DETAILED STUDY OF A TRANSITION POINT IN THE VENEZIANO–WOSIEK MODEL OF PLANAR QUANTUM MECHANICS

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Following a model recently investigated by Veneziano and Wosiek we briefly introduce Planar Quantum Mechanics (PQM). Then, we present high precision numerical results in the sectors with two and three fermions. We confirm, that the transition point in the 't Hooft's coupling constant λ occurs in these sectors at $\lambda_c = 1$, as was expected in this model.

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The exact numerical spectra of a branch of reduced supersymmetric theories can be calculated in a cut Fock basis by a method proposed recently by Wosiek [1]. In a series of papers [2,3] he studied some models of Supersymmetric Yang–Mills Quantum Mechanics (SYMQM). These systems result from a dimensional reduction of full dimensional ($D = d+1$) supersymmetric Yang–Mills quantum field theories to a single point in space $(0+1)$. The method provided a good understanding of the $D = 2$ and $D = 4$, $N = 2$ spectra [2,3]. The goal of such analysis is to reach, on one hand the system with $D = 4, N = 3$ which could shade some light on the real QCD, and on the other, the $D = 10$, SU($N \to \infty$) model, which is conjectured to be in relation with the M-theory [4]. The latter case needs to incorporate into the scheme the large N limit [5], which is expected to provide a remarkable simplification. It should reduce considerably the number of basis vectors to be taken into account and allow to disregard all non-planar contributions. The above was investigated by Veneziano and Wosiek in [6–8] with a simple supersymmetric model belonging to the class of Planar Quantum Mechanics (PQM).

This paper presents high precision numerical results on the sectors with two and three fermions in the model from Ref. [6]. We investigate the transition point in the 't Hooft's coupling λ, and show that it takes place at $\lambda_c = 1$. Therefore, we give a numerical confirmation of the analytical results obtained by Beccaria in [9].

The paper is constructed as follows. We start with an introductory part to PQM, then we present the studied model, and finally we discuss our results.

1. Supersymmetric Yang–Mills quantum mechanics

Now we will introduce systems called SYMQM in the Hamiltonian formulation. Let us consider a quantum mechanical system with N^2 bosonic and N^2 fermionic degrees of freedom. As was already mentioned, it can be regarded as a remainder after dimensional reduction of supersymmetric field theory with U(N) gauge symmetry to one point in space. During such procedure, a local gauge symmetry becomes a global one. Thus, our system should be invariant under the global U(N) rotation. Let T^a_{ij} be the generators of the U(N) group in the fundamental representation, thus, they are $N \times N$ matrices. We introduce bosonic and fermionic matrix-valued annihilation and creation operators

$$a_{ij} = a^b T^b_{ij}\,, \qquad a^\dagger_{ij} = a^{\dagger b} T^b_{ij}\,, \tag{1}$$

$$f_{ij} = f^b T^b_{ij}\,, \qquad f^\dagger_{ij} = f^{\dagger b} T^b_{ij}\,, \tag{2}$$

where the sum over $b = 1, \ldots, N^2$ is assumed, and $i, j = 1, \ldots, N$. The invariance of the system is assured by taking the Hamiltonian as a trace of a polynomial of the above operators. The creation and annihilation operators satisfy the following commutation and anticommutation relations

$$[a_{ij}, a^\dagger_{kl}] = \delta_{il}\delta_{jk}\,, \tag{3}$$

$$\{f_{ij}, f^\dagger_{kl}\} = \delta_{il}\delta_{jk}\,. \tag{4}$$

The Fock basis is composed of eigenstates of the occupation number operators, $B = \mathrm{Tr}(a^\dagger a)$ and $F = \mathrm{Tr}(f^\dagger f)$, which are explicitly U(N)-invariant. The construction of the basis starts from the Fock vacuum denoted by $|0\rangle$. We act on the latter with invariant "bricks" *i.e.* creation operators contracted with U(N) invariant tensors. For the U(2) group we have two such tensors, δ_{ij} and ϵ_{ijk}. The basis states are obtained by action of any combination of powers of these bricks.

