

ABELIAN CONNECTION IN FEDOSOV
DEFORMATION QUANTIZATION.
I. THE 2-DIMENSIONAL PHASE SPACE

JAROMIR TOSIEK

Institute of Physics, Technical University of Łódź
Wólczanska 219, 93-005 Łódź, Poland
tosiek@p.lodz.pl

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General properties of an Abelian connection in Fedosov deformation quantization are investigated. The definition and the criterion of being a finite formal series for an Abelian connection are presented. A proof that in 2-dimensional (2D) case the Abelian connection is an infinite formal series is done.

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1. Introduction

Deformation quantization on phase space \mathbb{R}^{2n} was proposed by Moyal [1]. In his paper ideas of Weyl [2], Wigner [3] and Groenewold [4] were developed. The article contained not only general considerations but also explicit formulas defining the $*$ -product of observables and so called Moyal bracket being the counterpart of the commutator of operators.

The first successful generalization of Moyal's results in case of a phase space different from \mathbb{R}^{2n} was presented 28 years later, when Bayen *et al.* [5,6] proposed an axiomatic version of deformation quantization. In these articles quantum mechanics became a deformed version of classical physics. Unfortunately, in contrast to Moyal, quoted authors did not present universal "computable" model of their idea. Since that their results were applicable only in some special cases like a harmonic oscillator.

One of the realizations of quantization programme is so called Fedosov deformation quantization [7,8]. The Fedosov construction is algebraic and it can be applied easily to solve some problems like harmonic oscillator [9,10] or 2D symplectic space with constant curvature tensor [11].

In Fedosov quantization we work with formal series. There is no general method to write these series in a compact form. Hence so important is to find cases, in which the form of these series may be predicted. Series of compact form appear for example when they contain finite number of terms. In that case $*$ -product of functions can be calculated exactly.

Fedosov deformation quantization is based on two recurrent equations, which generate formal series. The first one is the formula defining an Abelian connection, the second — relation introducing a series representing an observable. In this paper we deal with the Abelian connection.

Because the answer, when the Abelian connection is a finite series, depends on the dimension of the phase space of a system, we divide our considerations into two papers. In this one we present the necessary and sufficient condition for the Abelian connection to be a finite series. We prove that in 2D case the Abelian connection cannot be represented by such a series. In the second part we will deal with more dimensional phase spaces.

Our considerations are devoted only to Abelian connections determined by the iteration process proposed by Fedosov. Another kind of Abelian connection on Kähler symmetric manifolds can be found in [12].

In all formulas in which summation limits are obvious we use Einstein summation convention.

2. Foundations of Fedosov deformation quantization

All facts presented in this section have been published in some books or papers. Bibliography of symplectic geometry is given below. Foundations of Fedosov quantization may be seen in [7, 8]. We set this knowledge to simplify following our considerations. Moreover, we modify notation a bit in comparison to the original Fedosov article.

The starting point of the deformation quantization according to the Fedosov rules is a symplectic manifold equipped with some connection known as “symplectic”. Reader interested in details of symplectic geometry is pleased to look them up in [8, 10, 13, 14].

Let (\mathcal{M}, ω) be a $2n$ -D symplectic manifold and $\mathcal{A} = \{(\mathcal{U}_z, \phi_z)\}_{z \in J}$ an atlas on \mathcal{M} . By ω we mean the symplectic 2-form. Since we work only with symplectic manifolds, in our paper we will denote a manifold (\mathcal{M}, ω) just by \mathcal{M} .

Definition 2.1 *A symplectic connection Γ on \mathcal{M} is a torsion free connection locally satisfying conditions*

$$\omega_{ij;k} = 0, \quad 1 \leq i, j, k \leq 2n, \quad (2.1)$$

where a semicolon “;” stands for the covariant derivative.

In any Darboux coordinates the system of equations (2.1) reduces to

$$\omega_{ij;k} = -\Gamma_{ik}^l \omega_{lj} - \Gamma_{jk}^l \omega_{il} = -\Gamma_{jik} + \Gamma_{ijk} = 0, \quad (2.2)$$

where $\Gamma_{ijk} \stackrel{\text{def.}}{=} \Gamma_{jk}^l \omega_{li}$.

As can be seen from (2.2), the coefficients Γ_{ijk} are symmetric with respect to indices $\{i, j, k\}$. The number of independent elements Γ_{ijk} is $\binom{2n+2}{2n-1}$. A symplectic connection exists on any symplectic manifold.

Definition 2.2 A symplectic manifold \mathcal{M} equipped with a symplectic connection Γ is called a **Fedosov manifold** (\mathcal{M}, Γ) .

In Darboux coordinates the symplectic curvature tensor R_Γ is defined

$$(R_\Gamma)_{ijkl} = \frac{\partial \Gamma_{ilj}}{\partial q^k} - \frac{\partial \Gamma_{ijk}}{\partial q^l} + \omega^{mp} \Gamma_{plj} \Gamma_{ikm} - \omega^{mp} \Gamma_{pj k} \Gamma_{ilm}.$$

The tensor ω^{ij} and the symplectic form ω_{jk} are related by $\omega^{ij} \omega_{jk} = \delta_k^i$.

The symplectic curvature tensor R_Γ is symmetric in two first indices $(R_\Gamma)_{ijkl} = (R_\Gamma)_{jikl}$ and antisymmetric in two last indices $(R_\Gamma)_{ijkl} = -(R_\Gamma)_{ijlk}$. On a $2n$ -D Fedosov manifold (\mathcal{M}, Γ) a number of independent components of the tensor R_Γ is $\frac{1}{2}n(n+1)(2n+1)(2n-1)$.

