2u_4^2 \\ & - 180u_1^3u_3u_4^2 + 90u_2^2u_3u_4^2 + 80u_1u_3^2u_4^2 - 30u_1u_2u_4^3\,, \end{aligned}$$

provides a Casimir operator of degree 8. Observe that the degree of the monomials is always 8. Expressing this solution in terms of the x_i's, we obtain 59 terms. There exists a third solution with 59 terms in the u_i's generating an order 12 invariant[3], the explicit expression of which is given in Table II.

This example suggests that a similar argument can be valid in the general case, allowing to reduce system (8) to a unique equation. The first step is to show that the polynomials P_k transform adequately when inserted in equation (17).

Proposition 2 *Consider the differential operator $\widehat{X}$ defined by:*

$$\widehat{X} := \sum_{k=5}^{n+4} (k-4) x_k \frac{\partial}{\partial x_{k-1}} \,. \tag{21}$$

Then the following relations hold:

$$\widehat{X}(P_1) = \frac{3}{n} x_4^{-\frac{2}{n}} P_2 \,, \tag{22}$$

$$\widehat{X}(P_k) = x_4^{-\frac{2}{n}} \left(\frac{k+2}{n} P_{k+1} - \frac{2n-2k}{n} P_1 P_{k-1} \right), \quad 2 \leq k \leq n-3, \tag{23}$$

$$\widehat{X}(P_{n-2}) = x_4^{-\frac{2}{n}} \left(\frac{1}{n} P_{n-1} - \frac{4}{n} P_1 P_{n-3} \right), \tag{24}$$

$$\widehat{X}(P_{n-1}) = -2 x_4^{-\frac{2}{n}} P_1 P_{n-2} \,. \tag{25}$$

By this result, the Casimir operators of $\mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_n} (n+1) L_1$ follow from the solutions of the equation

$$\sum_{k=1}^{n-1} \widehat{X}(P_k) \frac{\partial F}{\partial P_k} = 0 \,. \tag{26}$$

Since all expressions $\widehat{X}(P_k)$ have the common factor $x_4^{-\frac{2}{n}}$, we can introduce the new variables $u_k = P_k$, obtaining an equation depending only on the u_k. This fact shows the second reduction of (8).

Proposition 3 *The obtainment of the Casimir operators of the affine Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_n} (n+1) L_1$ is reduced to the integration of the linear equation:*

$$\begin{aligned} 3u_2 \frac{\partial F}{\partial u_1} &+ \sum_{k=2}^{n-3} \left((k+2) u_{k+1} + 2(k-n) u_1 u_{k-1} \right) \frac{\partial F}{\partial u_k} \\ &+ (u_{n-1} - 4u_1 u_{n-3}) \frac{\partial F}{\partial u_{n-2}} - 2n u_1 u_{n-2} \frac{\partial F}{\partial u_{n-1}} = 0 \,. \end{aligned} \tag{27}$$

[3] Expressed over the variables $\{x_4, ..., x_9\}$, it contains 244 terms.

