RENORMALISABLE SO(10) MODELS AND NEUTRINO MASSES AND MIXING*

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We discuss some recent developments in SUSY Grand Unified Theories based on the gauge group SO(10). Considering renormalisable Yukawa couplings, we present ways to accommodate quark and lepton masses and mixings.

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1. Introduction

Grand unified theories (GUTs) based on the group SO(10) [1] are interesting because its 16-dimensional irreducible representation (irrep), the spinor representation, contains all chiral fermions included in a Standard Model (SM) family plus an additional SM gauge singlet, the right-handed neutrino. Moreover, such theories allow type I and type II seesaw mechanisms for generating light neutrino masses. The basis of the Lie algebra so(10) consists of 45 antisymmetric real matrices, usually taken to be

$$(M_{pq})_{jk} = \delta_{pj}\delta_{qk} - \delta_{qj}\delta_{pk} \quad (1 \leq p < q \leq 10)\,, \tag{1}$$

i.e.,

$$M_{12} = \begin{pmatrix} 0 & 1 & \cdots \\ -1 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad etc. \tag{2}$$

The commutation relations are

$$[M_{pq}, M_{rs}] = \delta_{ps}M_{qr} + \delta_{qr}M_{ps} - \delta_{pr}M_{qs} - \delta_{qs}M_{pr}\,. \tag{3}$$

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Defining a Clifford algebra with basis elements Γ_p and anticommutation relations

$$\{\Gamma_p, \Gamma_q\} = 2\delta_{pq}\mathbb{1} \quad (1 \leq p,\, q \leq 10)\,, \tag{4}$$

it is easy to show that the quantities [3,4]

$$\tfrac{1}{2}\sigma_{pq} \equiv \tfrac{1}{4}[\Gamma_p, \Gamma_q] \tag{5}$$

fulfil the so(10) commutation relations. Therefore, any representation of the Clifford algebra is at the same time a representation of so(10).

1.1. The 16-dimensional spinor representation

The Clifford algebra associated with so(10) has a 32-dimensional irrep. However, the associated so(10) representation obtained via (5) is reducible. It decays into the spinor irrep **16** and its complex conjugate $\overline{\mathbf{16}}$. The Pati–Salam group $G_{422} \equiv \mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, where $\mathrm{SU}(4)_C$ unifies colour and lepton number [2], is very useful for a classification of the fields contained in so(10) irreps. The decomposition of the **16** is given by

$$\mathbf{16} \stackrel{422}{=} (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\overline{\mathbf{4}},\mathbf{1},\mathbf{2})\,. \tag{6}$$

The further decomposition of the G_{422} multiplets with respect to the SM gauge group $G_{321} \equiv \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ is

$$(\mathbf{4},\mathbf{2},\mathbf{1}) \stackrel{321}{=} (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{2})_{-1/2}\,, \tag{7}$$

$$(\overline{\mathbf{4}},\mathbf{1},\mathbf{2}) \stackrel{321}{=} (\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1} \oplus (\mathbf{1},\mathbf{1})_{0}\,. \tag{8}$$

Thus the SM fermion field assignments (all fields are to be considered left-handed) are given by

$$(\mathbf{4},\mathbf{2},\mathbf{1}) : \begin{pmatrix} u_r & u_y & u_b & \nu \\ d_r & d_y & d_b & e \end{pmatrix}, \ (\overline{\mathbf{4}},\mathbf{1},\mathbf{2}) : \begin{pmatrix} d_r^c & d_y^c & d_b^c & e^c \\ u_r^c & u_y^c & u_b^c & \nu^c \end{pmatrix}. \tag{9}$$

1.2. Scalars for Yukawa couplings

For the Yukawa couplings one has two options: One option is to take into account only "low-dimensional" scalar irreps like **10** and **16**; in that case one has to resort to non-renormalizable interactions. The other option [3,4] is to take the scalar irreps which appear in

$$\mathbf{16} \otimes \mathbf{16} = \binom{10}{1} \oplus \binom{10}{3} \oplus \frac{1}{2}\binom{10}{5} = \mathbf{10} \oplus \mathbf{120} \oplus \mathbf{126}\,; \tag{10}$$

here one has rather high-dimensional irreps. Note that in $\mathbf{16}\otimes\mathbf{16}$ only totally antisymmetric tensors with uneven rank occur: the vector **10**, the totally antisymmetric 3-tensor **120** and the totally antisymmetric, selfdual 5-tensor **126**. Selfdual means that $*\mathbf{126} = i\mathbf{126}$; the star indicates the formation of the dual tensor with the epsilon tensor of rank 10. The **126** is a genuinely complex irrep. Therefore, the scalars for renormalisable Yukawa couplings are given by **10**, **120** and $\overline{\mathbf{126}}$.

