SIGNUM-GORDON WAVE EQUATION AND ITS SELF-SIMILAR SOLUTIONS

H. Arodź, P. Klimas, T. Tyranowski

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received July 30, 2007)

We investigate self-similar solutions of evolution equation of a (1+1) dimensional real, scalar field φ with V-shaped field potential $U(\varphi) = |\varphi|$. The equation contains a nonlinear term of the form sign(φ), and it has a scaling symmetry. It turns out that there are several families of the self-similar solutions with qualitatively different behaviour. We also discuss a rather interesting example of evolution with non self-similar initial data — the corresponding solution contains a self-similar component.

PACS numbers: 05.45.–a, 03.50.Kk, 11.10.Lm

1. Introduction

Signum-Gordon equation[1] has the form

$$\frac{\partial^2\varphi(x,t)}{\partial t^2} - \frac{\partial^2\varphi(x,t)}{\partial x^2} = -\mathrm{sign}(\varphi(x,t))\,, \tag{1}$$

where φ is a real scalar field, and x, t are dimensionless variables obtained by appropriate choice of units for the physical position and time coordinates. It follows from the Lagrangian

$$L = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - |\varphi|\,,$$

where $\mu = 0, 1,\ \partial_0 = \partial_t,\ \partial_1 = \partial_x$. The sign function has the values ± 1 when $\varphi \neq 0$ and 0 for $\varphi = 0$. Because of the sign(φ) term Eq. (1) is nonlinear in a rather interesting way. The corresponding field potential $U(\varphi) = |\varphi|$ has the minimum at $\varphi = 0$, and the field φ can oscillate around the equilibrium value $\varphi = 0$, but Eq. (1) cannot be linearised even if amplitude of the oscillations is arbitrarily small.

[1] We thank Benny Lautrup from NBI for suggesting the name.

The signum-Gordon model has rather sound physical justification. It has been obtained in a continuum limit (or in a large wavelength approximation) to certain easy to built mechanical system with a large number of degrees of freedom, see [1, 2] for details. The model can also be applied in the description of pinning of an elastic string, which represents a vortex, to a rectilinear impurity in the case both, the string and impurity, lie in one plane. Yet another interesting application is the dynamics of a system of two global strings in a plane. Such strings are represented by the real functions $\psi_1(x,t), \psi_2(x,t)$, and in certain approximations their dynamics can be summarised in the following Lagrangian

$$L_{\rm s}(\psi_1,\psi_2) = \frac{1}{2}\partial_\mu\psi_1\partial^\mu\psi_1 + \frac{1}{2}\partial_\mu\psi_2\partial^\mu\psi_2 - V(\psi_1-\psi_2)\,,$$

where V is an extrapolation of the well-known logarithmic interaction potential between separated global strings to small values of $|\psi_1-\psi_2|$:

$$V(\psi_1-\psi_2) = a\ln\left(1+\frac{|\psi_1-\psi_2|}{a}\right)\,,$$

where a is a positive constant (see [3] for a detailed discussion of the inter-string potential). For small values of $|\psi_1-\psi_2|/a$ we have

$$V(\psi_1-\psi_2) \approx |\psi_1-\psi_2|\,.$$

In this case Euler–Lagrange equations for ψ_1, ψ_2 obtained from the Lagrangian $L_{\rm s}$ imply that $\varphi = \frac{1}{2}(\psi_1-\psi_2)$ obeys the signum-Gordon equation. The solutions presented in our paper describe particular cases of the time evolution of all those systems. The present paper, however, is focused on the self-similar solutions of the signum-Gordon equation rather than on its applications.

The signum-Gordon equation possesses the exact scaling symmetry: if $\varphi(x,t)$ is its solution then

$$\varphi_\lambda(x,t) = \lambda^2\,\varphi\left(\frac{x}{\lambda},\frac{t}{\lambda}\right) \tag{2}$$

is a solution too, for any constant $\lambda > 0$[2]. This symmetry is one of the most interesting features of the model. As always when there is a symmetry, one may search for solutions which are invariant under the symmetry transformations. In the case of scaling they are called the self-similar ones. In paper [2]

[2] Transformations of the form (2) with $\lambda < 0$ are obtained as a product of the scaling transformation (2) with the reflections $x \to -x,\ t \to -t$. Eq. (1) is invariant with respect to such reflections, and also with respect to 1+1 dimensional Poincaré transformations.

an example of such solutions of the signum-Gordon equation has been given. Self-similar solutions of nonlinear equations play very important role in nonlinear dynamics, and they have plenty applications. Beautiful presentation of self-similarity with its applications is given in [4]. A shorter introduction to this topic can be found in [5].

The present paper is devoted to a thorough investigation of self-similar solutions of the signum-Gordon equation. Rather surprisingly, we have found that there are many classes of such solutions. The solutions presented in [2] form merely a measure zero subset in just one such class: the segment $S_0 = 0$, $-1/2 < \dot{S}_0 < 1/2$ in Fig. 1. Amusingly enough, the self-similar solutions are composed mainly from quadratic polynomials in x and t. The various types of solutions differ in particular by the number of isolated zeros of the field φ: they can have infinitely many, just one or no isolated zeros. We give explicit forms of all self-similar solutions corresponding to initial data (5) with two arbitrary, real parameters $S_0, \dot{S}_0$. These solutions form a two-dimensional subset of the four-dimensional set of the self-similar solutions of the signum-Gordon equation. It is depicted as Fig. 1, which can be regarded as the most interesting result of our paper. Let us stress that it is rather unusual that one can give explicit analytic forms of self-similar solutions of a nonlinear evolution equation for a wide class of initial data.

We have also investigated the evolution from certain particular initial data which are not self-similar, with $\partial_x\varphi$ discontinuous at a certain point. While the late time behaviour of the corresponding solution remains mysterious, we have noticed a very interesting phenomenon which occurs during the early stages of the evolution. At first, the point of the discontinuity of $\partial_x\varphi$ moves with a constant velocity equal to -1 ("the velocity of light") until $\partial_x\varphi$ becomes a continuous function of x. From this time on, the evolution of φ locally, that is in a vicinity of the former point of discontinuity, follows a self-similar solution with one isolated zero. This behaviour is universal in the sense that it does not depend on details of the initial data. Solutions of this class might be relevant for the description of the process of coalescence of two strings, see Fig. 9.

All our analytic solutions have been checked by comparisons with purely numerical solutions of Eq. (1). Actually, there was a very fruitful interplay of the two methods: certain solutions were seen first numerically and later obtained in the analytic form, whereas with others it was the other way round.

The plan of our paper is as follows. In Sec. 2 we discuss the Ansatz and initial data for the self-similar solutions, and we present a map of such solutions. Sec. 3 is devoted to a detailed presentation of all classes of the self-similar solutions: in Subsections 3.1–3.5 we give explicit forms of all solutions. In Sec. 4 we consider the initial stages of evolution in the particular case of non self-similar initial data. Sec. 5 contains a summary and remarks.