Explicit solutions of equation (27) for low values of n. TABLE II

N	Solutions
4	$F_1 = u_3 + \frac{4}{3}u_1^2$
	$F_2 = u_1^3 - \frac{189}{16}u_2^2 + \frac{63}{8}u_1u_3$
5	$F_1 = u_4^2 + u_1u_2u_4 + 6u_1^3u_3 + (8u_1u_3 - 3u_2^2)u_3 - 2u_1^2u_2^2$
	$F_2 = {u_4}^4 - 100{u_1}^3{u_2}^2{u_3}^2 - 135{u_2}^4{u_3}^2 + 400{u_1}^4{u_3}^3 + 720u_1{u_2}^2{u_3}^3 - 640{u_1}^2{u_3}^4 + 256{u_3}^5 + 80{u_1}^3{u_2}^3u_4 + 108{u_2}^5u_4 - 360u_1^4u_2u_3u_4$
	$-630u_1u_2^3u_3u_4 + 560u_1^2u_2u_3^2u_4 - 320u_2u_3^3u_4 + 108u_1^5u_4^2 + 165u_1^2u_2^2u_4^2 - 180u_1^3u_3u_4^2 + 90u_2^2u_3u_4^2 + 80u_1u_3^2u_4^2 - 30u_1u_2u_4^3$
	$F_3 = {u_4}^6 + \frac{100}{13}{u_1}^6{u_2}^6 + \frac{540}{13}{u_1}^3{u_2}^8 + \frac{729}{13}{u_2}^{10} - \frac{900}{13}{u_1}^7{u_2}^4u_3 - \frac{6030}{13}{u_1}^4{u_2}^6u_3 - \frac{9720}{13}u_1{u_2}^8u_3 + \frac{2025}{13}{u_1}^8{u_2}^2{u_3}^2$
	$+\frac{23880}{13}{u_1}^5{u_2}^4{u_3}^2 + \frac{49410}{13}{u_1}^2{u_2}^6{u_3}^2 - \frac{41460}{13}{u_1}^6{u_2}^2{u_3}^3 - \frac{117990}{13}{u_1}^3{u_2}^4{u_3}^3 - \frac{1215}{13}{u_2}^6{u_3}^3 + \frac{33120}{13}{u_1}^7{u_3}^4 + \frac{125760}{13}{u_1}^4{u_2}^2{u_3}^4$
	$+\frac{9720}{13}u_1{u_2}^4{u_3}^4 - \frac{27264}{13}{u_1}^5{u_3}^5 - \frac{18240}{13}{u_1}^2{u_2}^2{u_3}^5 + \frac{12800}{13}{u_1}^3{u_3}^6 + \frac{11520}{13}{u_2}^2{u_3}^6 - \frac{30720}{13}u_1{u_3}^7 + \frac{540}{13}{u_1}^8{u_2}^3u_4 + \frac{954}{13}{u_1}^5{u_2}^5u_4$
	$+\frac{1620}{13}{u_1}^2{u_2}^7u_4 - \frac{2430}{13}{u_1}^9u_2u_3u_4 - \frac{4140}{13}{u_1}^6{u_2}^3u_3u_4 - \frac{14940}{13}{u_1}^3{u_2}^5u_3u_4 - \frac{1620}{13}{u_1}^7u_2{u_3}^2u_4 + \frac{45540}{13}{u_1}^4{u_2}^3{u_3}^2u_4$
	$+\frac{1215}{13}u_1{u_2}^5{u_3}^2u_4 - \frac{60000}{13}{u_1}^5u_2{u_3}^3u_4 - \frac{24720}{13}{u_1}^2{u_2}^3{u_3}^3u_4 + \frac{50880}{13}{u_1}^3u_2{u_3}^4u_4 - \frac{23040}{13}{u_2}^3{u_3}^4u_4 + \frac{69120}{13}u_1u_2{u_3}^5u_4$