2. The cut-off method

As it is impossible to handle infinite matrices on a PC, one needs to cut them somehow. The most intuitive way to do this is to introduce some integer, $B_{\max}$, and to keep only those basis states for which the total bosonic occupation number does not exceed $B_{\max}$. The method of obtaining the spectrum simply consists of calculation the Hamiltonian matrix in such a cut

Fock basis, and then of its numerical diagonalization. This should be done for several different cut-offs and a limit of infinite cut-off should be extrapolated in order to obtain $B_{\max}$ independent, thus physical, results. The difficulty of such a program is hidden in the number of basis states growing exponentially with increasing $B_{\max}$ and N. Up to now, calculations have been made up to U(4) [10].

3. Large N limit and Planar Quantum Mechanics

The difficulties described in the preceding paragraph largely disappear in the 't Hooft limit, $N \to \infty$, $g^2 N = \text{const}$, where g is a coupling constant present in the system. The zeroth order approximation in the $1/N$ expansion consists in retaining only those contributions which correspond to planar graphs. It appears that this can be done already on the level of the Fock basis [6]. The main contribution will be given by basis states obtained by action of a single trace brick. Therefore, in the purely bosonic sector, $F = 0$, one needs to consider only the basis states of the form

$$|0, n\rangle = \frac{1}{\mathcal{N}_{0,n}} \operatorname{Tr}\left[(a^\dagger)^n\right] |0\rangle\,, \tag{5}$$

which are labeled by one integer, n. We can calculate an explicit expression for the normalization constant $\mathcal{N}_{0,n}$ [6]. The sector $F = 1$ contains one fermion and the basis states are given by

$$|1, n\rangle = \frac{1}{\mathcal{N}_{1,n}} \operatorname{Tr}\left[(a^\dagger)^n f^\dagger\right] |0\rangle. \tag{6}$$

With increasing fermionic occupation number F, things get complicated, because one has to use several integers to label basis states. For example, if $F = 2$ we need two integers, n_1 and n_2. The basis state is therefore, obtained by the action of a trace [7]

$$|2, n_1, n_2\rangle = \frac{1}{\mathcal{N}_{2,n_1,n_2}} \operatorname{Tr}\left[(a^\dagger)^{n_1} f^\dagger (a^\dagger)^{n_2} f^\dagger\right] |0\rangle. \tag{7}$$

Due to the cyclicity of the trace, we only need to deal with states with $n_1 \leq n_2$. Moreover, if $n_1 = n_2 = m$, the anticommutation of fermionic creation operators and the cyclicity of the trace imply that $|2, m, m\rangle = 0$. So, the basis is composed of states for which $n_1 < n_2$. Thus, for a given cut-off $B_{\max}$, we will have $\frac{1}{2} B_{\max}(B_{\max} - 1)$ states.

In the case of three fermions, we need three integers to label a basis state [7]

$$|3, n_1, n_2, n_3\rangle = \frac{1}{\mathcal{N}_{2,n_1,n_2,n_3}} \operatorname{Tr}\left[f^\dagger (a^\dagger)^{n_1} f^\dagger (a^\dagger)^{n_2} f^\dagger (a^\dagger)^{n_3}\right] |0\rangle\,. \tag{8}$$

Again, we can arrange them so that $n_1 < n_2, n_3$.

The large N limit of SYMQM systems in the zeroth order approximation is called Planar Quantum Mechanics.

4. Veneziano–Wosiek model

The model considered in Ref. [6–8] is given by the supersymmetry generators

$$Q = \mathrm{Tr}\left[f a^\dagger (1 + g a^\dagger)\right] , \qquad Q^\dagger = \mathrm{Tr}\left[f^\dagger (1 + g a) a\right] , \tag{9}$$

where g is the coupling constant. We can define the 't Hooft's coupling constant as $\lambda = g^2 N$, where N parameterizes the gauge group U(N). The Hamiltonian reads

$$H = \{Q, Q^\dagger\} = H_B + H_F ,$$

$$\begin{aligned} H_B &= \mathrm{Tr}\Big[a^\dagger a + g(a^{\dagger 2} a + a^\dagger a^2) + g^2 a^{\dagger 2} a^2\Big] , & (10)\\ H_F &= \mathrm{Tr}\Big[f^\dagger f + g(f^\dagger f \left(a^\dagger + a\right) + f^\dagger (a^\dagger + a) f\Big) \\ &\quad + g^2 \left(f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a\right)\Big] . & (11) \end{aligned}$$

It conserves the fermionic occupation number $F = \mathrm{Tr}[f^\dagger f]$, so we can analyze our model separately for each fixed F. The cases $F = 0$ and $F = 1$ were described in Ref. [6], whereas the sectors $F = 2$ and $F = 3$ in Ref. [7]. We will concentrate here exclusively on these higher-fermion-number sectors.