1.3. The Yukawa Lagrangian and fermion mass matrices

With the scalar irreps of the previous subsection the Yukawa Lagrangian reads [3]

$$\begin{aligned}\mathcal{L}_Y &= \frac{1}{2}\big(H_{ab}\mathbf{16}^T_{aL}C^{-1}\mathcal{B}\,\varGamma_p\mathbf{10}_p\mathbf{16}_{bL}\\ &\quad +G_{ab}\mathbf{16}^T_{aL}C^{-1}\mathcal{B}\,\varGamma_p\varGamma_q\varGamma_r\mathbf{120}_{pqr}\mathbf{16}_{bL}\\ &\quad +F_{ab}\mathbf{16}^T_{aL}C^{-1}\mathcal{B}\,\varGamma_p\varGamma_q\varGamma_r\varGamma_s\varGamma_t\overline{\mathbf{126}}_{pqrst}\mathbf{16}_{bL} + \text{H.c.}\big)\,. \end{aligned} \tag{11}$$

In this Lagrangian we have SO(10) indices $1 \le p,q,r,s,t \le 10$ and family indices $1 \le a,b \le 3$. The SO(10) "charge-conjugation matrix" $\mathcal{B}$ has the properties

$$\mathcal{B}^T = \mathcal{B}\,, \qquad \mathcal{B}^{-1}\varGamma_p^T\mathcal{B} = \varGamma_p\,. \tag{12}$$

Due to the structure of $\mathcal{L}_Y$, the Yukawa coupling matrices must fulfil

$$H^T = H\,, \qquad G^T = -G\,, \qquad F^T = F\,. \tag{13}$$

The vacuum expectation values (VEVs) $k_{d,u}$, $\kappa_{d,u,\ell,D}$, $v_{d,u}$ of the Higgs doublets contained in the scalars of $\mathcal{L}_Y$ determine the mass matrices of the fermions. For the charged fermions we have

$$M_d = k_d\,H + \kappa_d\,G + v_d\,F\,, \tag{14}$$

$$M_u = k_u\,H + \kappa_u\,G + v_u\,F\,, \tag{15}$$

$$M_\ell = k_d\,H + \kappa_\ell\,G - 3\,v_d\,F\,. \tag{16}$$

The -3 is a Clebsch–Gordan coefficient. In the neutrino sector also SU(2) triplet VEVs w_R and w_L, stemming from the $\overline{\mathbf{126}}$, occur. One needs the following matrices:

$$M_D = k_u\,H + \kappa_D\,G - 3\,v_u\,F\,, \qquad M_R = w_R\,F\,, \qquad M_L = w_L\,F\,, \tag{17}$$

where M_R with the large VEV w_R is the mass matrix of the heavy Majorana neutrinos. The mass matrix of the light neutrinos is determined by the seesaw mechanism:

$$\mathcal{M}_\nu = M_L - M_D^T M_R^{-1} M_D. \tag{18}$$

The VEV w_L is small according to the type II seesaw mechanism.

2. The Minimal SUSY SO(10) GUT

The Minimal SUSY SO(10) GUT [5] (MSGUT) is characterised by the following multiplets. Each fermion family resides in a **16**, the gauge bosons are, of course, in the adjoint representation **45** and the scalars are given by $\mathbf{10} \oplus \overline{\mathbf{126}} \oplus \mathbf{126} \oplus \mathbf{210}$. The **210** is the totally antisymmetric tensor of rank 4. The tasks of the different scalar multiplets are the following:

- $\mathbf{10} \oplus \overline{\mathbf{126}} \rightarrow$ Yukawa couplings,
- $\overline{\mathbf{126}} \oplus \mathbf{126} \oplus \mathbf{210} \rightarrow$ breaking SO(10) down to G_{321},
- $\mathbf{126} \rightarrow$ avoiding SUSY breaking at high scales.