	$+\frac{729}{13}{u_1}^{10}{u_4}^2 + \frac{1620}{13}{u_1}^7{u_2}^2{u_4}^2 + \frac{3810}{13}{u_1}^4{u_2}^4{u_4}^2 - \frac{15390}{13}{u_1}^5{u_2}^2u_3{u_4}^2 + \frac{7695}{13}{u_1}^2{u_2}^4u_3{u_4}^2 + \frac{24300}{13}{u_1}^6{u_3}^2{u_4}^2$
	$-\frac{10980}{13}{u_1}^3{u_2}^2{u_3}^2{u_4}^2 + \frac{14175}{13}{u_2}^4{u_3}^2{u_4}^2 - \frac{34080}{13}{u_1}^4{u_3}^3{u_4}^2 - \frac{44880}{13}u_1{u_2}^2{u_3}^3{u_4}^2 - \frac{6720}{13}{u_1}^2{u_3}^4{u_4}^2 + \frac{3072}{13}{u_3}^5{u_4}^2$
	$-\frac{5855}{13}{u_1}^3{u_2}^3{u_4}^3 - \frac{2160}{13}{u_2}^5{u_4}^3 + \frac{22500}{13}{u_1}^4u_2u_3{u_4}^3 + \frac{4950}{13}u_1{u_2}^3u_3{u_4}^3 + \frac{11760}{13}{u_1}^2u_2{u_3}^2{u_4}^3 - \frac{3840}{13}u_2{u_3}^3{u_4}^3 - \frac{2160}{13}{u_1}^5{u_4}^4$
	$-\frac{135}{13}{u_1}^2{u_2}^2{u_4}^4 - \frac{2070}{13}{u_1}^3u_3{u_4}^4 + \frac{1035}{13}{u_2}^2u_3{u_4}^4 + \frac{1080}{13}u_1{u_3}^2{u_4}^4 - \frac{345}{13}u_1u_2{u_4}^5$
6	$F_1 = u_5 - \frac{9}{10}u_2^2 + \frac{12}{5}u_1u_3$
	$F_2 = u_5^2 - \frac{156}{5}u_1u_2u_5 + \frac{369}{20}u_2^2u_5 + \frac{72}{5}u_1^3 + 90u_1u_4^4 - 54u_2u_3u_4 - \frac{108}{5}u_1^2u_2u_4 + \frac{72}{5}u_3^3 + \frac{27}{5}u_1u_2^2u_3 - \frac{81}{80}u_2^4$
	$F_3 = {u_5}^3 + \frac{729}{80}{u_2}^6 - \frac{729}{10}u_1{u_2}^4u_3 + \frac{729}{5}{u_1}^2{u_2}^2{u_3}^2 + \frac{729}{5}{u_1}^2{u_2}^3u_4 - \frac{2916}{5}{u_1}^3u_2u_3u_4 - \frac{729}{10}{u_2}^3u_3u_4 + \frac{1458}{5}u_1u_2{u_3}^2u_4$
	$+\frac{243}{2}u_1{u_2}^2{u_4}^2 - 972{u_1}^2u_3{u_4}^2 + 405{u_3}^2{u_4}^2 - \frac{2025}{2}u_2{u_4}^3 + \frac{3159}{40}{u_2}^4u_5 - \frac{3969}{10}u_1{u_2}^2u_3u_5 + \frac{1296}{5}{u_1}^2{u_3}^2u_5 - \frac{864}{5}{u_3}^3u_5$
	$+\frac{3321}{5}{u_1}^2u_2u_4u_5 + \frac{891}{2}u_2u_3u_4u_5 + 270u_1{u_4}^2u_5 - \frac{864}{5}{u_1}^3{u_5}^2 - \frac{1431}{80}{u_2}^2{u_5}^2 - \frac{504}{5}u_1u_3{u_5}^2 + \frac{2916}{5}{u_1}^4{u_4}^2$
	$F_4 = {u_5}^5 + 864{u_1}^3{u_2}^2{u_3}^2{u_4}^2 + 972{u_2}^4{u_3}^2{u_4}^2 - 3456{u_1}^4{u_3}^3{u_4}^2 - 5184u_1{u_2}^2{u_3}^3{u_4}^2 + 4608{u_1}^2{u_3}^4{u_4}^2 - 1536{u_3}^5{u_4}^2$