Following the rules of planar calculus, described in detail in Ref. [6], one can calculate the matrix elements of the Hamiltonian in the sectors with two and three fermions. We just recall here the explicit results [7].

Two fermion sector

We use the notation for the matrix element:

$$H_{n_1,n_2;m_1,m_2} \equiv \langle 2, n_1, n_2 | H | 2, m_1, m_2 \rangle .$$

Then

$$\begin{aligned} H_{n_1,n_2;n_1,n_2} &= (n_1+n_2+2)(1+\lambda) - \lambda(2 - \delta_{n_1,0} + 2\delta_{n_2,n_1+1}) , & (12)\\ H_{n_1+1,n_2;n_1,n_2} &= H_{n_1,n_2;n_1+1,n_2} = \sqrt{\lambda}(n_1 + 2) , & (13)\\ H_{n_1,n_2+1;n_1,n_2} &= H_{n_1,n_2;n_1,n_2+1} = \sqrt{\lambda}(n_2 + 2) , \\ H_{n_1+1,n_2-1;n_1,n_2} &= H_{n_1,n_2;n_1+1,n_2-1} = 2\lambda(1 - \delta_{n_2,n_1+1}) . & (14) \end{aligned}$$

Three fermion sector

Similarly, we denote the matrix element by

$$H_{n_1,n_2,n_3;m_1,m_2,m_3} \equiv \langle 3, n_1, n_2, n_3 | H | 3, m_1, m_2, m_3 \rangle .$$

We have

$$H_{n_1,n_2,n_3;n_1,n_2,n_3} = (n_1+n_2+n_3)(1+\lambda)-\lambda(3-\delta_{n_1,0}-\delta_{n_2,0}-\delta_{n_3,0})\,, \quad (15)$$

$$\begin{aligned} H_{n_1+1,n_2,n_3;n_1,n_2,n_3} &= H_{n_1,n_2,n_3;n_1+1,n_2,n_3} = \sqrt{\lambda}(n_1+2)\Delta\,, \\ H_{n_1,n_2+1,n_3;n_1,n_2,n_3} &= H_{n_1,n_2,n_3;n_1,n_2+1,n_3} = \sqrt{\lambda}(n_2+2)\Delta\,, \\ H_{n_1,n_2,n_3+1;n_1,n_2,n_3} &= H_{n_1,n_2,n_3;n_1,n_2,n_3+1} = \sqrt{\lambda}(n_3+2)\Delta\,, \end{aligned} \quad (16)$$

$$\begin{aligned} H_{n_1+1,n_2-1,n_3;n_1,n_2,n_3} &= H_{n_1,n_2,n_3;n_1+1,n_2-1,n_3} = \lambda\Delta\,, \\ H_{n_1,n_2+1,n_3-1;n_1,n_2,n_3} &= H_{n_1,n_2,n_3;n_1,n_2+1,n_3-1} = \lambda\Delta\,, \\ H_{n_1-1,n_2,n_3+1;n_1,n_2,n_3} &= H_{n_1,n_2,n_3;n_1-1,n_2,n_3+1} = \lambda\Delta\,, \end{aligned} \quad (17)$$

where Δ is defined by

$$\Delta = \begin{cases} \frac{1}{\sqrt{3}} & \text{if for the right state } n_1 = n_2 = n_3\,, \\ \sqrt{3} & \text{if for the left state } n_1 = n_2 = n_3\,, \\ 1 & \text{otherwise}\,. \end{cases} \quad (18)$$

Previous investigations show the existence of a transition point in the 't Hooft's coupling constant λ at $\lambda_{\rm c} = 1$. On one hand, it appears as a critical slow down of the convergence of eigenenergies as a function of $B_{\rm max}$, and on the other, the spectrum becomes continuous, whereas it was discrete away from λ_c. It was possible to derive the existence of this transition point analytically in the sectors with none or one fermion. Numerical results strongly suggest that the transition also occurs at $\lambda_{\rm c} = 1$ in the higher-fermion-number sectors. The aim of the present paper is to confirm this by new high precision results from larger cut-offs calculations.