Let \hbar denote some positive parameter and $X_{\mathbf{p}}^1, \dots, X_{\mathbf{p}}^{2n}$ components of an arbitrary vector $\mathbf{X}_{\mathbf{p}}$ belonging to the tangent space $T_{\mathbf{p}}\mathcal{M}$ to the symplectic manifold \mathcal{M} at the point \mathbf{p} . The components $X_{\mathbf{p}}^1, \dots, X_{\mathbf{p}}^{2n}$ are written in the natural basis $\left(\frac{\partial}{\partial q^i}\right)_{\mathbf{p}}$ determined by the chart (\mathcal{U}_z, ϕ_z) such that $\mathbf{p} \in \mathcal{U}_z$.

In the point \mathbf{p} we introduce a formal series

$$a \stackrel{\text{def.}}{=} \sum_{l=0}^{\infty} \hbar^l a_{k, i_1 \dots i_l} X_{\mathbf{p}}^{i_1} \dots X_{\mathbf{p}}^{i_l}, \quad 0 \leq k. \quad (2.3)$$

For $l = 0$ we put $a = \hbar^k a_k$. By $a_{k, i_1 \dots i_l}$ we denote components of a covariant tensor symmetric with respect to indices $\{i_1, \dots, i_l\}$ taken in the basis $dq^{i_1} \odot \dots \odot dq^{i_l}$.

The part of the series a standing at \hbar^k and containing l components of the vector $\mathbf{X}_{\mathbf{p}}$ will be denoted by $a[k, l]$ so that $a = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hbar^k a[k, l]$. The **degree** $\deg(a[k, l])$ of the component $a[k, l]$ is the sum $2k + l$. The degree of the series a is the maximal degree of its nonzero components $a[k, l]$.

Let $P_{\mathbf{p}}^* \mathcal{M}[[\hbar]]$ be the set of all elements a of the kind (2.3) at the point \mathbf{p} . The set $P_{\mathbf{p}}^* \mathcal{M}[[\hbar]]$ is a linear space over \mathbb{C} .

Definition 2.3 The product $\circ : P_{\mathbf{p}}^* \mathcal{M}[[\hbar]] \times P_{\mathbf{p}}^* \mathcal{M}[[\hbar]] \rightarrow P_{\mathbf{p}}^* \mathcal{M}[[\hbar]]$ of two elements $a, b \in P_{\mathbf{p}}^* \mathcal{M}[[\hbar]]$ is the mapping

$$a \circ b \stackrel{\text{def.}}{=} \sum_{t=0}^{\infty} \frac{1}{t!} \left(\frac{i\hbar}{2} \right)^t \omega^{i_1 j_1} \dots \omega^{i_t j_t} \frac{\partial^t a}{\partial X_{\mathbf{p}}^{i_1} \dots \partial X_{\mathbf{p}}^{i_t}} \frac{\partial^t b}{\partial X_{\mathbf{p}}^{j_1} \dots \partial X_{\mathbf{p}}^{j_t}}. \quad (2.4)$$

The pair $(P_{\mathbf{p}}^* \mathcal{M}[[\hbar]], \circ)$ is a noncommutative associative algebra called the **Weyl algebra**. The \circ -product does not depend on the chart. Moreover, for all $a, b \in (P_{\mathbf{p}}^* \mathcal{M}[[\hbar]], \circ)$ the relation holds

$$\deg(a \circ b) = \deg(a) + \deg(b).$$

Definition 2.4 A **Weyl bundle** is a triplet $(\mathcal{P}^* \mathcal{M}[[\hbar]], \pi, \mathcal{M})$, where

$$\mathcal{P}^* \mathcal{M}[[\hbar]] \stackrel{\text{def.}}{=} \bigcup_{\mathbf{p} \in \mathcal{M}} (P_{\mathbf{p}}^* \mathcal{M}[[\hbar]], \circ)$$

is a differentiable manifold called the total space, \mathcal{M} is the base space and $\pi : \mathcal{P}^* \mathcal{M}[[\hbar]] \rightarrow \mathcal{M}$ the projection.

A Weyl bundle is a vector bundle in which the typical fibre is also an algebra.

Definition 2.5 An m -differential form with value in the Weyl bundle is a form written locally

$$a = \sum_{l=0}^{\infty} \hbar^l a_{k, i_1 \dots i_l, j_1 \dots j_m}(q^1, \dots, q^{2n}) X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_m}, \quad (2.5)$$

where $0 \leq m \leq 2n$. Now $a_{k, i_1 \dots i_l, j_1 \dots j_m}(q^1, \dots, q^{2n})$ are components of smooth tensor fields on \mathcal{M} and $C^\infty(\mathcal{T}\mathcal{M}) \ni \mathbf{X} \stackrel{\text{locally}}{=} X^i \frac{\partial}{\partial q^i}$ is a smooth vector field on \mathcal{M} .

For simplicity we will omit variables (q^1, \dots, q^{2n}) .

Let Λ^m be a smooth field of m -forms on the symplectic manifold \mathcal{M} . Forms of the kind (2.5) are smooth sections of the direct sum $\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda \stackrel{\text{def.}}{=} \oplus_{m=0}^{2n} (\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^m)$.

The **projection** $\sigma(a)$ of $a \in C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^0)$ means $a|_{\mathbf{X}=\mathbf{0}}$.

Definition 2.6 The **commutator** of forms $a \in C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{m_1})$ and $b \in C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{m_2})$ is the form $[a, b] \in C^\infty(\mathcal{P}^* \mathcal{M}[[\hbar]] \otimes \Lambda^{m_1+m_2})$ defined by

$$[a, b] \stackrel{\text{def.}}{=} a \circ b - (-1)^{m_1 \cdot m_2} b \circ a. \quad (2.6)$$

A form $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$ is called **central**, if for every $b \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$ the commutator $[a, b]$ vanishes. Only forms not containing X^i 's are central in the Weyl algebra.