The idea is that SUSY breaking is accomplished by soft terms at G_{321} stage. A further condition is that of *minimal finetuning* [6]: At the electroweak scale there are only two light Higgs doublets H_d, H_u, just the ones which appear in the Minimal Supersymmetric Standard Model (MSSM). This is a non-trivial condition because each of the scalar multiplets contains two SM doublets, one for each hypercharge $\pm 1/2$.

However, the minimal SUSY SO(10) is GUT ruled out [7,8]. It is amazing that this theory is so constrained that one can falsify it. In essence the reason for the failure of the MSGUT can be formulated in the following way:

> The MSSM gauge coupling unification occurs at the scale M_{GUT} *without* intermediate scale, but neutrino masses via the seesaw mechanism require a scale $M_{\mathrm{seesaw}} < M_{\mathrm{GUT}}$.

Identifying the electroweak scale with the SM VEV $v \simeq 174$ GeV and using $\Delta m^2_{\mathrm{atm}} \sim 2.5 \times 10^{-3}$ eV^2 for the atmospheric neutrino mass-squared difference, we find

$$v^2/M_{\mathrm{seesaw}} \gtrsim \sqrt{\Delta m^2_{\mathrm{atm}}} \;\Rightarrow\; M_{\mathrm{seesaw}} \lesssim 6 \times 10^{14} \text{ GeV}. \tag{19}$$

Thus, M_{seesaw} is more than one order of magnitude below $M_{\mathrm{GUT}} \simeq 2 \times 10^{16}$ GeV from the MSSM gauge coupling unification. This is the source of the problem.

Let us present some details. In the MSGUT, $G = 0$ in the mass formulas (14)–(18). Suppose the VEVs $k_{d,u}$, $v_{d,u}$, $w_{R,L}$ are free parameters, then one obtains an excellent fit to known fermion masses and mixings [8]. The number of independent parameters in the system of mass matrices is 21, 13 absolute values and 8 phases, whereas the number of observables is 18: nine charged-fermion masses, two neutrino mass-squared differences, six mixing angles and one CKM phase. This fit is done by minimising [8]

$$\chi^2(p) = \sum_{i=1}^{n} \left(\frac{f_i(p) - \bar{O}_i}{\sigma_i} \right)^2 , \tag{20}$$

where $p = \{p_1, \ldots, p_r\}$ is the set of parameters (in the MSGUT $r = 21$) and the functions $f_i(p)$ are the predictions for the observables ($n = 18$ in our case). In (20) the input values $\bar{O}_i \pm \sigma_i$ have to be taken at the GUT scale, extrapolated via the renormalisation group equations of the MSSM from the values of the observables at the electroweak scale [9]. A suitable numerical procedure for finding the minimum of χ^2 is the downhill simplex method [8].

Though the fit to fermion masses and mixings turns out to be excellent if the VEVs are considered as free parameters, this is not the whole story. In the MSGUT the number of terms in the scalar potential is limited due to SUSY and the SO(10) multiplet content. Therefore, the VEV ratios are not free but functions of $\tan\beta = \langle H_u^0 \rangle / \langle H_d^0 \rangle$ and the parameters of the scalar potential. Thus the fit in the fermionic sector constrains the scalar potential, resulting in a light scalar G_{321} multiplet $(\mathbf{8}, \mathbf{3})_0$ which destroys gauge coupling unification [7, 8].

3. The MSGUT plus the scalar 120-plet

The addition of the scalar **120** can release the strain from the mass and mixing fit, allowing a "small" Yukawa coupling matrix F of the $\overline{\mathbf{126}}$ in order to enhance the value of the neutrino masses via the type I seesaw mechanism [10] with the inverse of F. In references [11, 12] we have put forward the idea to add the **120** and to make the identification $w_R \equiv M_{\mathrm{GUT}}$, in order to avoid light scalar multiplets except the MSSM Higgs doublets. This identification was done by hand, therefore, fits to fermion masses and mixings by leaving the other VEVs as free parameters are only "generic" fits, without taking into account the full theory. With the 120-plet no new heavy VEVs appear compared to the MSGUT; the **120** has *two* SM Higgs doublets for each hypercharge $\pm 1/2$ and now each MSSM Higgs doublet $H_{d,u}$ is a linear combination of *six* doublets.