5. High cut-off results and the transition point

Our main goal here is to study in detail the location of the transition point $\lambda_{\rm c}$ in the sectors with two or three fermions. Since the bases in these sectors are much bigger than the ones in lower-fermion-number sectors, one needs another tool for more quantitative analysis. We used `ARPACK`, a `Fortran77` library for spare matrices, to diagonalize our Hamiltonian matrix. In this way we were able to reach cut-offs $B_{\rm max} = 500\,(110)$, respectively for $F = 2$ (3), corresponding to the sizes of basis up to 100 000 vectors, compared to $B_{\rm max} = 40\,(30)$ attained in Ref. [7].

Two fermion sector

We will find the transition point by examining the dependence of the energy of the ground state on the coupling constant λ. Fig. 1 shows this energy, which in the following will be called $E_{B_{\rm max}}(\lambda)$, for a given cut-off $B_{\rm max}$ and in some interval around $\lambda = 1$. Suggestions from previous work are confirmed. Namely, for $\lambda > \lambda_{\rm c}$ the ground energy vanishes, and thus

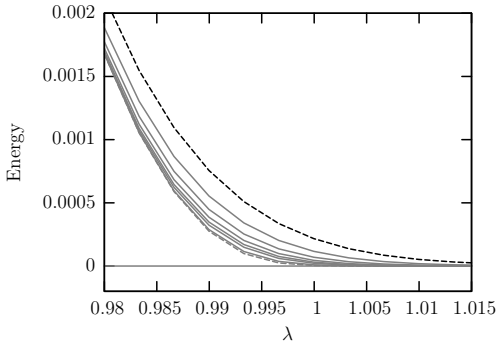

Fig. 1. The dependence of the ground state of the sector with two fermions on the coupling constant λ for different cut-offs $B_{\max}$. The highest dashed curve represents the results for the smallest $B_{\max} = 100$, whereas the lowest one corresponds to the highest $B_{\max} = 400$.

constitutes one of two SUSY ground states, which are present in this sector. For $\lambda < \lambda_c$, $E_{B_{\max}}(\lambda)$ is non-null and has a nontrivial dependence on λ. The determination of the transition point is carried out by fitting to $E_{B_{\max}}(\lambda)$ a polynomial in λ for several fixed $B_{\max}$. This polynomial is chosen to be positive for $\lambda < \lambda_0$ and equal to zero at $\lambda = \lambda_0$

$$w(\lambda) = w_1(\lambda - \lambda_0) + w_3(\lambda - \lambda_0)^3 + w_5(\lambda - \lambda_0)^5.$$

The fitted curves, together with the polynomial roots $\lambda_0(B_{\max})$, are shown in Fig. 2. In order to obtain the value of the physical transition point λ_c, we extrapolate $\lambda_0(B_{\max})$ to the limit $B_{\max} \to \infty$. We do this by fitting two types of decreasing functions

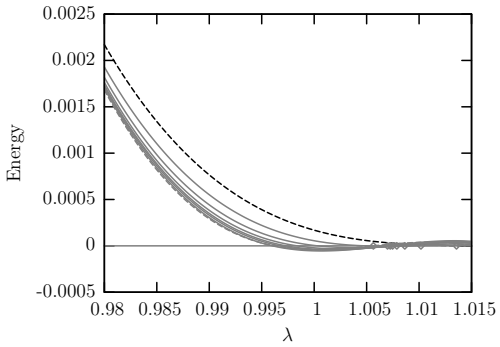

Fig. 2. Polynomials fitted to the dependence of the energy of the ground state in sector with two fermions on the coupling λ for different $B_{\max}$ as well as their zero points. The highest dashed curve represents the results for the smallest $B_{\max} = 100$, whereas the lowest one corresponds to the highest $B_{\max} = 400$.

$$\lambda_0(B_{\max}) = \lambda_c + b(B_{\max})^c,$$
$$\lambda_0(B_{\max}) = \lambda_c + b\exp(cB_{\max}).$$