Definition 2.7 *The antiderivation operator δ : $C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m) \rightarrow C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m+1})$ is defined by*

$$\delta a \stackrel{\text{def.}}{=} dq^k \wedge \frac{\partial a}{\partial X^k}. \quad (2.7)$$

The operator δ lowers the degree $\deg(a)$ of the elements of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$ by 1.

Every two forms $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m_1})$ and $b \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$ satisfy

$$\delta(a \circ b) = (\delta a) \circ b + (-1)^{m_1} a \circ (\delta b). \quad (2.8)$$

Definition 2.8 *The operator $\delta^{-1} : C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m) \rightarrow C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m-1})$ is defined by*

$$\delta^{-1}a = \begin{cases} \frac{1}{l+m} X^k \frac{\partial}{\partial q^k} \Big| a & \text{for } l+m > 0, \\ 0 & \text{for } l+m = 0, \end{cases} \quad (2.9)$$

where l is the degree of a in X^i 's i.e. the number of X^i 's.

δ^{-1} raises the degree of the forms of $\mathcal{P}^*\mathcal{M}[[\hbar]]\Lambda$ in the Weyl algebra by 1.

The linear operators δ and δ^{-1} do not depend on the choice of local coordinates and have the following properties:

$$(i) \quad \delta^2 = (\delta^{-1})^2 = 0;$$

(ii) let us assume that indices i_1, \dots, i_l and j_1, \dots, j_m are arbitrary but fixed. For the monomial

$X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_m}$ we have

$$(\delta\delta^{-1} + \delta^{-1}\delta)X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_m} = X^{i_1} \dots X^{i_l} dq^{j_1} \wedge \dots \wedge dq^{j_m}.$$

The straightforward consequence of the linearity and the decomposition of monomials is the Hodge decomposition of the form $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$ as shows the next theorem.

Theorem 2.1 [7, 8] *For every $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$*

$$a = \delta\delta^{-1}a + \delta^{-1}\delta a + a_{00}, \quad (2.10)$$

where a_{00} is a smooth function on the symplectic manifold \mathcal{M} .

Definition 2.9 The exterior covariant derivative ∂_γ of the form $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m)$ determined by a connection 1-form $\gamma \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^1)$ is the linear operator $\partial_\gamma : C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^m) \rightarrow C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^{m+1})$ defined in a Darboux chart by the formula

$$\partial_\gamma a \stackrel{\text{def.}}{=} da + \frac{1}{i\hbar}[\gamma, a]. \quad (2.11)$$

In case when γ represents the symplectic connection, we use a symbol Γ instead of γ i.e.

$$\gamma \stackrel{\text{denoted}}{=} \Gamma = \frac{1}{2}\Gamma_{ijk}X^iX^jdq^k. \quad (2.12)$$

The curvature form R_γ of a connection 1-form γ in a Darboux chart can be expressed by the formula

$$R_\gamma = d\gamma + \frac{1}{2i\hbar}[\gamma, \gamma] = d\gamma + \frac{1}{i\hbar}\gamma \circ \gamma. \quad (2.13)$$

Hence the second covariant derivative $\partial_\gamma(\partial_\gamma a) = (1/i\hbar)[R_\gamma, a]$.

The crucial role in the Fedosov deformation quantization is played by an Abelian connection $\tilde{\Gamma}$. By definition, by an **Abelian** connection we mean a connection $\tilde{\Gamma}$ whose curvature form $R_{\tilde{\Gamma}}$ is central so that $\partial_{\tilde{\Gamma}}(\partial_{\tilde{\Gamma}}a) = 0$ for every $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$.

The Abelian connection proposed by Fedosov is of the form

$$\tilde{\Gamma} = \omega_{ij}X^iX^jdq^j + \Gamma + r. \quad (2.14)$$

Its curvature

$$R_{\tilde{\Gamma}} = -\frac{1}{2}\omega_{j_1j_2}dq^{j_1} \wedge dq^{j_2} + R_\Gamma - \delta r + \partial_\Gamma r + \frac{1}{i\hbar}r \circ r. \quad (2.15)$$

The requirement that the central curvature 2-form $R_{\tilde{\Gamma}} = -\frac{1}{2}\omega_{j_1j_2}dq^{j_1} \wedge dq^{j_2}$ means that we look for the solution of the equation

$$\delta r = R_\Gamma + \partial_\Gamma r + \frac{1}{i\hbar}r \circ r. \quad (2.16)$$

Fedosov proved (see [7, 8]) the following theorem.

Theorem 2.2 The equation (2.16) has a unique solution

$$r = \delta^{-1}R_\Gamma + \delta^{-1}\left(\partial_\Gamma r + \frac{1}{i\hbar}r \circ r\right) \quad (2.17)$$

fulfilling the following conditions

$$\delta^{-1}r = 0, \quad 3 \leq \deg(r). \quad (2.18)$$

We work only with the Abelian connection of the form (2.14) with the correction r defined by (2.17) and fulfilling (2.18). The general solution of (2.16) was published in [15]. The Abelian connection on Kähler symmetric manifolds proposed by Tamarkin [12] does not fulfill the condition $\delta^{-1}r = 0$.