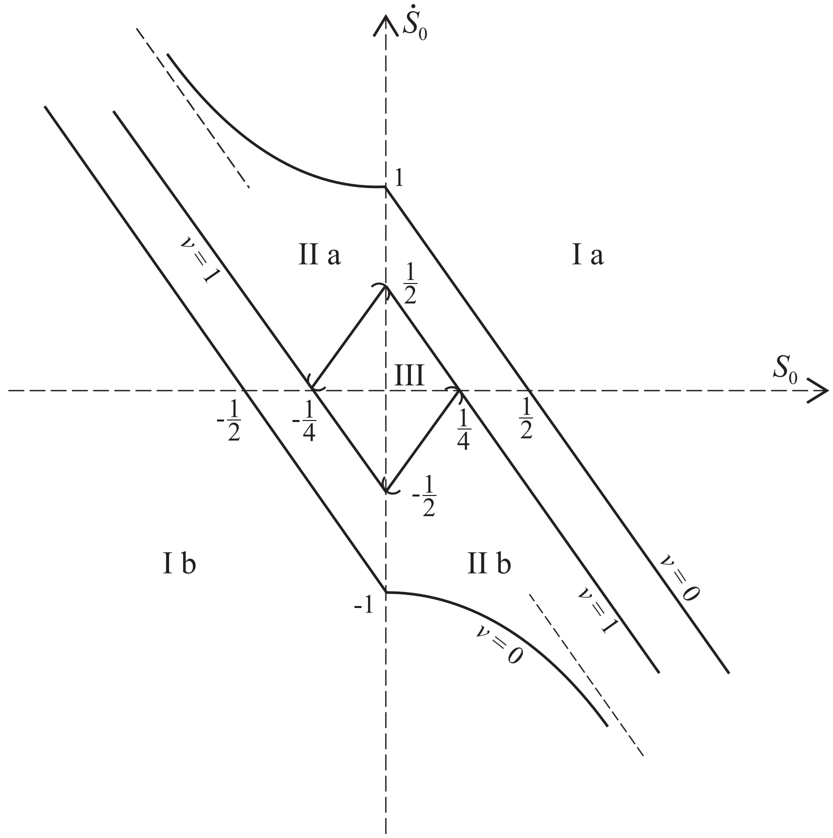

Fig. 1. The map of the self-similar solutions of the signum-Gordon Eq. (1). Each point in the $(S_0, \dot{S}_0)$ plane represents one solution. The symmetry of the picture with respect to the reflection $(S_0, \dot{S}_0) \rightarrow (-S_0, -\dot{S}_0)$ is related to the reflection symmetry $\varphi \rightarrow -\varphi$ of the signum-Gordon equation. Because of this symmetry it is sufficient to discuss solutions from the half-plane $S_0 \geq 0$. The dashed lines are auxiliary: two coordinate lines, and the two lines $\dot{S}_0 = -2S_0$ which are asymptotically tangent to the lines $v = 0$. $v \in (-1, 1]$ is a certain velocity defined in the text. It is a function of the parameters $S_0, \dot{S}_0$ for the given solution. The continuous lines are border lines between the classes of solutions, but their points also represent some special solutions discussed in subsections 3.1 (the $v = 0$, $S_0 \geq 0$ rectilinear half-line), 3.2 (the $v = 0, S_0 \geq 0$ curved line), 3.4 (the open segment connecting the point $(0, -1/2)$ with the point $(1/4, 0)$), and 3.5 (the $v = 1, S_0 \geq 0$ half-line). The lower, curved $v = 0$ line is a cubic curve given by Eq. (16), the other one is obtained by the reflection. The rounded brackets, which enclose two line segments — the edges of the rhomb III, indicate that these segments do not contain their ends — the end points belong to the two $v = 1$ half-lines.

2. The Ansatz and initial data

The physical context in which the signum-Gordon equation appears implies that the most interesting solutions are such that $\varphi(x,t)$ is a continuous function on the (x,t) plane. For the first order partial derivatives $\partial_t\varphi$, $\partial_x\varphi$ only discontinuities across the lines $x = x_0 \pm t$ (the characteristics) are allowed. For example, a discontinuity of φ as a function of x would mean that the string is broken. A discontinuity of $\partial_t\varphi(x,t)$ at a certain instant t_1 would mean that the velocity of a certain part of a continuous mechanical system (*e.g.*, of a string) suddenly jumps in spite of the fact that the force, given by the $\mathrm{sign}(\varphi)$ term, is always finite. Hence, such a jump is possible only if simultaneously also $\partial_x\varphi(x,t)$ is not continuous (then $\partial_x^2\varphi(x,t)$ becomes infinite). The well-known reasoning [6] gives Rankine–Hugoniot condition for matching the continuous pieces of the solutions. We do not quote it here because it just implies that the partial derivatives can be discontinuous only across the characteristics, which is a well-known fact for hyperbolic evolution equations.

On the other hand, Eq. (1) implies that at least one of the second order derivatives $\partial_t^2\varphi$, $\partial_x^2\varphi$ has to be discontinuous when the piecewise constant function $F(x,t) = \mathrm{sign}(\varphi(x,t))$ changes its value from 0 to ±1. At such a point $\varphi = 0$, and in a certain neighbourhood of it $\varphi > 0$ or $\varphi < 0$. Then, the left and right second derivatives of φ can have different values — if this is the case then the ordinary second derivative of φ does not exist, strictly speaking. Nevertheless, such solutions are physically admissible. The proper mathematical framework for discussing them is based on the notion of weak solutions, see, *e.g.* [6,7]. The reader interested in the mathematical aspects of the weak solutions, including the uniqueness of solution of Cauchy problem, should consult the literature.

In the case of transformation (2) the scale invariant Ansatz can be taken in the form

$$\varphi(x,t) = x^2 S(y)\,, \qquad y = \frac{t}{x}\,. \tag{3}$$

We will consider solutions of Eq. (1) for $t > 0$ with initial data specified at $t = 0$. Solutions for $t < 0$ can be easily obtained with the help of the time reflection symmetry. It will turn out that $S(y)$ can be a quadratic, linear or constant function of y. Therefore, φ will be continuous at $x = 0$ in spite of the fact that the scale invariant variable y is singular at that point. Equivalent Ansatz

$$\varphi(x,t) = t^2\, T\left(\frac{x}{t}\right)\,,$$

avoids the superficial singularity at $x = 0$, but it is less convenient for incorporating the initial data.