The resulting fits are shown in Fig. 3, whereas Table I contains the values of fitted parameters. We can read off the infinite-cut-off limit λ_c equal to

$$\lambda_c = 1.0061 \pm 0.0005,$$

where the uncertainty is given by the difference between the two values of λ_c coming from the fits of the two functions. Finally, we also check the convergence of the ground energy $E_{B_{\max}}(\lambda)$ at the suspected value of transition point $\lambda = \lambda_c = 1.0$. To this end, we calculate the extrapolation of the values $E_{B_{\max}}(\lambda = 1.0)$, obtained for some specific cut-offs, by fitting a function

$$E_{B_{\max}}(\lambda = 1.0) = E_c + b(B_{\max})^c.$$

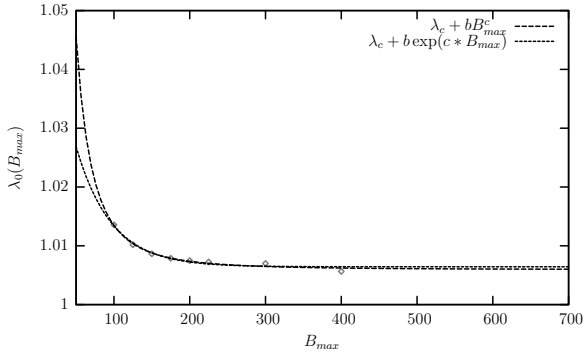

Fig. 3. The fitted dependencies of $\lambda_0(B_{\max})$ on $B_{\max}$.

TABLE I

Numerical values of fitted parameters for the $\lambda_0(B_{\max})$ dependence.

Fitted function	obtained parameters
$\lambda_0(B_{\max}) = \lambda_c + b(B_{\max})^c$	$\lambda_c = 1.0059$
	$b = 471.1$
	$c = -2.40$
$\lambda_0(B_{\max}) = \lambda_c + b\exp(cB_{\max})$	$\lambda_c = 1.0064$
	$b = 0.060$
	$c = -0.022$

Results for the fitted parameters are shown in Table II, and the curve is plotted in Fig. 4. We can conclude, that

$$E_{\rm c} = -1.08\ 10^{-6} \pm 4.5\ 10^{-7},$$

where the error is given by the difference between the two results with highest cut-off.

TABLE II

Numerical values of fitted parameters for the $E_{B_{\rm max}}(\lambda = 1.0)$ dependence.

Fitted function	obtained parameters
$E_{B_{\rm max}}(\lambda = 1.0) = E_{\rm c} + b(B_{\rm max})^c$	$E_{\rm c} = -1.08 \times 10^{-6}$
	$b = 62$
	$c = -2.73$

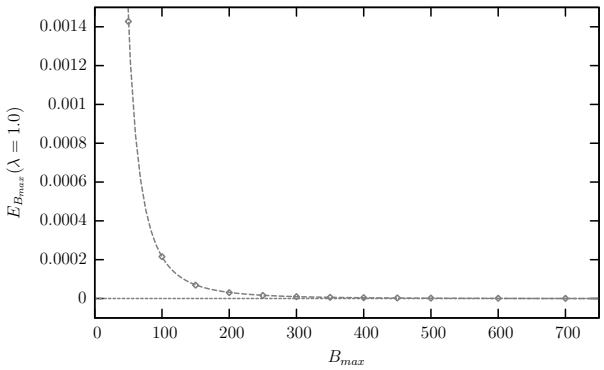

Fig. 4. The fitted dependence of $E_{B_{\rm max}}(\lambda = 1.0)$ on $B_{\rm max}$.

To summarize this paragraph we have shown, that in the sector with two fermions, the transition point occurs for $\lambda_{\rm c} = 1.0$ and that the energy of the ground state at the conjectured transition point $\lambda = 1.0$ converges to zero.

Three fermion sector

The analysis of the transition point in this sector follows the lines of the preceding paragraph. Similarly, we will examine the dependence of the energy of the ground state, called $E_{B_{\rm max}}(\lambda)$, on the coupling λ. Fig. 5 demonstrates the numerical curves for different cut-offs. The highest, dashed, curve represents calculations for $B_{\rm max} = 45$, and the lowest one for $B_{\rm max} = 90$. We see that the convergence is very good for λ away from $\lambda_{\rm c}$ *i.e.* $\lambda < 0.90$ and $\lambda > 1.05$. The transition takes place for λ between these values, and can be seen on this plot as a slow down of the numerical method. Let us denote, for each $B_{\rm max}$, the minimal energy of the ground state by $E_{\rm min}(B_{\rm max})$ and