	$-3456{u_1}^3{u_2}^3{u_4}^3 - 3888{u_2}^5{u_4}^3 + 15552{u_1}^4u_2u_3{u_4}^3 + 22680u_1{u_2}^3u_3{u_4}^3 - 20160{u_1}^2u_2{u_3}^2{u_4}^3 + 9600u_2{u_3}^3{u_4}^3 + 64{u_1}^3{u_5}^4$
	$-29700{u_1}^2{u_2}^2{u_4}^4 + 32400{u_1}^3u_3{u_4}^4 - 13500{u_2}^2u_3{u_4}^4 - 12000u_1{u_3}^2{u_4}^4 + 22500u_1u_2{u_4}^5 - 3125{u_4}^6 - 576{u_1}^3{u_2}^2{u_3}^3u_5$
	$-648{u_2}^4{u_3}^3u_5 + 2304{u_1}^4{u_3}^4u_5 + 3456u_1{u_2}^2{u_3}^4u_5 - 3072{u_1}^2{u_3}^5u_5 + 1024{u_3}^6u_5 + 2592{u_1}^3{u_2}^3u_3u_4u_5 + 2916{u_2}^5u_3u_4u_5$
	$-11520{u_1}^4u_2{u_3}^2u_4u_5 - 16848u_1{u_2}^3{u_3}^2u_4u_5 + 14976{u_1}^2u_2{u_3}^3u_4u_5 - 6912u_2{u_3}^4u_4u_5 - 864{u_1}^4{u_2}^2{u_4}^2u_5 - 972u_1{u_2}^4{u_4}^2u_5$
	$+20736{u_1}^5u_3{u_4}^2u_5 + 27216{u_1}^2{u_2}^2u_3{u_4}^2u_5 - 28896{u_1}^3{u_3}^2{u_4}^2u_5 + 9720{u_2}^2{u_3}^2{u_4}^2u_5 + 10560u_1{u_3}^3{u_4}^2u_5 + 720{u_1}^3u_2{u_4}^3u_5$
	$+1350{u_2}^3{u_4}^3u_5 - 19800u_1u_2u_3{u_4}^3u_5 - 1500{u_1}^2{u_4}^4u_5 + 3750u_3{u_4}^4u_5 - 648{u_1}^3{u_2}^4{u_5}^2 - 729{u_2}^6{u_5}^2 + 3456{u_1}^4{u_2}^2u_3{u_5}^2$
	$+4860u_1{u_2}^4u_3{u_5}^2 - 3072{u_1}^5{u_3}^2{u_5}^2 - 8208{u_1}^2{u_2}^2{u_3}^2{u_5}^2 + 4352{u_1}^3{u_3}^3{u_5}^2 + 1440{u_2}^2{u_3}^3{u_5}^2 - 1536u_1{u_3}^4{u_5}^2 + 64{u_3}^3{u_5}^3$
	$-4608{u_1}^5u_2u_4{u_5}^2 - 5832{u_1}^2{u_2}^3u_4{u_5}^2 + 5760{u_1}^3u_2u_3u_4{u_5}^2 - 3564{u_2}^3u_3u_4{u_5}^2 + 576u_1u_2{u_3}^2u_4{u_5}^2 + 192{u_1}^4{u_4}^2{u_5}^2$
	$+4590u_1{u_2}^2{u_4}^2{u_5}^2 + 1080{u_1}^2u_3{u_4}^2{u_5}^2 - 1200{u_3}^2{u_4}^2{u_5}^2 - 750u_2{u_4}^3{u_5}^2 + 1024{u_1}^6{u_5}^3 + 1440{u_1}^3{u_2}^2{u_5}^3 + 243{u_2}^4{u_5}^3$
	$-1536{u_1}^4u_3{u_5}^3 - 108u_1{u_2}^2u_3{u_5}^3 + 480{u_1}^2{u_3}^2{u_5}^3 + 64{u_3}^3{u_5}^3 - 23328{u_1}^5{u_4}^4 - 1296{u_1}^2u_2u_4{u_5}^3 + 360u_2u_3u_4{u_5}^3$
	$+150u_1{u_4}^2{u_5}^3 - 27{u_2}^2{u_5}^4 - 48u_1u_3{u_5}^4$