Definition 2.10 *The subalgebra $\mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{F}} \subset C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^0)$ consists of flat sections, i.e. such that $\partial_{\tilde{F}}a = 0$.*

Theorem 2.3 [7, 8] *For any $a_0 \in C^\infty(\mathcal{M})$ there exists a unique section $a \in \mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{F}}$ such that $\sigma(a) = a_0$.*

Applying the operator δ^{-1} it follows from the Hodge decomposition (2.10) that

$$a = a_0 + \delta^{-1} \left(\partial_{\Gamma} a + \frac{1}{i\hbar} [r, a] \right). \quad (2.19)$$

Using the one-to-one correspondence between $\mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{F}}$ and $C^\infty(\mathcal{M})$ we introduce an associative star product “ $*$ ” of functions $a_0, b_0 \in C^\infty(\mathcal{M})$

$$a_0 * b_0 \stackrel{\text{def.}}{=} \sigma(\sigma^{-1}(a_0) \circ \sigma^{-1}(b_0)). \quad (2.20)$$

The $*$ -product (2.20) fulfills axioms of the star product in deformation quantization and is interpreted as the quantum multiplication of observables.

3. Properties of the Abelian connection

In this section we present general features of the Abelian connection constructed according to Fedosov procedure and consider conditions under which the correction r is a finite formal series.

3.1. Connections in the Weyl bundle

Let $\mathcal{P}^*\mathcal{M}[[\hbar]]$ be a Weyl algebra bundle equipped with some connection determined by 1-form γ . We do not assume that γ is an Abelian or symplectic.

Proposition 3.1 *Every connection $\gamma \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^1)$ such that $\delta\gamma = 0$ satisfies $\delta R_\gamma = 0$.*

Proof

For every $a \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda)$

$$d(\delta a) + \delta(da) = 0. \quad (3.1)$$

Hence the condition $\delta\gamma = 0$ and the property (3.1) gives

$$\delta(d\gamma) = 0. \quad (3.2)$$

Using formula (2.13) describing the curvature form we obtain

$$\delta R_\gamma = \delta(d\gamma + \frac{1}{i\hbar}\gamma \circ \gamma) \stackrel{(2.8)}{=} \delta(d\gamma) + \frac{1}{i\hbar}(\delta(\gamma) \circ \gamma - \gamma \circ \delta(\gamma)).$$

From the assumption $\delta\gamma=0$ and Eq. (3.2) we see that indeed $\delta R_\gamma=0$. ■

A straightforward consequence of Proposition 3.1 and decomposition (2.10) is the following corollary.

Corollary 3.1 *If the connection form γ fulfills the condition $\delta\gamma = 0$ then its curvature 2- form $R_\gamma = \delta\delta^{-1}R_\gamma$.*

Hence, for a connection γ such that $\delta\gamma = 0$ we have $R_\gamma = 0$ if and only if $\delta^{-1}R_\gamma = 0$.

Let us apply the above corollary to the symplectic connection represented by the 1-form Γ (see (2.12)). Since the fact that coefficients Γ_{ijk} are symmetric in indices $\{i, j, k\}$, we obtain that $\delta\Gamma = 0$.

Applying Corollary 3.1 we conclude that

Proposition 3.2 *Two symplectic curvature forms R_Γ and $R_{\Gamma'}$ defined by symplectic connections Γ and Γ' respectively, are equal if and only if $\delta^{-1}R_\Gamma = \delta^{-1}R_{\Gamma'}$.*

From Proposition 3.2 we see that the geometry of a symplectic space can be characterized by a tensor $(R_\Gamma)_{ijkl}$ symmetric in indices $\{i, j\}$ and anti-symmetric in $\{k, l\}$ or, equivalently, by a tensor $(\delta^{-1}R_\Gamma)_{ijkl}$ symmetric in indices $\{i, j, k\}$.

Let us consider the structure of equation (2.17). Its solution fulfilling conditions (2.18) can be found by the iteration method [7, 8]

$$r_0 \stackrel{\text{def.}}{=} 0, \quad r_s = \delta^{-1} \left(R_\Gamma + \partial_\Gamma r_{s-1} + \frac{1}{i\hbar} r_{s-1} \circ r_{s-1} \right), \quad s = 1, 2, \dots, \quad (3.3)$$

The component of r of the lowest degree is $\delta^{-1}R_\Gamma$, $\deg(\delta^{-1}R_\Gamma) = 3$ and it is the only term of that degree. Hence, the solution of (2.17) can be written in the form

$$r = \delta^{-1}R_\Gamma + \sum_{z=4}^{\infty} \sum_{k=0}^{[\frac{z}{2}]} \hbar^k r_m[k, z-2k] dq^m. \quad (3.4)$$

By $[\frac{z}{2}]$ we denote the maximal integral number not bigger than $\frac{z}{2}$. The symbol $r_m[k, z-2k]dq^m$ means a 1-form containing $(z-2k)$ X^i 's and standing at \hbar^k .

From Proposition 3.2 we conclude that if $R_\Gamma \neq R_{\Gamma'}$ then corrections r determined by connections Γ and Γ' , respectively, are different. We deduce that $z - 2k \geq 1$, because each component of r_s contains one or more X 's. Moreover, the product $r_{s-1} \circ r_{s-1}$ generates only odd powers of \hbar , so the index k in (3.4) is even.

Therefore, formula (3.4) can be written in the following form

$$r = \delta^{-1} R_\Gamma + \sum_{z=4}^{\infty} \sum_{k=0}^{[\frac{z-1}{4}]} \hbar^{2k} r_m[2k, z-4k] dq^m. \quad (3.5)$$

In the case when $\deg(r) = d$, $d \in \mathbb{N}$ we say that r is a **finite formal series**. For $\deg(r) = \infty$ we deal with an infinite series.