The self-similar initial data have the form

$$\varphi(x,0) = \begin{cases} R_0 x^2 & \text{for } x \le 0\,, \\ S_0 x^2 & \text{for } x \ge 0\,, \end{cases} \qquad \partial_t \varphi(x,t)|_{t=0} = \begin{cases} \dot{R}_0 x & \text{for } x \le 0\,, \\ \dot{S}_0 x & \text{for } x \ge 0\,, \end{cases} \tag{4}$$

where $R_0, \dot{R}_0, S_0, \dot{S}_0$ are constants. Such initial data are self-similar because the point $x = 0$ is not shifted by the rescaling $x \to x/\lambda$. In general, the $\varphi(x,0)$ is of $C^1(R)$ class, while the $\partial_t \varphi(x,t)|_{t=0}$ is of class $C^0(R)$. One can easily show that formula (4) gives the most general self-similar initial data.

We shall restrict our considerations to the case $R_0 = \dot{R}_0 = 0$. The other choice: $S_0 = \dot{S}_0 = 0$ and $R_0, \dot{R}_0$ arbitrary is related to the previous one by the spatial reflection $x \to -x$, which is a symmetry of Eq. (1). Thus, in the main part of our paper we assume that

$$\varphi(x,0) = \begin{cases} 0 & \text{for } x \le 0\,, \\ S_0 x^2 & \text{for } x \ge 0\,, \end{cases} \qquad \partial_t \varphi(x,t)|_{t=0} = \begin{cases} 0 & \text{for } x \le 0\,, \\ \dot{S}_0 x & \text{for } x \ge 0\,. \end{cases} \tag{5}$$

Such initial data correspond to the following conditions for the function $S(y)$

$$\lim_{y\to 0+} S(y) = S_0\,, \quad \lim_{y\to 0-} S(y) = 0\,, \quad \lim_{y\to 0+} S'(y) = \dot{S}_0\,, \quad \lim_{y\to 0-} S'(y) = 0\,. \tag{6}$$

Let us remark that solutions for more general self-similar initial data (4) can be obtained in certain cases just by combining solutions obeying (5) with the ones obtained from them by applying the spatial reflection, but there are also cases in which such approach does not work.

The Ansatz (3) reduces Eq. (1) to the following ordinary differential equation for the function $S(y)$

$$(1-y^2)\, S'' + 2y\, S' - 2\, S = -\text{sign}(S)\,, \tag{7}$$

where $S' = dS/dy$, and the sign(S) function has the values $+1$ or -1 for $S > 0$ or $S < 0$, respectively, and sign$(0) = 0$. Notice that at the points $y = \pm 1$ (which correspond to characteristics of Eq. (1)) the coefficient in front of the second derivative term in Eq. (7) vanishes. This has the consequence that the first derivative dS/dy of the solution does not have to be continuous at these points. However, one can easily show that the discontinuity at $y = 1$ can occur only if sign(S) is not constant at that point. In order to obtain weak solutions of this equation we first solve it assuming that $S > 0$ or $S < 0$ or $S = 0$. Such partial solutions have a rather simple form of quadratic polynomials in y:

$$\text{when} \quad S > 0: \quad S_+(y) = -\tfrac{\beta}{2}(y^2+1) + \tfrac{\alpha}{2} y + \tfrac{1}{2}\,,$$

$$\text{when} \quad S < 0: \quad S_-(y) = \tfrac{\beta'}{2}(y^2+1) - \tfrac{\alpha'}{2} y - \tfrac{1}{2}\,,$$

where $\alpha, \beta, \alpha', \beta'$ are arbitrary constants. The polynomial solutions are valid on appropriate intervals of the y-axis determined by the conditions $S > 0$ or $S < 0$, respectively. There is also the trivial solution

$$S_0(y) = 0 \, .$$

Next, we match such partial solutions requiring that S and S' are continuous functions of y, except at the point $y = 1$, where only continuity of S is required. It turns out that the continuity of only S at $y = 1$ is sufficient to ensure the implied by wave equation (1) continuity of

$$\frac{\partial \varphi}{\partial x_+} = \frac{1}{2} \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t} \right) ,$$

where $x_+ = x + t$, across the characteristic line $x = t$. The matching conditions together with the initial data (5) determine the constants $\alpha, \beta, \alpha', \beta'$ from the partial solutions. In this manner we have constructed exact, self-similar weak solutions of the Cauchy problem (5) for all values of parameters $S_0, \dot{S}_0$. Because $x \in (-\infty, \infty)$, the variable y can take all real values. The factor x^2 in (3) makes the solutions finite at $x = 0$.

Depending on the values of $S_0, \dot{S}_0$ the self-similar solutions have quite different forms. As one can see from the "map" presented in Fig. 1, the set of such solutions of the signum-Gordon equation is surprisingly rich.

3. Self-similar solutions

Below we give the analytic forms of the self-similar solutions with initial data (5). We assume that $S_0 \geq 0$. Solutions with negative values of S_0 can be obtained by the reflection $\phi(x,t) \to -\phi(x,t)$, which in particular means that $S_0 \to -S_0, \ \dot{S}_0 \to -\dot{S}_0$.

3.1. Solutions of types: I a, II a, $v = 0$

These solutions lie in the open region above the rhomb III and above the $v = 1, \ S_0 \geq 0$ line, see Fig. 1. The solutions are obtained by matching the trivial solution with the partial solution S_+ at the point $y = 1/v$ which corresponds to the line $x = vt$. The velocity v has values in the interval $(-1, 1)$, and it is given by formula (8). It turns out that one has to use a second solution of the type S_+ which matches the previous one at the point $y = 1$ corresponding to the characteristic line $x = t$. That latter S_+ solution has the form $S_0 + \dot{S}_0 y - (1/2 - S_0) y^2$, where $S_0 \geq 0, \ \dot{S}_0$ are the parameters specifying the initial data according to formula (5).

The velocity v is given by the formula

$$v = \frac{1}{2S_0 + \dot{S}_0} - 1 \, , \tag{8}$$

which follows from the matching condition at the point $y = 1$. In the region I a the velocity has values from the interval $(-1, 0)$, while in the region II a $v \in (0, 1)$. On the line $2S_0 + \dot{S}_0 = 1$, which separates the two regions, $v = 0$. The full solution has the form

$$\varphi(x,t) = \begin{cases} \varphi_0 = 0\,, & \text{for } x \leq vt\,, \\ \varphi_1 = \frac{(x-vt)^2}{2(1-v^2)}\,, & \text{for } vt \leq x \leq t\,, \\ \varphi_2 = S_0 x^2 + \dot{S}_0 tx + (S_0 - \frac{1}{2})t^2\,, & \text{for } x \geq t\,. \end{cases} \tag{9}$$

It is depicted in Fig. 2.