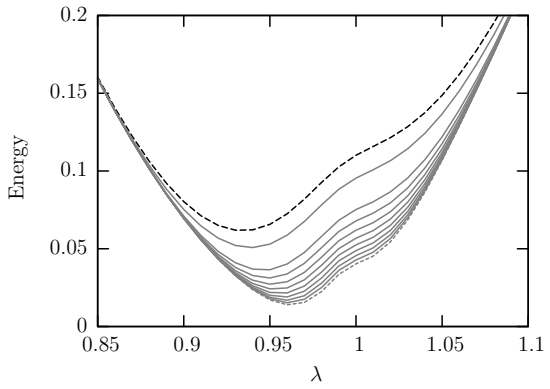

Fig. 5. The dependence of the ground state of the sector with three fermions on the coupling constant λ for different cut-offs $B_{\max}$. The highest dashed curve represents the results for the smallest $B_{\max} = 45$, whereas the lowest one corresponds to the highest $B_{\max} = 90$.

its position by $\lambda_{\min}(B_{\max})$. The physical results, *i.e.* cut-off independent, are thus the limiting quantities $E_{\rm c}$ and $\lambda_{\rm c}$ such that $E_{\min}(B_{\max}) \to E_{\rm c}$ and $\lambda_{\min}(B_{\max}) \to \lambda_{\rm c}$ as $B_{\max} \to \infty$. The results from sectors $F = 0, 1$ suggest that the energy of all states collapses to zero and we get a continuous spectrum at the speculated transition point $\lambda_{\rm c} = 1$.

We determine $\lambda_{\min}(B_{\max})$ and $E_{\min}(B_{\max})$ by two methods. First of them consists in fitting a fourth order polynomial,

$$w(\lambda) = w_0 + w_1\lambda + w_2\lambda^2 + w_4\lambda^4,$$

to $E_{B_{\max}}(\lambda)$ for each $B_{\max}$. Then, $\lambda_{\min}(B_{\max})$ and $E_{\min}(B_{\max})$ are calculated analytically given the fitted parameters. The fitted curves are shown in Fig. 6, together with the calculated minima. The second method uses the cubic spline to transform $E_{B_{\max}}(\lambda)$ into a continuous curve. The approximated values of the minima are then found numerically by bracketing. Differences between the results coming from these two methods will be later used as an estimate of the uncertainty of the calculated quantities. To get $E_{\rm c}$ we extrapolate $E_{\min}(B_{\max})$ to $B_{\max} \to \infty$, and to this end, we fit a polynomial function

$$E_{\min}(B_{\max}) = E_{\rm c} + b(B_{\max})^c\,.$$

The obtained fit is shown in Fig. 7, whereas the values of parameters are presented in Table III. We thus have

$$E_{\rm c} = -0.00094 \pm 0.00021\,.$$

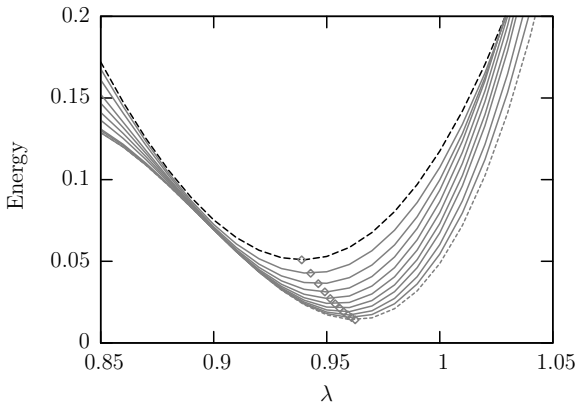

Fig. 6. Polynomials fitted to the dependence of the energy of the ground state in sector with three fermions on the coupling λ for different $B_{\max}$. The highest dashed curve corresponds to $B_{\max} = 45$ and the lowest one to $B_{\max} = 90$. The calculated minima are shown as well.

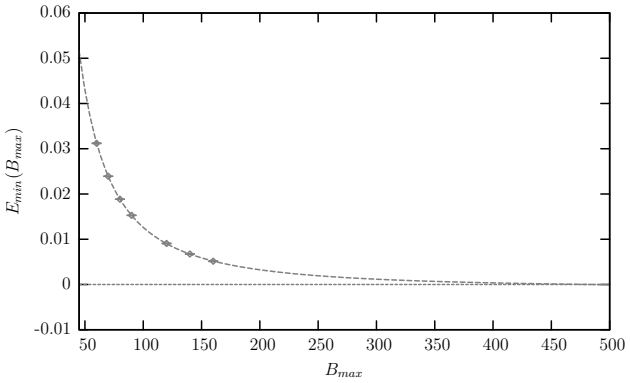

Fig. 7. The fitted dependence of $E_{\min}(B_{\max})$ on $B_{\max}$.