If in an arbitrary chart the term $r_m[2k, z-4k]dq^m$ for fixed k and z does not disappear, the same happens in any other chart. This statement follows from the fact that the 1-form $r_m[2k, z-4k]dq^m$ is determined by the tensor components $r_{i_1 \dots i_{z-4k}, m}$, $1 \leq i_1, \dots, i_{z-4k}, m \leq \dim \mathcal{M}$. Moreover, from the same reason

Corollary 3.2 *At an arbitrary point $\mathbf{p} \in \mathcal{M}$ the fact that the series r is finite does not depend on the chart.*

Corollary 3.3 *At an arbitrary point $\mathbf{p} \in \mathcal{M}$ the inequalities $\partial_\Gamma r_m[l, u]dq^m \neq 0$, $r_m[l, u]dq^m \circ r_j[p, k]dq^j \neq 0$ are true in each chart.*

3.2. Finite Abelian connection

Now we consider in which cases an Abelian connection is a finite formal series. This situation is eligible because then, Fedosov method may lead to compact elements of the Weyl algebra representing observables. By $r[z]$ we will denote the component $r[z] \stackrel{\text{def.}}{=} \sum_{k=0}^{[\frac{z-1}{4}]} \hbar^{2k} r_m[2k, z-4k]dq^m$, $3 \leq z$ of r of the degree z .

As it was proved [16, 17]

$$r[3] = \delta^{-1} R_\Gamma, \\ r[z] = \delta^{-1} \left(\partial_\Gamma r[z-1] + \frac{1}{i\hbar} \sum_{j=3}^{z-2} r[j] \circ r[z+1-j] \right), \quad 4 \leq z. \quad (3.6)$$

From Proposition 3.2 we see that for curvature $R_\Gamma \neq 0$ it must hold that $\delta^{-1} R_\Gamma \neq 0$. So on any nonflat Fedosov manifold (\mathcal{M}, Γ) the term $r[3]$ is different from 0.

Assume that r is a finite formal series of the degree $m - 1$, $4 \leq m$. It means that $r[m - 1]$ is the last nonzero term of the r series. Hence, from (3.6)

$$\begin{aligned} \delta^{-1} \left(\partial_I r[m - 1] + \frac{1}{i\hbar} \sum_{j=3}^{m-2} r[j] \circ r[m + 1 - j] \right) &= 0, \\ \delta^{-1} \left(\frac{1}{i\hbar} \sum_{j=3}^{m-1} r[j] \circ r[m + 2 - j] \right) &= 0, \\ &\vdots \\ \delta^{-1} \left(\frac{1}{i\hbar} (r[m - 2] \circ r[m - 1] + r[m - 1] \circ r[m - 2]) \right) &= 0, \\ \delta^{-1} \left(\frac{1}{i\hbar} r[m - 1] \circ r[m - 1] \right) &= 0. \end{aligned} \quad (3.7)$$

According to Theorem 2.2 the series r is the only one solution of equation (2.16). Therefore, relations $\delta r[z] = 0$, $m \leq z$ imply

$$\partial_I r[m - 1] + \frac{1}{i\hbar} \sum_{j=3}^{m-2} r[j] \circ r[m + 1 - j] = 0, \quad (3.8a)$$

$$\sum_{j=3}^{m-1} r[j] \circ r[m + 2 - j] = 0, \quad (3.8b)$$

$$\vdots$$

$$r[m - 2] \circ r[m - 1] + r[m - 1] \circ r[m - 2] = 0, \quad (3.8c)$$

$$r[m - 1] \circ r[m - 1] = 0. \quad (3.8d)$$

Conversely, let components $r[z]$, $3 \leq z \leq m - 1$, where $4 \leq m$ of the Abelian correction r fulfill the system of equations (3.8a)–(3.8d). Then, applying formula (3.6) to (3.8a) we see that $r[m] = 0$. Substituting this result and relation (3.8b) to (3.6) we obtain $r[m + 1] = 0$. Repeating this procedure we find that $r[z] = 0$ for any $m \leq z$. Hence $\deg(r) \leq m - 1$ so r is the finite formal series.

To conclude,

Theorem 3.1 *An Abelian connection $\tilde{\Gamma} = \omega_{ij}X^i dq^j + \Gamma + r$ of the symplectic connection Γ with the curvature 2-form $R_\Gamma \neq 0$ is a finite formal series if there exists a natural number $4 \leq m$ such that the components $r[z]$, $3 \leq z \leq m-1$, of r fulfill the system of equations (3.8a)–(3.8d).*

That theorem yields the following statements:

Corollary 3.4 *A sufficient condition for series r to be infinite is that for every $3 \leq z$ the product $r[z] \circ r[z] \neq 0$.*

This fact is the straightforward consequence of equation (3.8d).

Corollary 3.5 *A sufficient condition for series r to be infinite is that for every $3 \leq z$ the commutator $[r[z], r[z+1]] \neq 0$.*

The latter conclusion comes from (3.8c).

Corollary 3.6 *Let $4 \leq d$ be the minimal value of parameter m for which the system of equations (3.8a)–(3.8d) holds. Then $\deg(r) = d-1$.*

To illustrate how Theorem 3.1 works, we consider an example of a finite Abelian connection on some symplectic manifold \mathcal{M} , $4 \leq \dim \mathcal{M}$. Detailed analysis of this case will be presented in the next paper.