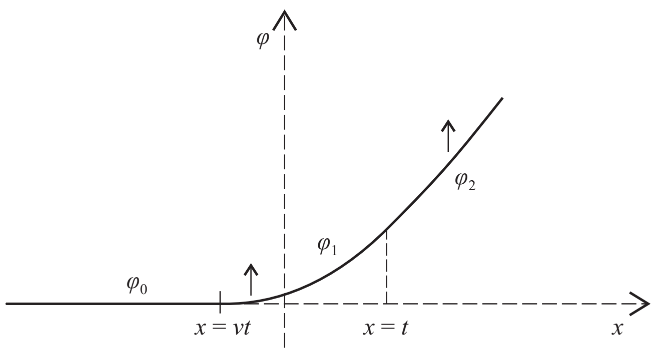

Fig. 2. Picture of a solution of type I a for which $-1 < v < 0$. v is the velocity of the point on the x axis at which φ_1 merges with the trivial solution $\varphi = 0$. The two vertical arrows indicate that values of φ increase with time — the $\varphi > 0$ part moves to the left. Solutions of the type II a have a similar shape, but they move to the right $(0 < v < 1)$.

The half-line $2S_0 + \dot{S}_0 = 1,\ S_0 \geq 0, v = 0$, which separates the classes I a and II a, includes the static solution

$$\varphi_{s+}(x) = \begin{cases} 0 & \text{for } x \leq 0\,, \\ x^2/2 & \text{for } x \geq 0\,, \end{cases} \tag{10}$$

for which $S_0 = 1/2,\ \dot{S}_0 = 0$. Other solutions from that line are time dependent. They coincide with the static one in the growing with time interval $0 \leq x \leq t$, while at the points $x > t$ they have the form $\varphi_2(x,t)$, see the second and third lines in formula (9). Asymptotically, as t grows to infinity, such φ entirely covers the static solution.

The signum-Gordon equation is invariant under the Lorentz transformations

$$t' = \gamma(t - wx), \quad x' = \gamma(x - wt)\,,$$

where $\gamma = 1/\sqrt{1-w^2}$, $|w| < 1$. Such transformations preserve self-similarity of solutions. One can see that the static solution (10) is transformed into the solution with $S_0 = \gamma^2/2$, $\dot{S}_0 = -w\gamma^2$. In this case formula (8) gives $v = w$. For these particular solutions the functions φ_1 and φ_2 in formula (9) have the same form.

3.2. Solutions of types: I b, II b, $v = 0$.

These solutions lie in the open region below the rhomb III and the $v = 1$, $S_0 > 0$ line, see Fig. 1. Solutions of this type have the form shown in Fig. 3.

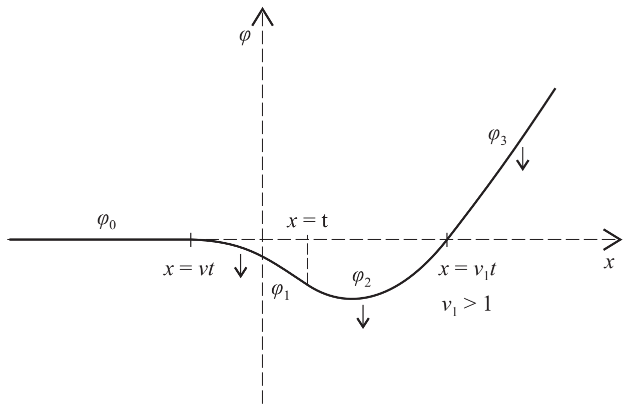

Fig. 3. Picture of a solution of type I b for which $-1 < v < 0$. The arrows indicate that the values of φ decrease with time — the $\varphi < 0$ part expands in both directions. The solutions of type II b have a similar shape, but they move to the right $(0 < v < 1)$.

Solutions from the half-line $S_0 = 0$, $\dot{S}_0 \leq -1/2$ are not considered here because they can be obtained from the solutions lying on the $S_0 = 0$, $\dot{S}_0 \geq 1/2$ half-line with the help of the reflection $\phi \to -\phi$. The solutions shown in Fig. 3 are composed of the trivial solution $S = 0$ in the region $y > 1/v > 1$, one solution S_+ in the region $y < 1/v_1 < 1$, and two solutions S_- in the region $1/v_1 < y < 1/v$ which continuously match each other at the point $y = 1$. The pertinent formula for φ has the following form

$$\varphi(x,t) = \begin{cases} \varphi_0 = 0 & \text{for } x \leq vt\,, \\ \varphi_1 = -\frac{(x-vt)^2}{2(1-v^2)} & \text{for } vt \leq x \leq t\,, \\ \varphi_2 = \frac{\beta_2}{2}\left(x^2+t^2\right) - \frac{\alpha_2}{2}xt\frac{1}{2}x^2 & \text{for } t \leq x \leq v_1 t\,, \\ \varphi_3 = S_0 x^2 + \dot{S}_0 tx + \left(S_0 - \frac{1}{2}\right)t^2 & \text{for } x \geq v_1 t\,. \end{cases} \tag{11}$$

Here $S_0, \dot{S}_0$ are given in the initial data (5). The velocity v_1 is obtained from the condition $\varphi_3(v_1 t, t) = 0$, and α_2, β_2 from the matching conditions at $x = v_1 t$, *i.e.* at $y = 1/v_1 < 1$. Straightforward calculations give

$$v_1 = \frac{-\dot{S}_0 + \sqrt{\dot{S}_0^2 - 4S_0^2 + 2S_0}}{2S_0}, \tag{12}$$

$$\alpha_2 = 2\frac{v_1 + (1+v_1^2)\sqrt{\dot{S}_0^2 - 4S_0^2 + 2S_0}}{v_1^2 - 1}, \tag{13}$$

$$\beta_2 = \frac{v_1^2 + 2v_1\sqrt{\dot{S}_0^2 - 4S_0^2 + 2S_0}}{v_1^2 - 1}. \tag{14}$$

Finally, the velocity v is obtained from the condition of continuity of $\varphi(x,t)$ at the point $x = t$

$$v = \frac{2\beta_2 - \alpha_2}{2 + \alpha_2 - 2\beta_2}. \tag{15}$$

Formulas (12–15) imply that $v_1 > 1$ and $-1 < v < 1$.

The condition $v = 0$ is satisfied by $(S_0,\ \dot{S}_0)$ such that

$$\eta = \frac{1}{2} + \frac{1}{2\xi^2}, \tag{16}$$

where

$$\eta = 2S_0 - \dot{S}_0, \qquad \xi = 2S_0 + \dot{S}_0.$$

These points form the lower curved line in Fig. 1. Solutions of this kind are related to another static solution of Eq. (1), namely to

$$\varphi_{s-}(x) = \begin{cases} 0 & \text{for } x \le 0, \\ -x^2/2 & \text{for } x \ge 0, \end{cases} \tag{17}$$

for which $S_0 = -1/2,\ \dot{S}_0 = 0$. These $v = 0$ solutions are time dependent. Their φ_1 part, see formula (11), coincides with the static one in the growing with time interval $0 \le x \le t$, while at the points $x > t$ their form is given by $\varphi_2(x,t),\ \varphi_3(x,t)$. Asymptotically, as t grows to infinity, we recover the static solution.