TABLE III

Numerical values of fitted parameters for the $E_{\min}(B_{\max})$ dependence.

Fitted function	obtained parameters
$E_{\min}(B_{\max}) =$	$E_{\rm c} = -0.00094 \pm 0.00021$
$E_{\rm c} + b(B_{\max})^c$	$b = 33.2 \pm 2.4$
	$c = -1.694 \pm 0.019$

In order to extrapolate $\lambda_{\min}(B_{\max})$ we fit three slowly growing functions:

- $\lambda_{\min}(B_{\max}) \sim (B_{\max})^c$,
- $\lambda_{\min}(B_{\max}) \sim (B_{\max})^{-1/2}$
- $\lambda_{\min}(B_{\max}) \sim \ln(B_{\max})^{-1}$.

Table IV contains the obtained values of the fitted parameters, and the fitted curves are shown in Fig. 8. Eventually, we can assume that the value of the constant coefficient $\lambda_{\rm c}$ is equal to the mean of the values obtained from the three fits, and its error is the standard deviation. Therefore,

$$\lambda_{\rm c} = 1.034 \pm 0.016\,.$$

TABLE IV

Numerical values of fitted parameters for the $\lambda_{\min}(B_{\max})$ dependencies.

Fitted functions	obtained parameters
$\lambda_{\min}(B_{\max}) = \lambda_{\rm c} + b(B_{\max})^c$	$\lambda_{\rm c} = 1.0155 \pm 0.0053$
	$b = -0.474 \pm 0.045$
	$c = -0.480 \pm 0.043$
$\lambda_{\min}(B_{\max}) = \lambda_{\rm c} + b(B_{\max})^{-1/2}$	$\lambda_{\rm c} = 1.01304 \pm 0.00038$
	$b = -0.4964 \pm 0.0033$
$\lambda_{\min}(B_{\max}) = \lambda_{\rm c} + b\ln(c * B_{\max})^{-1}$	$\lambda_{\rm c} = 1.073 \pm 0.010$
	$b = -0.478 \pm 0.084$
	$c = 0.80 \pm 0.28$

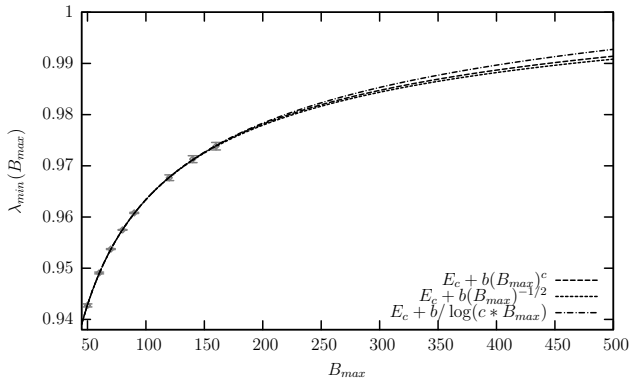

Fig. 8. The fitted dependencies of $\lambda_{\min}(B_{\max})$ on $B_{\max}$.

One also notes, that the general fit of a power function gave approximately the same results as the fit of the inverse of the square root.

As a conclusion of this section we recapitulate our results for the sector with three fermions. Namely, we showed that the transition point occurs at $\lambda_c = 1.0$ and that at this value of coupling constant the ground energy converges to zero.

6. Discussion and conclusions

In this paper we used high precision numerical results in order to check the transition point in the 't Hooft's coupling constant λ in the Veneziano–Wosiek model. We investigated the sectors with two and three fermions. By fitting some specific functions we extrapolated from the numerical data the physical, *i.e.* cut-off independent, values of the transition point and ground energy at $\lambda = 1.0$. We confirmed that in both sectors this transition point occurs nearly at $\lambda = \lambda_c = 1.0$, and that the ground energy at this value of coupling constant converges almost to zero. The uncertainty given with these results is not a true statistic error since it was not calculated from any statistical ensemble. It should be only interpreted as an indication of the real uncertainty.

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