Example

Assume that in some Darboux chart (\mathcal{U}, ϕ) on \mathcal{M} nonzero symplectic connection coefficients $\Gamma_{l_1 l_2 l_3}(q^{l_4}, \dots, q^{l_s})$, $1 \leq l_1, \dots, l_s \leq \dim \mathcal{M}$ are these for which Poisson brackets $\{q^{l_i}, q^{l_j}\}_P = 0$, $1 \leq i, j \leq s$. Such a connection can be curved only if $4 \leq \dim \mathcal{M}$. In the considered case all the products $r[z] \circ r[k]$, disappear. Applying (3.6) we see that

$$r[z] = (\delta^{-1} \partial_\Gamma)^{z-3} \delta^{-1} R_\Gamma.$$

From Theorem 3.1 for $R_\Gamma \neq 0$ a sufficient and necessary condition for r to be a finite series is that there exists a natural number $4 \leq z$ satisfying

$$(\partial_\Gamma \delta^{-1})^{z-3} R_\Gamma = 0. \quad (3.9)$$

The minimal number z for which (3.9) holds, is the degree of r .

4. An Abelian connection on a 2D phase space

In this section we prove that every Abelian connection on a curved 2D Fedosov space (\mathcal{M}, Γ) is an infinite formal series.

Proposition 4.1 *Let (\mathcal{M}, Γ) be a 2D Fedosov manifold and $F \in C^\infty(\mathcal{P}^*\mathcal{M}[[\hbar]] \otimes \Lambda^2)$ be a 2-form defined on (\mathcal{M}, Γ) fulfilling conditions:*

1. *F contains only terms of the same degree so $\exists_{0 \leq z} F = F[z]$,*
2. *F contains only even powers \hbar^{2k} , $0 \leq k$ of the deformation parameter \hbar .*

Then $\delta^{-1}F \circ \delta^{-1}F = 0$ iff $F = 0$.

Proof

Computations presented below were also tested in the Mathematica 5.2 by Wolfram Research.

‘ \Leftarrow ’

It is obvious, that if $F = 0$ then $\delta^{-1}F = 0$ and $\delta^{-1}F \circ \delta^{-1}F = 0$.

‘ \Rightarrow ’

We perform our computations locally in a Darboux chart (U, ϕ) and denote canonically conjugated coordinates by q and p . Their Poisson bracket $\{q, p\}_P = 1$.

In the chart (U, ϕ) the most general form F of the degree $(z - 1)$ and fulfilling conditions from Proposition 4.1 is

$$F = \sum_{k=0}^{\left[\frac{z-1}{4}\right]} \sum_{l=0}^{z-1-4k} \hbar^{2k} a_{2k,l,z-1-4k-l}(q,p) (X^1)^l (X^2)^{z-1-4k-l} dq \wedge dp, \quad 1 \leq z.$$

By $a_{2k,l,z-1-4k-l}(q,p)$ we denote some smooth functions. To shorten our notation we omit variables q and p .

Hence,

$$\begin{aligned} \delta^{-1}F &= \sum_{k=0}^{\left[\frac{z-1}{4}\right]} \sum_{l=0}^{z-1-4k} \frac{\hbar^{2k}}{z-4k+1} \\ &\times \left(a_{2k,l,z-1-4k-l}(X^1)^{l+1} (X^2)^{z-1-4k-l} dp - a_{2k,l,z-1-4k-l}(X^1)^l (X^2)^{z-4k-l} dq \right). \end{aligned} \quad (4.1)$$

Let us define

$$b_{2k,l} \stackrel{\text{def.}}{=} \frac{1}{z-4k+1} a_{2k,l,z-1-4k-l}.$$

Therefore

$$\begin{aligned} \delta^{-1}F &= \sum_{k=0}^{\left[\frac{z-1}{4}\right]} \sum_{l=0}^{z-1-4k} \hbar^{2k} b_{2k,l}(X^1)^{l+1}(X^2)^{z-1-4k-l} dp \\ &\quad - \sum_{k=0}^{\left[\frac{z-1}{4}\right]} \sum_{l=0}^{z-1-4k} \hbar^{2k} b_{2k,l}(X^1)^l(X^2)^{z-4k-l} dq. \end{aligned} \quad (4.2)$$

To find the square of (4.2) we need first to compute the product:

$$\begin{aligned} (X^1)^r(X^2)^j \circ (X^1)^s(X^2)^k &= \sum_{t=0}^{\min[r,k]+\min[j,s]} \frac{1}{t!} \left(\frac{i\hbar}{2}\right)^t \sum_{a=0}^t (-1)^a \binom{t}{a} \\ &\quad \times \frac{r! j! s! k!}{(r-t+a)! (j-a)! (s-a)! (k-t+a)!} (X^1)^{r+s-t} (X^2)^{k+j-t}. \end{aligned} \quad (4.3)$$

In fact the sum (4.3) over a different from 0 can be only elements from $a = \max[t-r, t-k, 0]$ until $a = \min[j, s, t]$. Simplifying (4.3) we see that

$$\begin{aligned} (X^1)^r(X^2)^j \circ (X^1)^s(X^2)^k &= r! j! s! k! \sum_{t=0}^{\min[r,k]+\min[j,s]} \left(\frac{i\hbar}{2}\right)^t (X^1)^{r+s-t} (X^2)^{k+j-t} \\ &\quad \times \sum_{a=\max[t-r, t-k, 0]}^{\min[j, s, t]} (-1)^a \frac{1}{a! (t-a)! (r-t+a)! (j-a)! (s-a)! (k-t+a)!} \end{aligned} \quad (4.4)$$

$$= \sum_{t=0}^{\min[r,k]+\min[j,s]} (i\hbar)^t (X^1)^{r+s-t} (X^2)^{k+j-t} f(r, j, s, k, t), \quad (4.5)$$

where

$$\begin{aligned} f(r, j, s, k, t) &\stackrel{\text{def.}}{=} \frac{(-1)^{\text{mx}}}{2^t} \frac{r! j! s! k!}{(j-\text{mx})! (s-\text{mx})! (t-\text{mx})! (\text{mx}-t+r)! (\text{mx}-t+k)! \text{mx}!} \\ &\quad \times {}_4F_3\left(\{1, \text{mx}-j, \text{mx}-s, \text{mx}-t\}; \{1-t+r+\text{mx}, 1-t+k+\text{mx}, 1+\text{mx}\}; 1\right). \end{aligned} \quad (4.6)$$

In the upper formula

$$\text{mx} \stackrel{\text{def.}}{=} \max[t-r, t-k, 0].$$

By ${}_4F_3(\{a_1, a_2, a_3, a_4\}; \{b_1, b_2, b_3\}; x)$ is denoted the generalized hypergeometric function.