3.3. Solutions of type III

Let us try to put together a number of the partial solutions S_+, S_- in order to cover as large as possible interval of the $y \ge 0$ half-axis, see Fig. 4, where $S_k,\ k = 1, 2, \ldots$, denote the consecutive partial solutions $S_\pm$.

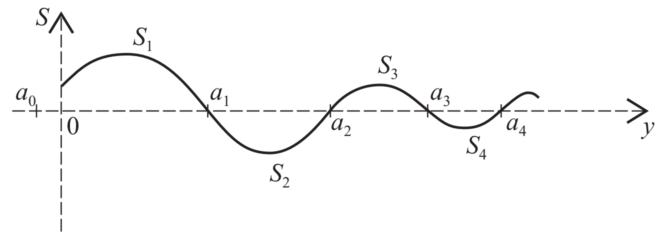

Fig. 4. The polynomials S_k and the matching points a_k, $k = 1, 2, \ldots$

The restriction to $y \geq 0$, see Fig. 4, is quite natural. The point is that $y = 0_-$ corresponds to $x = -\infty$, while $x = 0_+$ to $x = +\infty$. Hence, there is no physical reason for demanding that $S(y)$ is continuous at $y = 0$ — the continuity at $y = 0$ would mean that we arbitrarily impose the restriction $\varphi(\infty, t) = \varphi(-\infty, t)$. Therefore, the solution S_1 is terminated at $y = 0$, and in the region $y < 0$ we take the trivial solution $S_0(y) = 0$. On the other hand, both $y = -\infty$ and $y = +\infty$ correspond to $x = 0$, where the solution φ is required to be a continuous and differentiable function of x.

Let us write the partial solutions in the form

$$S_k(y) = \frac{1}{2}(-1)^k \left[\beta_k(y^2 + 1) - \alpha_k y - 1\right] .$$

Equation (1) implies that the following matching conditions at the points $y = a_k, k \geq 1$, have to be satisfied:

$$S_k(a_k) = 0 = S_{k+1}(a_k) , \qquad S'_k(a_k) = S'_{k+1}(a_k) , \tag{18}$$

provided that $a_k \neq \pm 1$. When $y = \pm 1$, only continuity of $S(y)$ has to be required.

The matching conditions (18) yield the recurrence relations:

$$\alpha_{k+1} = \frac{4a_k}{1-a_k^2} - \alpha_k , \quad \beta_{k+1} = \frac{2}{1-a_k^2} - \beta_k , \quad a_{k+1} = \frac{2a_k - (1+a_k^2)a_{k-1}}{1+a_k^2 - 2a_k a_{k-1}} , \tag{19}$$

where $\alpha_1 = a_1 > 0, a_0 \leq 0$, and

$$\beta_1 = \frac{1}{1 - a_0 a_1} , \qquad \alpha_1 = \frac{a_0 + a_1}{1 - a_0 a_1} . \tag{20}$$

The relations (20) follow from the fact that a_0, a_1 are the zeros of the polynomial $S_1(y)$. It is clear from Fig. 4 that $a_0 \leq 0, a_1 > 0$.

The solution is single-valued when $a_{k+1} \geq a_k$. Simple calculations show that this condition taken for $k = 1, 2$ implies that $a_1 \leq 1$, $a_2 \leq 1$, and that $a_0 \geq -1$. Furthermore, we notice that $a_0 = -1$ or $a_1 = 1$, always give

$a_2 = 1$, what implies that $a_k = 1$ for all $k > 2$. Such solutions, *i.e.*, with $a_2 = 1$, belong to the classes of solutions discussed in Subsections 3.4, 3.5, except for the case $a_0 = -1, a_1 = 0$ in which we obtain the trivial solution $S_0 = 0$. Therefore, in the remaining part of this subsection we assume that

$$-1 < a_0 \leq 0\,, \qquad 0 < a_1 < 1\,.$$

Recurrence relations (19) have the following general solutions:

$$\alpha_k \;=\; \frac{1}{1+r}\left[\frac{1}{pr^{k-1}} - pr^k + (-1)^k(\frac{1}{p} - q)\right] - (-1)^k\alpha_1\,, \tag{21}$$

$$\beta_k \;=\; \frac{1}{2}+\frac{1}{2(1+r)}\left[\frac{1}{pr^{k-1}}+pr^k+(-1)^k(1+\frac{1}{p})(1+q)\right]-(-1)^k\beta_1\,, \tag{22}$$

and

$$a_k = \frac{p^{k-1} - q^k}{p^{k-1} + q^k}\,, \tag{23}$$

where

$$q = \frac{1-a_1}{1+a_1}\,, \quad p = \frac{1-a_0}{1+a_0}\,, \quad r = \frac{q}{p}\,. \tag{24}$$

Note that $0 \leq q < 1$ and $p > 1$.

Solutions (21)–(23) have been conjectured after seeing several first iterations of recurrence relations (19), and checked by substitution into these relations. In paper [6] we were able to find only solutions with $a_0 = 0, a_1 < 1$. They lie on the segment $S_0 = 0, 0 < \dot{S}_0 < 1/2$ inside the rhomb III in Fig. 1.

Formula (23) implies that $a_k < 1$, and that $a_k \to 1$ when $k \to \infty$. Therefore, the piecewise polynomial solution constructed above covers only the interval [0, 1). For this reason we introduce a special notation for it, namely $S_{pp}(y)$. Calculations show that $S_{pp}(y)$ vanishes when $y \to 1$. On the other hand, the first derivative $S'_{pp}(y)$ does not vanish in that limit, but it remains finite.

This solution can be extended to the full range of y. We have already taken the trivial solution $S_0(y) = 0$ in the region $y < 0$. In the region $y \geq 1$ we also take the trivial solution which is consistent with the continuity of $S(y)$ at $y = 1$. The first derivative dS/dy in general is not continuous at $y = 1$, but this is not forbidden by Eq. (7) because the coefficient $1 - y^2$ in front of S'' vanishes when $y = 1$. Let us recall that $y = \pm 1$ correspond to the characteristics $x = \pm t$ of the signum-Gordon equation.