This equation has $n-2$ independent solutions, which can moreover chosen to be polynomials in the u_k. Replacing these variables by the rational functions P_k, we obtain invariants of the Lie algebra $\mathfrak{g}_n$ in terms of the generators. Explicit solutions of the equation for $n \leq 6$ are given in Table II. Even if it looks relatively simple, it is not possible to integrate equation (27) explicitly for arbitrary n. However, in this form we can obtain some information about the degrees of Casimir operators of $\mathfrak{g}_n$.

5. Solutions of minimal degree

We now analyze some results concerning the polynomial solutions of (27) of minimal degree, which will provide the Casimir operators of lowest degree of the algebras $\mathfrak{g}_n$.

Proceeding like in the example, for each of the new variables $u_k = P_k$ we define its degree as $\deg u_k := (2k+2)/n$, *i.e.* its degree as rational function. We observe that in particular deg $u_{n-1} = 2$ for all n. With this definition, the degree d of a monomial $u_1^{\alpha_1}...u_{n-1}^{\alpha_{n-1}}$ is given by

$$d = \frac{4\alpha_1 + ... + 2n\alpha_{n-1}}{n}\,. \tag{28}$$

It is not difficult to see that a polynomial F in $\{u_1, ..., u_{n-1}\}$ that satisfies equation (27) has the property that all summands have the same degree. This common integer will be called the degree of F. Thus, for example, a polynomial of order 2 would have the shape

$$F = \sum_{i,j} a_{ij} u_i^{\alpha_i} u_j^{\alpha_j}\,, \tag{29}$$

where $a_{ij} \in \mathbb{R}$ and the condition

$$(2i+2)\alpha_i + (2j+2)\alpha_j = 2n \tag{30}$$

is satisfied for any pair (i,j) appearing in (29). Using these new variables, we can deduce the existence and explicit shape of quadratic Casimir operators for even n.

Proposition 4 *For $n = 2m \geq 4$ the equation (27) admits the solution*

$$C_2 = u_{2m-1} + \sum_{l=1}^{m-1} \lambda_l u_l u_{2m-2-l}\,, \tag{31}$$

where

$$\lambda_k = (-1)^{k+1}\frac{2m(k+1)!(2m-1-k)!}{(2m-1)(2m-2)!}\,, \qquad 1 \le k \le m-2 \quad (32)$$

$$\lambda_{m-1} = (-1)^m\frac{m(m!)^2}{(2m-1)(2m-2)!}\,. \quad (33)$$

Proof. The proof follows by direct insertion. Consider a function $F(u_1,...,u_{2m-1}) = u_{2m-1} + \sum_{l=1}^{m-1}\lambda_l u_l u_{2m-2-l}$ for parameters $\lambda_l \in \mathbb{R}$. We first observe that, according to (28), F is a polynomial of degree 2. Further it satisfies $\frac{\partial F}{\partial u_{2m-2}} = 0$. Inserting F into equation (27) we obtain the condition:

$$\begin{aligned} 3\lambda_2 u_2 u_{2m-4} &+ \sum_{k=2}^{m-1}\lambda_k\left((k+2)u_{k+1} + 2(k-2m)u_1u_{k-1}\right)u_{2m-2-k} \\ &+ \sum_{k=m}^{2m-3}\lambda_{2m-2-k}\left((k+2)u_{k+1} + 2(k-2m)u_1u_{k-1}\right)u_{2m-2-k} \\ &- 4mu_1u_{2m-2} = 0\,. \end{aligned} \quad (34)$$

In particular, any parameter λ_k appears twice in (34), with the exception of λ_{m-1}. Reordering the monomials in (34) we obtain the linear system:

$$\begin{gathered} (2m-1)\lambda_1 - 4m = 0\,, \\ (k+2)\lambda_k + (2m-1-k)\lambda_{k+1} = 0\,, \\ 1 \le k \le m - 2m\lambda_{m-2} + 2(m+1)\lambda_{m-1}\,, \end{gathered} \quad (35)$$

the explicit solution of which is given by (32) and (33). ■

It follows from (16) that $\deg P_k P_{2m-2-k} = 2$, therefore, C_2 has degree 2. Thus we obtain

Corollary 1 *The Lie algebras $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_R(2n+1)L_1$ with $n = 2m$ have the quadratic Casimir operator*

$$\begin{aligned} C_2 = P_{2m-1} &+ \sum_{k=1}^{m-2}(-1)^{k+1}\frac{2m(k+1)!(2m-1-k)!}{(2m-1)(2m-2)!} \\ &\times P_kP_{2m-2-k} + \frac{(-1)^m m(m!)^2}{(2m-1)(2m-2)!}P_{m-1}^2\,. \end{aligned} \quad (36)$$

We now prove the nonexistence of invariants of odd order for the even dimensional algebras $\mathfrak{g}_n$.