Therefore

$$\begin{aligned} \delta^{-1}F \circ \delta^{-1}F &= \sum_{k=0}^{\left[\frac{z-1}{4}\right]} \sum_{w=0}^{\left[\frac{z-1}{4}\right]} \sum_{l=0}^{z-1-4k} \sum_{r=0}^{z-1-4w} \sum_{u=0}^{\min[l+1, z-4w-r] + \min[z-1-4k-l, r]} i^u \hbar^{2k+2w+u} \\ &\times b_{2k,l} b_{2w,r} (X^1)^{l+r+1-u} (X^2)^{2z-4k-4w-l-r-u-1} \\ &\times \left\{ f(l+1, z-1-4k-l, r, z-4w-r, u) - f(r, z-4w-r, l+1, z-1-4k-l, u) \right\} dq \wedge dp. \end{aligned} \quad (4.7)$$

Remember that in $f(l+1, z-1-4k-l, r, z-4w-r, u)$ we put (see (4.6)) $\text{mx} = \max[u-l-1, u-z+4w+r, 0]$ and in $f(r, z-4w-r, l+1, z-1-4k-l, u)$ by mx we mean $\max[u-r, u-z+1+4k+l, 0]$.

From Definition (4.6) of $f(r, j, s, k, t)$ we see that in the sum (4.7) only terms with odd u are different from 0. Indeed,

$$f(l+1, z-1-4k-l, r, z-4w-r, u) = (-1)^u f(r, z-4w-r, l+1, z-1-4k-l, u).$$

Hence

$$\begin{aligned} \delta^{-1}F \circ \delta^{-1}F &= \sum_{k=0}^{\left[\frac{z-1}{4}\right]} \sum_{w=0}^{\left[\frac{z-1}{4}\right]} \sum_{l=0}^{z-1-4k} \sum_{r=0}^{z-1-4w} \\ &\times \sum_{u=0}^{\left\lceil \frac{\min[l+1, z-4w-r] + \min[z-1-4k-l, r]}{2} - \frac{1}{2} \right\rceil} 2i(-1)^u \hbar^{2k+2w+2u+1} \\ &\times b_{2k,l} b_{2w,r} (X^1)^{l+r-2u} (X^2)^{2z-4k-4w-l-r-2u-2} \\ &\times f(l+1, z-1-4k-l, r, z-4w-r, 2u+1) dq \wedge dp. \end{aligned} \quad (4.8)$$

The degree of the product $\delta^{-1}F \circ \delta^{-1}F$ is $2z$ and each term of (4.8) is determined by powers of \hbar^{2A+1} and $(X^1)^B$ only. It is obvious that $4A+B+2 \leq 2z$.

Formula (4.8) can be written as

$$\delta^{-1}F \circ \delta^{-1}F = \hbar^{2A+1} (X^1)^B (X^2)^{2z-4A-B-2} g_{2A+1,B}, \quad (4.9)$$

where $g_{2A+1,B}$ are some coefficients computed below.

Comparing (4.8) and (4.9) we obtain the system of equations

$$2k+2w+2u+1 = 2A+1, \quad l+r-2u = B. \quad (4.10)$$

Its solutions are

$$u = A - k - w, \quad r = 2A + B - 2k - 2w - l. \quad (4.11)$$

So

$$g_{2A+1,B} \sim \sum_{k=0}^{\lfloor \frac{z-1}{4} \rfloor} \sum_{w=0}^{\lfloor \frac{z-1}{4} \rfloor} \sum_{l=0}^{z-1-4k} 2i(-1)^{A-k-w} b_{2k,l} b_{2w,2A+B-2k-2w-l} \\ \times f(l+1, z-1-4k-l, 2A+B-2k-2w-l, z-2A-B+2k-2w+l, 2A-2k-2w+1).$$

We did not write the equality symbol because not for all A, B this formula works. The reason is that parameters u and r fulfill conditions

$$0 \leq 2u \leq \min[l+1, z-2A-B+2k-2w+l] \\ + \min[z-1-4k-l, 2A+B-2k-2w-l] - 1, \\ 0 \leq r \leq z-1-4w \quad (4.12)$$

which were not taken into account in sums appearing in the definition of g_{AB} . From inequalities (4.12) we obtain that

$$k+w \leq A, \quad |k-w| \leq \left\lfloor \frac{z-1}{2} \right\rfloor - A \quad (4.13)$$

and

$$2A+B-z+1-2k+2w \leq l \leq 2A+B-2k-2w. \quad (4.14)$$

Finally

$$g_{2A+1,B} = \sum_{k=0}^{\min[A, \lfloor \frac{z-1}{4} \rfloor]} \sum_{w=\max[0, A+k-\lfloor \frac{z-1}{2} \rfloor]}^{\min[\lfloor \frac{z-1}{4} \rfloor, A-k, k-A+\lfloor \frac{z-1}{2} \rfloor]} \sum_{l=\max[0, 2A+B-z-2k+2w+1]}^{\min[z-1-4k, 2A+B-2k-2w]} \\ \times 2i(-1)^{A-k-w} b_{2k,l} b_{2w,2A+B-2k-2w-l} \\ \times f(l+1, z-1-4k-l, 2A+B-2k-2w-l, \\ z-2A-B+2k-2w+l, 2A-2k-2w+1). \quad (4.15)$$

We look for solutions $\delta^{-1}F$ of the equation $\delta^{-1}F \circ \delta^{-1}F = 0$. This equation is equivalent to the system of equations

$$g_{2A+1,B} = 0, \quad 4A+B+2 \leq 2z. \quad (4.16)$$

for functions $b_{2k,r}$.