The solution we have just constructed has the following form:

$$\varphi(x,t) = \begin{cases} \varphi_0 = 0 & \text{for } x \leq t\,, \\ \varphi_{pp} = x^2 S_{pp}\left(\frac{t}{x}\right) & \text{for } x > t\,. \end{cases} \tag{25}$$

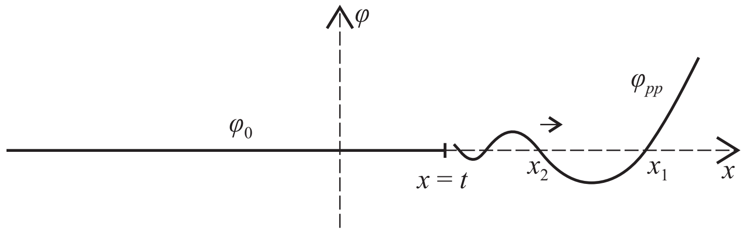

Fig. 5. The self-similar solution of type III. The zeros $x_k = v_k t$ of φ accumulate at $x = t$ because $v_k = 1/a_k$, where $a_k \to 1$ when $k \to \infty$. The arrow indicates that the whole structure moves to the right.

Snapshot of the solution is presented in Fig. 5.

The zeros of ϕ are given by the formula $x_k(t) = t/a_k$. They move with the constant velocities $v_k = 1/a_k > 1$.

The parameters $S_0, \dot{S}_0$, which specify initial data (5) for these solutions, enter the first polynomial S_1. Hence,

$$S_0 = S_1(0) = \frac{1}{2}(1-\beta_1)\,, \qquad \dot{S}_0 = S_1'(0) = \frac{\alpha_1}{2}\,, \tag{26}$$

where α_1, β_1 have the form (20). Because a_0, a_1 are the zeros of this polynomial, they are related to the initial data by the following formulas

$$a_0 = \frac{\dot{S}_0 - \sqrt{\Delta}}{1-2S_0}\,, \qquad a_1 = \frac{\dot{S}_0 + \sqrt{\Delta}}{1-2S_0}\,,$$

where $\Delta = \dot{S}_0^2 - 4S_0^2 + 2S_0$. By the assumption, for the considered solutions $-1 < a_0 \le 0$, $0 < a_1 < 1$, see Fig. 4. It follows that

$$2S_0 - \frac{1}{2} < \dot{S}_0 < \frac{1}{2} - 2S_0\,, \qquad 0 \le S_0 < \frac{1}{4}\,. \tag{27}$$

These inequalities correspond to the $S_0 \ge 0$ part of the interior of rhomb III in Fig. 1.

3.4. Solutions from the bottom-right open edge of rhomb III

In the present case we take the trivial solution in the regions $y \le 0$ and $y \ge 1$. At the point $y = 1$ the trivial solution matches the partial solution S_-. At this point we demand only continuity of $S(y)$. The solution S_- in turn matches a partial solution S_+ at a point y_1 from the interval $0 < y_1 < 1$. Here we also demand continuity of derivatives dS/dy. Of course, both $S_-(y_1)$ and $S_+(y_1)$ are equal to 0. The calculations are straighforward. It turns out that

$$v_1 = \frac{1}{2S_0} - 1\,. \tag{28}$$

This formula implies that $v_1 > 1$. The solution φ has the following form

$$\varphi(x,t) = \begin{cases} \varphi_0 = 0 & \text{for } x \leq t\,, \\ \varphi_1 = \frac{1}{2(v_1-1)}\,(x-t)(x-v_1 t) & \text{for } t \leq x \leq v_1 t\,, \\ \varphi_2 = (x+t)\left[S_0(x+t) - \frac{t}{2}\right] & \text{for } x \geq v_1 t\,, \end{cases} \tag{29}$$

see Fig. 6.

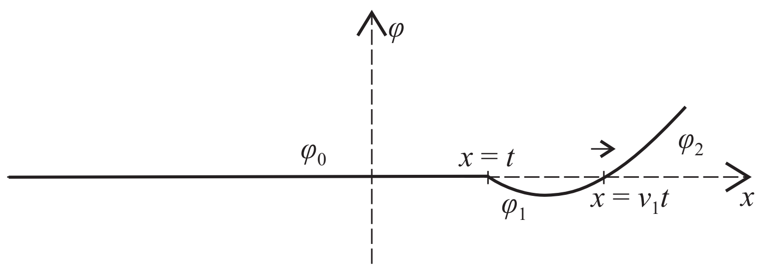

Fig. 6. The self-similar solution (29).

These solutions have the following initial data

$$\dot{S}_0 = 2S_0 - \frac{1}{2}\,, \qquad 0 < S_0 < \frac{1}{4}\,. \tag{30}$$

They lie on the open segment which is the interior of the $S_0 > 0$, $\dot{S}_0 < 0$ edge of rhomb III in Fig. 1.

Let us note that these solutions can be regarded as a limiting case of solutions discussed in the previous subsection obtained by putting $a_0 = -1$ and keeping a_1 in the open interval $(0, 1)$.

3.5. Solutions with $v = 1$

In this case we combine the trivial solution and an S_+ solution. They match continuously at the point $y = 1$.

$$\varphi(x,t) = \begin{cases} \varphi_0 = 0 & \text{for } x \leq t\,, \\ \varphi_1 = (x-t)\left[S_0(x-t) + \frac{t}{2}\right] & \text{for } x > t\,, \end{cases} \tag{31}$$

see Fig. 7. In particular, for $S_0 = 1/4$ we have $\varphi(x,t) = (x^2 - t^2)/4$ for $x \geq t$, and $\varphi = 0$ for $x \leq t$.

The parameters $S_0, \dot{S}_0$ for these solutions obey the relation

$$\dot{S}_0 = -2S_0 + \frac{1}{2}\,, \tag{32}$$

where $S_0 \geq 0$.

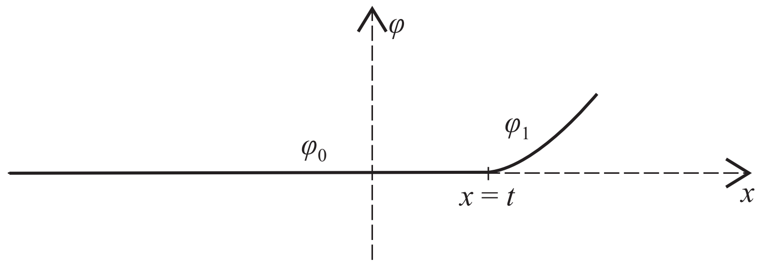

Fig. 7. The self-similar solution (31).

Some of these solutions are related to the ones discussed in Subsection 3.3: if we take $a_1 = 1$ and $-1 \leq a_0 \leq 0$, then only the first polynomial S_1 is present and it leads to the solution of the form (31). However, the restriction $-1 \leq a_0 \leq 0$ implies that $0 \leq S_0 \leq 1/4$ because $S_0 = -a_0/2(1 - a_0)$. This corresponds to that part of the $v = 1$, $S_0 \geq 0$ half-line, see Fig. 1, which is the upper-right edge of rhomb III.