Proposition 5 *If $n = 2m+1$, then equation (27) has no polynomial solutions of odd order.*

Proof. If $n = 2m+1$, the degree of u_k is given by $(2k+2)/(2m+1)$. Now let $F = \sum \lambda_{i_1,\dots,i_{2m}} u_1^{\alpha_{i_1}} \dots u_{2m}^{\alpha_{i_{2m}}}$ be a polynomial of odd degree l. By (28), the powers α_{i_j} must satisfy the linear equation:

$$2\alpha_{i_1} + \dots + (4m+2)\alpha_{i_{2m}} = (2m+1)l\,, \tag{37}$$

for any $(i_1, \dots, i_{2m})$ such that $\lambda_{i_1,\dots,i_{2m}} \neq 0$. However, equation (37) has no integer solutions since the left side is always an even number, while the left one is odd. ■

Corollary 2 *For odd n the Lie algebras $\mathfrak{g}_n = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_n} (n+1)L_1$ have no Casimir operators of odd order.*

Actually it can be shown with a similar argument that the only Lie algebra $\mathfrak{g}_n$ admitting a third order Casimir operator is $\mathfrak{g}_4$. The existence of this invariant follows from the fact that this algebra is a contraction of the non-compact real forms of A_2.

Proposition 6 *For odd n the minimal degree of a Casimir operator of the Lie algebra $\mathfrak{g}_n = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{D_n} (n+1)L_1$ is four.*

The proof of this result involves a large amount of computations, for which reason we only sketch the argument here. Let $n = 2m+1$. In this case, the monomials $u_1^{\alpha_1} \dots u_{2m}^{\alpha_{2m}}$ of a polynomial of degree $d = 4$ have to satisfy the linear equation

$$\sum_{j=1}^{2m} (2+2j)\alpha_j = 8m+4\,. \tag{38}$$

If P has the form

$$P = \sum_{i_1,\dots,i_{2m}} \lambda_{\alpha_{i_1},\dots,\alpha_{i_{2m}}} u_1^{\alpha_{i_1}} \dots u_{2m}^{\alpha_{i_{2m}}}\,, \tag{39}$$

we further impose the two following conditions:

$$0 \leq \alpha_{i_j} \leq 2\,, \qquad 1 \leq j \leq 2m, \qquad j \neq m-1, m \tag{40}$$

$$0 \leq \alpha_{i_j} \leq 3\,, \qquad j = m-1, m \tag{41}$$

for all coefficients $\lambda_{\alpha_{i_1},\dots,\alpha_{i_{2m}}}$. These conditions must in general not to be satisfied by an order four solution, but provide solutions with a specific structure. Evaluating such a polynomial (39) in equation (27) gives rise to a linear system with $\frac{1}{2}(3m^2-m+2)$ variables with a unique solution (up to multiples). For sake of simplicity, the solution with $\lambda_{0,\dots,0,2}=1$ has been chosen. The shape of such a polynomial is given by

$$\begin{aligned} P \;=\; & u_{2m}^2 + \sum_{l=1}^{m-1} a_l u_{2m} u_l u_{2m-1-l} + \sum_{l=1}^{m} b_l u_{2m-1} u_l u_{2m-l} \\ & + \sum_{i_1,\dots,i_4} \lambda^{i_1,\dots,i_4} u_{i_1}^{\alpha_{i_1}} \dots u_{i_4}^{\alpha_{i_4}} \,, \end{aligned} \tag{42}$$

where the condition

$$\sum_{j=1}^{4} i_j \alpha_{i_j} = 4m-2, \forall \lambda^{i_1,\dots,i_4} \neq 0\,, \tag{43}$$

is satisfied.

As observed, the preceding construction does not exclude the possibility of other polynomial solutions of degree 4. For example, in $n=9$ we find a second solution given by