We start solving (4.16) from the equation parametrized by $A=B=0$. Since the only one possibility is $k=w=l=0$, from (4.15) we immediately obtain

$$2i b_{0,0}^2 f(1, z-1, 0, z, 1) = 0.$$

As $f(1, z-1, 0, z, 1) = \frac{z}{2} \neq 0$ we deduce that $b_{0,0} = 0$.

The next equation we choose to solve is for $A = 0, B = 1$. It contains only a product $b_{0,0}b_{0,1}$ with some factor so, due to the fact that $b_{0,0} = 0$, it is fulfilled automatically. The next equation determined by $A = 0, B = 2$ with the condition $b_{0,0} = 0$ reduces to

$$g_{1,2} = 2i b_{0,1}^2 f(2, z-2, 1, z-1, 1) = 0.$$

Because $f(2, z-2, 1, z-1, 1) = \frac{z}{2} \neq 0$ we conclude that $b_{0,1} = 0$.

Repeating that procedure for $A = 0, B \leq 2z-2$ we see that all

$$b_{0,l} = 0, \quad 0 \leq l \leq z-1. \quad (4.17)$$

Putting $A = 1$ we do not obtain any further conditions for coefficients. But for $A = 2, B = 0$ (that implies $z \geq 5$) we see that if relations (4.17) hold then

$$g_{5,0} = 2i b_{2,0}^2 f(2, z-5, 0, z-4, 1) = 0.$$

As $f(2, z-5, 0, z-4, 1) = z-4 \neq 0$, we see that $b_{2,0} = 0$. Following this way we solve the system of equations (4.16) completely obtaining all of $b_{2k,r} = 0$.

Thus we showed that if $\delta^{-1}F \circ \delta^{-1}F = 0$ then $F = 0$. ■

As it has been said in Subsection 3.2, on a 2D Fedosov manifold (\mathcal{M}, Γ) the relation $R_\Gamma \neq 0$ yields $r[3] \neq 0$. Hence $r[3] = \delta^{-1}R_\Gamma$. From Proposition 4.1 we obtain that $r[3] \circ r[3] \neq 0$. Using Theorem 3.1 we conclude that there exists at least one nonzero component $r[z]$ of the correction r of degree $3 < z$. Remembering that (see (3.6)) $r[z] = \delta^{-1}(F[z-1])$, where

$$F[z-1] \stackrel{\text{def.}}{=} \partial_\Gamma r[z-1] + \frac{1}{i\hbar} \sum_{j=3}^{z-2} r[j] \circ r[z+1-j],$$

and applying Proposition 4.1 to $r[z]$ we see that $r[z] \circ r[z] \neq 0$. Hence, from Theorem 3.1 there exists $r[z_1] \neq 0$ such that $z_1 > z$. The Definition (3.6) of $r[z_1]$ plus Proposition 4.1 guarantee that $r[z_1] \circ r[z_1] \neq 0$. Theorem 3.1 implies $z_1 < \deg(r)$. Following this pattern we arrive at the following

Theorem 4.1 *On 2D phase space with nonvanishing symplectic curvature 2-form R_Γ any Abelian connection is an infinite series.*

We stress that Theorem 4.1 holds for 2D real symplectic manifolds with the correction r determined by formula (2.17) fulfilling (2.18). On Kähler locally symmetric manifolds of complex dimension 1 there exists a finite Abelian connection (see [12]) but for it $\delta^{-1}r \neq 0$.

5. Conclusions

The Fedosov quantization method is based on recurrent formulas (2.17) and (2.19). The first one defines the correction r to the Abelian connection, the second defines a flat section $\sigma^{-1}(a_0) \in \mathcal{P}^*\mathcal{M}[[\hbar]]_{\tilde{F}}$ representing a quantum observable a_0 .

The general form of the series $\sigma^{-1}(a_0)$ and the formula expressing the product of observables $a_0 * b_0$ are not known. Hence we search cases, in which these objects can be written in a compact form. Such a situation happens for example when both iterations (2.17) and (2.19) generate finite formal series. To know a complete general form of $\sigma^{-1}(G)$ for G belonging to some class of functions is extremely useful for example when we look for a solution of an eigenvalue equation for an observable a_0 ,

$$a_0 * W = A \cdot W.$$

By W we mean a Wigner eigenfunction corresponding to an eigenvalue A . We assume only that W is some smooth real function defined on the symplectic manifold \mathcal{M} so the general formula for $\sigma^{-1}(W)$ representing a Wigner function W is required.

Another advantage of working with finite series $\sigma^{-1}(a_0)$ is connected with computer calculations. After the finite number of steps we obtain complete results.

As we have mentioned, in the Fedosov quantization method there are two iterative formulas. In current paper we have considered the question, when the Abelian connection on a Fedosov manifold (\mathcal{M}, Γ) described in Theorem 2.2 is a finite formal series. We find a system of equations determining the sufficient and necessary condition for r to be finite.

Then we apply the result quoted above to the case of 2D phase space with nonvanishing curvature. We have shown that the series r on such spaces is always infinite. By the way we have found an explicit formula describing the \circ -product in 2D case.

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