4. Non self-similar solutions with a self-similar component

In this section we would like to present a certain interesting solutions of the signum-Gordon equation which are not self-similar. The corresponding initial data for them are different from (5); they have the form

$$\varphi(x, 0) = \begin{cases} 0 & \text{for} \quad x \leq 0\,, \\ \alpha x & \text{for} \quad x \geq 0\,, \end{cases} \qquad \partial_t \varphi(x, t)|_{t=0} = 0\,, \tag{33}$$

where $\alpha > 0$ is a free parameter. The solution φ was first obtained numerically in a context which is not relevant here. Computer simulations showed that, rather surprisingly, certain important quantitative characteristics of the evolution of φ apparently did not depend on the parameter α at all. Such numerical findings have motivated us to study these solutions in more detail. It has turned out that the solutions have a self-similar component during a certain finite time interval. For this reason we would like to present them here.

Because the signum term in Eq. (1) remains constant until φ becomes equal to zero, the analytic solution of Eq. (1) with the initial data (33) can be constructed piecewise. In each interval on the x axis such that φ has a constant sign in it, one can use the well-known formula for the general solution of the one-dimensional wave equation, suitably modified to incorporate the constant $+1$ (or -1) term in the equation:

$$\varphi(x, t) = h(x - t) + g(x + t) + c_1 t^2 - c_2 x^2\,,$$

where $2(c_1+c_2) = \pm 1 (= \mathrm{sign}(\varphi))$. The functions h, g and the constants c_1, c_2 are determined from the initial conditions and from the matching conditions. The resulting solution has a relatively simple form when the time t is not too large. The evolution of φ can be divided into distinct stages. In the first one time t changes from 0 to α. Then

$$\varphi_a(x,t) = \begin{cases} \varphi_0(x,t) = 0 & \text{for } x \leq -t\,, \\ \varphi_{a1}(x,t) = \frac{1}{8}(x+t)(4\alpha + x - 3t) & \text{for } -t \leq x \leq t\,, \\ \varphi_{a2}(x,t) = \alpha x - \frac{t^2}{2} & \text{for } x \geq t\,. \end{cases} \tag{34}$$

The function φ_{a1} remains positive in the interval $x \in (-t, t]$ until $t = \alpha$. Notice that the border point between the supports of the functions φ_0, φ_{a1} moves with the velocity -1 which does not depend on the slope α of the initial shape (33) of φ. This fact is related to the jump of the value of $\partial_x\varphi_a$ at the point $x = -t$ — such discontinuity is allowed for by the wave equation only on characteristics. Hence the border point between the supports of the functions φ_0, φ_{a1} has to move with velocity -1 until $\partial_x\varphi_{a1}|_{x=-t}$ vanishes. This happens at the moment $t = \alpha$.

At that moment

$$\varphi_a(x,\alpha) = \begin{cases} 0 & \text{for } x \leq -\alpha\,, \\ \frac{1}{8}(x+\alpha)^2 & \text{for } -\alpha \leq x \leq \alpha\,, \\ \alpha\left(x - \frac{\alpha}{2}\right) & \text{for } x \geq \alpha\,, \end{cases} \tag{35}$$

$$\text{and} \quad (\partial_t\varphi_a)(x,\alpha) = \begin{cases} 0 & \text{for } x \leq -\alpha\,, \\ -\frac{1}{4}(x+\alpha) & \text{for } -\alpha \leq x \leq \alpha\,, \\ -\alpha & \text{for } x \geq \alpha\,. \end{cases} \tag{36}$$

These values of $\varphi, \partial_t\varphi$ constitute the initial data for the second stage of the evolution of φ which lasts until $t = 2\alpha$. Here the crucial observation is that $\varphi_a, \partial_t\varphi_a$ at the time α have the shape which coincides with the self-similar initial data (5) in which x is replaced by $\xi = x + \alpha$. The corresponding solution belongs to the class discussed in Subsection 3.4. It has $S_0 = 1/8, \dot{S}_0 = -1/4$, and $v_1 = 3$. Note that the velocity v_1 does not depend on the parameter α. The φ_1 part, see Fig. 6, begins to appear at the point $x = -\alpha$ at the time $t = \alpha$. The right end of its support moves to the right with velocity $+3$, while the left end moves slower, with velocity $+1$. Thus, we expect that in the interval $x \in (-\infty, t]$ the solution has the form of a time and space translated self-similar solution (29) ($t \to \tau = t - \alpha, x \to \xi = x + \alpha$). At the point $x = t$ it continuously matches the function φ_{a2} from formula (34). With this in mind one can easily construct the analytic solution.

In the time interval $t \in [\alpha, 2\alpha]$ we have $\varphi(x,t) = \varphi_b(x,t)$, where

$$\varphi_b(x,t) = \begin{cases} 0 & \text{for } x \le t - 2\alpha\,, \\ \varphi_{b1} = \frac{1}{4}(x - x_0(t))(x - 3t + 4\alpha) & \text{for } t - 2\alpha \le x \le 3t - 4\alpha\,, \\ \varphi_{b2} = \frac{1}{8}(x+t)(x - 3t + 4\alpha) & \text{for } 3t - 4\alpha \le x \le t\,, \\ \varphi_{b3} = \alpha x - \frac{1}{2}t^2 & \text{for } x \ge t\,. \end{cases} \tag{37}$$

Here $x_0(t) = t - 2\alpha$.

Note that φ_{b3} describes the "freely falling" half-line $\varphi = \alpha x$, which is the remnant of the initial data (33).

At the time $t = 2\alpha$ another interesting thing happens. At this moment the support of φ_{b2} becomes reduced to the point $x = 2\alpha$, and simultaneously $\varphi_{b3}(2\alpha, 2\alpha) = 0$. The right end of the support of φ_{b1}, which moves with the velocity $+3$, hits the support of φ_{b3}. At this time the length of the support of φ_{b1} is equal to 2α.

In the next time interval, $t \in [2\alpha, 3\alpha]$, the solution $\varphi = \varphi_c$ has the following form, see also Fig. 8:

$$\varphi_c(x,t) = \begin{cases} \varphi_0 = 0 & \text{for } x \le t - 2\alpha\,, \\ \varphi_{c1} = -\frac{1}{4}(x - x_0(t))(3t - x - 4\alpha) & \text{for } t - 2\alpha \le x \le 4\alpha - t\,, \\ \varphi_{c2} & \text{for } 4\alpha - t \le x \le t\,, \\ \varphi_{c3} & \text{for } t \le x \le \frac{t^2}{2\alpha}\,, \\ \varphi_{c4} = \alpha x - \frac{1}{2}t^2 & \text{for } x \ge \frac{t^2}{2\alpha}\,, \end{cases} \tag{38}$$

where

$$\varphi_{c2} = \frac{3}{8}x^2 - \frac{3}{4}xt + \frac{7}{8}t^2 + \alpha^2 - \frac{1}{2}\alpha t + \frac{7}{2}\alpha x - \frac{\sqrt{\alpha}}{3}\,[2(x+t) + \alpha]^{3/2}\,,$$

$$\varphi_{c3} = \frac{t^2}{2} + 3\alpha x - \frac{\sqrt{\alpha}}{3}\,[2(x+t) + \alpha]^{3/2} + \frac{\sqrt{\alpha}}{3}\,[2(x-t) + \alpha]^{3/2} + \frac{2\alpha^2}{3}\,.$$