$$\begin{aligned} F_2 = {} & u_8^2 + \left(\frac{270}{637}u_2^3 - \frac{135}{91}u_1u_2u_3 + \frac{27}{13}u_1^2u_4 - \frac{27}{91}u_3u_4 + \frac{135}{182}u_2u_5 + \frac{9}{26}u_1u_6\right)u_8 \\ & + \frac{216}{13}u_1u_7^2 + \left(-\frac{405}{637}u_2^2u_3 + \frac{270}{91}u_1u_3^2 - \frac{243}{91}u_1u_2u_4 + \frac{135}{91}u_4^2 - \frac{108}{91}u_3u_5 - \frac{81}{13}u_2u_6\right)u_7 \\ & + \left(-\frac{1215}{2548}u_2u_3^2 + \frac{891}{637}u_2^2u_4 - \frac{27}{52}u_1u_3u_4 - \frac{2025}{728}u_1u_2u_5 - \frac{135}{91}u_4u_5\right)u_6 - \frac{486}{4459}u_3^2u_4^2 \\ & + \left(\frac{675}{208}u_1^2 + \frac{27}{13}u_3\right)u_6^2 + \left(\frac{1215}{4459}u_3^3 - \frac{12393}{17836}u_2u_3u_4 - \frac{1215}{2548}u_1u_4^2\right)u_5 + \frac{1215}{4459}u_2u_4^3 \\ & + \left(-\frac{243}{71344}u_2^2 + \frac{891}{637}u_1u_3\right)u_5^2 + \frac{270}{637}u_5^3 \,. \end{aligned} \tag{44}$$

The corresponding replacement of u_i by P_i gives a second fourth order Casimir operator of $\mathfrak{sl}(2,\mathbb{R}) \oplus_{D_9} 10L_1$.

Resuming, for even values of n the affine algebras $\mathfrak{sl}(2,\mathbb{R}) \oplus_{D_n} (n+1)L_1$ have a quadratic Casimir invariant, while for odd n the minimal degree of such an operator is four. Table III gives the minimal solutions of $\mathfrak{g}_n$ for $n \leq 12$.

TABLE III

Solutions of equation (27) of minimal degree for $N \leq 12$.

N	Solution
4	$F_1 = u_3 + \frac{4}{3}u_1^2$
5	$F_1 = u_4^2 + u_1u_2u_4 + 6u_1^3u_3 + (8u_1u_3 - 3u_2^2)u_3 - 2u_1^2u_2^2$
6	$F_1 = u_5 - \frac{9}{10}u_2^2 + \frac{12}{5}u_1u_3$
7	$F_1 = u_6^2 + (2u_1u_4 - \frac{2}{5}u_2u_3)u_6 + (8u_1u_3 - \frac{16}{5}u_2^2)u_1u_5 + (\frac{8}{5}u_1u_3 - \frac{3}{5}u_2^2)u_3^2$
	$+\frac{8}{5}u_2^3u_4 + (8u_1u_5 - 4u_2u_4 + \frac{8}{5}u_3^2)u_5 + (u_1u_4 - \frac{22}{5}u_2u_3)u_1u_4$
8	$F_1 = u_7 + \frac{16}{7}u_1u_5 - \frac{8}{7}u_2u_4 + \frac{16}{35}u_3^2$
9	$F_1 = u_8^2 + (\frac{5}{2}u_1u_6 - \frac{9}{14}u_2u_5 + \frac{1}{7}u_3u_4)u_8 + (6u_1u_5 - \frac{24}{7}u_2u_4 + \frac{10}{7}u_3^2)u_1u_7$
	$+(\frac{9}{7}u_2u_4 - \frac{15}{28}u_3^2)u_2u_6 + (\frac{15}{49}u_3^2 - \frac{153}{196}u_2u_4)u_3u_5 + (\frac{9}{7}u_1u_3 + \frac{81}{784}u_2^2)u_5^2$
	$+(-\frac{15}{28}u_1u_5 + \frac{15}{49}u_2u_4 - \frac{6}{49}u_3^2)u_4^2 + (8u_1u_7 - 3u_2u_6 + \frac{12}{7}u_3u_5 - \frac{5}{7}u_4^2)u_7$
10	$F_1 = u_9 + \frac{20}{9}u_1u_7 - \frac{5}{6}u_2u_6 + \frac{10}{21}u_3u_5 - \frac{25}{126}u_4^2$
11	$F_1 = u_{10}^2 + (\frac{14}{5}u_1u_8 - \frac{2}{3}u_2u_7 + \frac{1}{5}u_3u_6 - \frac{1}{21}u_4u_5)u_{10} + (\frac{16}{25}u_1u_3 + \frac{1}{9}u_3^2)u_7^2$
	$(\frac{24}{5}u_1u_7 - \frac{32}{15}u_2u_6 + \frac{4}{3}u_3u_5 - \frac{4}{7}u_4^2)u_1u_9 + (\frac{6}{35}u_1u_7 - \frac{8}{105}u_2u_6 + \frac{1}{21}u_3u_5 - \frac{5}{252}u_4^2)u_5^2$
	$(8u_1u_9 - \frac{12}{5}u_2u_8 + \frac{16}{7}u_3u_7 - \frac{2}{3}u_4u_6 + \frac{2}{7}u_5^2)u_9 + (\frac{-79}{225}u_2u_6 + \frac{8}{45}u_3u_5 - \frac{8}{105}u_4^2)u_3u_7$
	$(-\frac{2}{5}u_1u_7 + \frac{8}{45}u_2u_6 - \frac{73}{630}u_3u_5 + \frac{1}{21}u_4^2)u_4u_6 + (\frac{16}{25}u_2u_6 - \frac{2}{5}u_3u_5 + \frac{6}{35}u_4^2)u_2u_8$
	$(\frac{49}{25}u_1u_8 - \frac{178}{75}u_2u_7 + \frac{7}{25}u_3u_6 - \frac{1}{15}u_4u_5)u_1u_8 + \frac{1}{100}u_3^2u_6^2 + \frac{1}{63}u_2u_4u_5u_7$
12	$F_1 = u_{11} + \frac{24}{11}u_1u_9 - \frac{36}{55}u_2u_8 + \frac{16}{55}u_3u_7 - \frac{2}{11}u_4u_6 + \frac{6}{77}u_5^2$