The aim of [11, 12] was to show that the idea presented above works for fermion masses and mixings. We used the method of [8] for the numerics. However, due to the addition of the **120** there is the numerical problem that the number of parameters is large and the downhill simplex method becomes time consuming and less trustworthy. Therefore, it is necessary to reduce the number of parameters, motivated by physical reasons. First of all, type I seesaw will be dominant and one can neglect type II contributions to $\mathcal{M}_\nu$. Furthermore, one can assume real Yukawa couplings, motivated by spontaneous CP violation. In [11] we reduced the number of parameters in addition by postulating a $\mathbb{Z}_2$ symmetry: $\mathbf{16}_2 \to -\mathbf{16}_2$, $\mathbf{120} \to -\mathbf{120}$; with complex VEVs there are 21 parameters, including 6 phases (Case A). In [12], dispensing with a family symmetry, we assumed the VEVs of **10** and $\overline{\mathbf{126}}$ to be real, but the VEVs of the **120** to be imaginary; this leads to 18 parameters (Case B).

We obtained excellent fits in both cases. *E.g.* for Case B the minimal χ^2 is 0.33 with normal and 0.011 with inverted neutrino mass spectrum. Unfortunately we could not find striking predictions, except that both cases strongly prefer a hierarchical spectrum—see Fig. 1 where χ^2 is plotted as a function the minimal neutrino mass over the solar mass-squared difference.

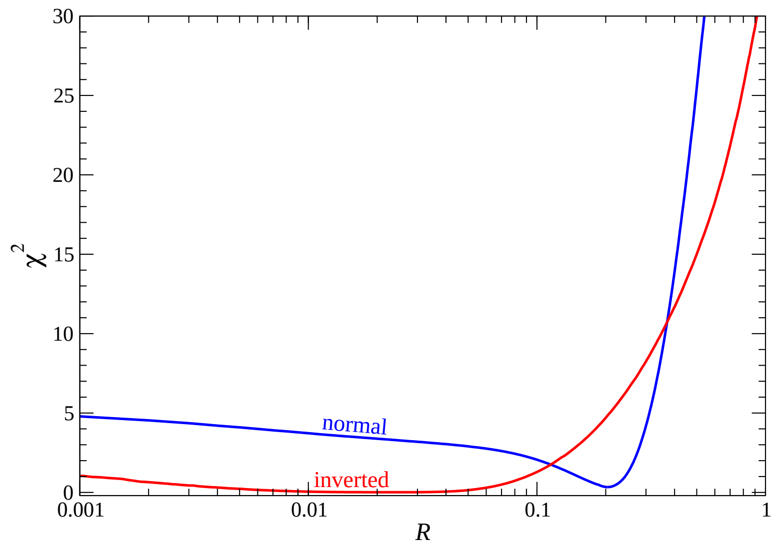

Fig. 1. The χ^2 of Case B as a function of $R = m_{\min}/\Delta m^2_{\rm sol}$.

4. Conclusions

The MSGUT together with the scalar **120** can excellently reproduce all known fermion masses and mixings, while identifying the triplet VEV w_R (the "seesaw scale") with the GUT scale—at least by doing a generic fit without taking possible relations among the VEVs into consideration. A more complete discussion has been started in [13], dubbing the model "New MSGUT" or NMSGUT. Hopefully in the NMSGUT the VEVs necessary for the fit to fermion masses and mixings are compatible with the scalar potential on the one hand and gauge coupling unification on the other hand.

This consistency has yet to be checked. While in the fit to the fermionic sector spontaneous CP violation mainly played a role in reducing the number of parameters in the Yukawa couplings [11, 12], there are indications that it could play a much more important role in SO(10) GUTs [13, 14]. In particular, in the NMSGUT the requirement of spontaneous CP violation pushes the GUT scale closer to the Planck scale [13]. In conclusion, we want to note, however, that still there is no idea how to obtain suitable simple fermion mass matrices with *explanatory power* in SO(10) GUTs; the discussion still centres on accommodation of fermion masses and mixings and consistency of the theory.

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