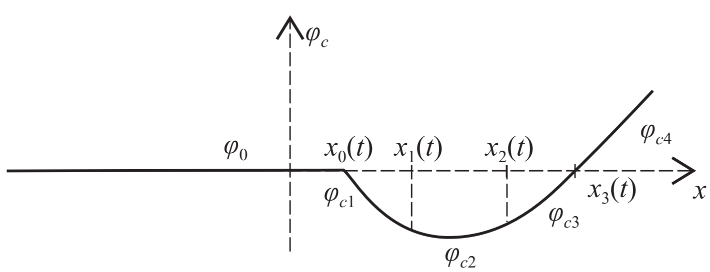

Fig. 8. The solution φ_c. Here $x_0(t) = t - 2\alpha$, $x_1(t) = 4\alpha - t$, $x_2(t) = t$, $x_3(t) = t^2/(2\alpha)$, and $t \in [2\alpha, 3\alpha]$.

Note that φ_{c1} coincides with φ_{b1} from formula (37). Thus, the point $x_3(t)$ (see Fig. 8) accelerates from the initial velocity $+2$ to $\dot{x}_3(3\alpha) = 9/2$. At the time $t = 3\alpha$ the points x_0, x_1 meet each other, and the support of φ_{c1} vanishes. It is the beginning of the fourth stage of evolution of φ.

When considered in the context of the interacting strings mentioned in the Introduction, the solution discussed above provides an interesting picture of the initial stages of a merging of the strings. In the initial state at $t = 0$, frame (a) in Fig. 9, the strings form a "Y" shape with the vertex at the point $x = 0$, and the angle α between the two rectilinear pieces of strings in the region $x > 0$. In the region $x < 0$ the strings have already merged. In the time interval $[0, \alpha)$ the vertex moves with the velocity -1 until a cusp is formed, see frame (c) in Fig. 9 — this happens at the time $t = \alpha$. At this moment a bubble appears, which grows and moves along the strings in accordance with the self-similar solution of the signum-Gordon equation. Such smooth evolution goes on until $t = 2\alpha$. In the third time interval, *i.e.* when $t \in [2\alpha, 3\alpha)$, the bubble grows and still moves along the strings, but now it is distorted by the two wave fronts $x_1(t), x_2(t)$ travelling along it, as follows from the solution φ_c given by formula (38). The evolution can be calculated also for times $t > 3\alpha$, but we shall not dwell on it.

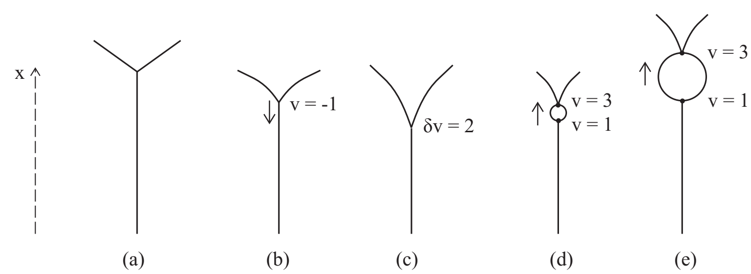

Fig. 9. The process of merging of two strings as described by solutions (34), (37). The "bubble" visible in frames (d), (e) is a part of the self-similar solution. Frame (c) shows the cusp formed at the time $T_1 = \alpha$. Frames (d), (e) show snapshots of the two strings at the times t_1, t_2 such that $\alpha < t_1 < t_2 < 2\alpha$.

5. Summary and remarks

1. The signum-Gordon equation appears in several problems of classical physics. It has a peculiar discontinuous nonlinearity, but this does not mean that the equation is intractable. We have constructed the full set of its self-similar solutions with initial data (5). The manifold of these solutions, presented as the map in Fig. 1, is amazingly rich. Equally astonishing is the fact that it has been possible to find the exact analytic forms of all solutions.

The self-similar solutions may also appear as a component of a globally non self-similar solution. This possibility is illustrated by the example presented in Sec. 4. The pattern of evolution of φ does not depend on the slope α in the initial data (33) — this property is related to the scaling symmetry of the signum-Gordon equation: the slope can be changed just by the scaling transformation (2). It is interesting to see that a self-similar solution plays a role in the evolution which starts from initial data which are not self-similar.

2. There are several obvious directions for extending our results. First, one may consider the four-parameter set of self-similar initial data (4) instead of the two-parameter set (5). It seems that not always the corresponding solutions can be obtained as a straightforward combination of solutions we have given above. Therefore, the map of such solutions will be truly four-dimensional.

3. One may also be interested in the stability of our solutions. For that matter, we have not seen any instability in the numerical investigations of the self-similar solutions. This seems to indicate that the unstable modes, if present at all, grow rather slowly. The main obstacle in the investigations of the stability is the fact that the signum-Gordon equation can not be linearised around $\varphi = 0$.

4. Probably the most interesting topic is related to the dynamics of topological compact solitons (compactons) found in [8]. These solitons have a piece-wise parabolic shape, and the pertinent field-theoretic model contains sectors which are described by the signum-Gordon equation. Therefore, one may expect that certain self-similar waves will appear in processes such as the scattering of the solitons, or the relaxation of a single excited soliton. Perhaps one can even provide the exact analytic description of certain stages of such processes. This would give the much desired counterbalance to purely numerical investigations which are dominant in the literature so far.

This work was supported in part by the Programme "COSLAB" of the European Science Foundation. H.A. gratefully acknowledges hospitality and support during his stays at the Niels Bohr Institute and the Yukawa Institute of Theoretical Physics, where parts of this work were done.

REFERENCES

[1] H. Arodź, P. Klimas, T. Tyranowski, *Acta Phys. Pol. B* **36**, 3861 (2005).

[2] H. Arodź, P. Klimas, T. Tyranowski, *Phys. Rev.* **E73**, 046609 (2006).

[3] L. Perivolaropoulos, *Nucl. Phys.* **B375**, 665 (1992).

[4] G.I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press 1996.

[5] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser, Boston–Basel–Berlin 2005, Chapter 8.

[6] L.C. Evans, *Partial Differential Equations*, American Math. Society, 1998.

[7] R.D. Richtmyer, *Principles of Advanced Mathematical Physics*, Springer-Verlag, New York–Heidelberg–Berlin 1978, Section 17.3.

[8] H. Arodź, *Acta Phys. Pol. B* **33**, 1241 (2002).