6. Conclusions

We have shown that up to the six, eight, ten and fourteen dimensional case, the Lie algebras $\mathfrak{g}_n$ cannot be obtained by a non-trivial contraction. This implies that the Casimir operators of these algebras must be computed by some other method. Although the invariants of $\mathfrak{g}_n$ depend only on the variables of the Abelian radical, its integration is far from being trivial due to the fact that the generator X_2 of the Levi part $\mathfrak{sl}(2,\mathbb{R})$ acts as a lowering operator, while the generator X_3 acts as a raising operator.

In order to integrate the system, we have first considered a maximal solvable subalgebra $\mathfrak{b}_n$ of co-dimension one and obtained a complete set of invariants in terms of rational functions. These are shown to transform in a specific way by the action of the differential operator associated to the raising operator X_3 of $\mathfrak{sl}(2,\mathbb{R})$, and allows to reduce the problem to one equation. The latter has been used to prove some results concerning the existence and form of Casimir operators of various degrees. In all, we can resume the method to obtain the Casimir invariants of $\mathfrak{g}_n$ in the following steps:

(i) Consider the maximal solvable subalgebra $\mathfrak{b}_n$.

(ii) Identify the $(n–1)$-independent invariants P_k of $\mathfrak{b}_n$ as new variables u_k.

(iii) Look for polynomial solutions P of the equation (27) of even order.

(iv) Replace the variables u_k by P_k and symmetrize the homogeneous polynomial to recover the Casimir operator.

The method also provides an answer to an interesting question, namely, which is the maximal order of a Casimir operator in a fundamental system of invariants. For $n = 6$, the nine dimensional Lie algebra $\mathfrak{g}_6$ has a fundamental basis formed by Casimir operators of orders 4, 8 and 12, the order of the latter exceeding the dimension. This fact suggests that looking for explicit formulae for the Casimir operators of these algebras is a hopeless task, since even the degrees of the Casimir operators forming a basis of invariants cannot be predicted.

It remains the question whether the method developed here can be enlarged to decomposable representations. However, in this case the stability theorem of [24] is not sufficient, since the radical is not necessarily Abelian, and in general the exact number of invariants is not known [12, 29]. The solution of this case is necessarily based on the problem of classification of derivations of Lie algebras, which at present has no satisfactory solution since solvable Lie algebras are classified only up to dimension six (see *e.g.* [30] for an actualized review on the classification). Finally, it would be desirable to compare these results with the geometric method of [19], in order to know to which extent the existence of a Levi subalgebra affects the possibility of solving explicitly the corresponding algebraic system.

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