STABILITY OF THICK BRANE CONFIGURATIONS*

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We study higher dimensional models with a cutoff Λ and determine conditions under which brane configurations can be generated by dynamics at scales below Λ. Then we study the stability of these configurations.

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1. Introduction and motivation

Models in > 4 dimensions have become very popular in recent time due to their very interesting phenomenology and their ability to tame or solve the hierarchy problem [1]. These models assume that space time has the topology of $\mathbb{R}^4 \times (\mathbb{R}^k/\mathcal{G})$ where $\mathbb{R}^4$ correspond to Minkowski space and $\mathcal{G}$ is a discrete "lattice" group composed of a set of translations, rotations and reflections in $\mathbb{R}^k$; $\mathbb{R}^k/\mathcal{G}$ is then compact. Depending on the modality of the theory all of the fields propagate throughout the $k+4$-dimensional space [2] or, in other cases, only gravity propagates throughout and the rest of the fields are confined to some subspaces [1] or "branes". None of these models is viable for arbitrarily large energies, they are understood to be the low-energy effective theories of yet more fundamental theory whose interactions become apparent at some cutoff scale Λ.

Despite their promise, the simplest of these theories (those with $k = 1$ and without tree-level branes) are not viable phenomenologically [3]: $\sin^2\theta_W$ and/or ρ are off the observed values. The simplest fix is then to assume the

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existence of branes, but this raises several questions, for example: What mechanism confines the non-gravitational fields to these subspaces? What dynamics is responsible for brane creation? Under what circumstances are these subspaces stable? In this talk we will assume that branes are present and created by some dynamics at scale below Λ, we will give an example of such dynamics and then study the stability for the configurations obtained.

2. Obstacles

Tough constructing brane-like configurations is relatively straightforward in flat space, the gravitational interactions considerably complicate the enterprise. This is best summarized by a very useful set of sum rules derived in Ref. [4] which we apply to the following situation. Consider a 5-dimensional space with metric[1]

$$g_{MN} = \begin{pmatrix} A(y)\ {}^{(4)}g_{\mu\nu}(x) & 0 \\ 0 & 1 \end{pmatrix}, \qquad M,N = 0,1,2,3,4 \tag{1}$$

then the Einstein equations imply

$$A^{-\ell}\left(A^{\ell}\, A'\right)' = \frac{4\pi k}{3} A\left(T^{\mu}_{\mu} + 4\ell T^{4}_{4}\right) + \frac{2\ell+1}{6}\ {}^{(4)}R\,, \tag{2}$$

where T^M_N denotes the energy-momentum tensor, $2\ell =$ integer, ${}^{(4)}R$ is the curvature scalar generated by ${}^{(4)}g_{\mu\nu}$ and a prime indicates a derivative with respect to the argument.

If we then assume that T^M_N is generated by a set of scalar fields $\boldsymbol{\Phi}$ whose Lagrangian contains both self interactions and brane terms,

$$\mathcal{L}_{\boldsymbol{\Phi}} = \sqrt{-g}\Big[-\tfrac{1}{2} g^{MN}\partial_M\boldsymbol{\Phi}\partial_N\boldsymbol{\Phi} - V(\boldsymbol{\Phi}) - \sum_b \lambda_a(\boldsymbol{\Phi})\delta(y-y_b)\Big] \tag{3}$$

(y_b denotes the brane position) and if we assume that the fields are periodic in y then integrating

$$\oint dy\Big[|\boldsymbol{\Phi}'|^2 + \sum_b \lambda_b(\boldsymbol{\Phi})\delta(y-y_b)\Big] = 0\,. \tag{4}$$

In particular $\boldsymbol{\Phi} =$ constant in absence of brane terms[2].

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[1] $x^\mu(\mu = 0,1,2,3)$ denote the non-compact coordinates; y the coordinate of the compact direction.

[2] The Randall–Sundrum model [1] satisfies this constraint.

3. Model

The above obstacle can be overcome by an appropriate generalization of the scalar Lagrangian

$$\mathcal{L}_\phi = \sqrt{-g}\left[-\tfrac{1}{2}g^{MN}\partial_M\phi\,\partial_N\phi - V(\phi) - \frac{\xi}{2}\phi^2 R\right], \tag{5}$$

where we have taken a single scalar field for simplicity. In this case (4) is modified:

$$\oint dy\,\phi'\left[\phi' + 4\,\xi\frac{A'}{A}\phi\right] = 0\,, \tag{6}$$

so that a periodic field configuration is no longer necessarily constant.

4. Classical solutions

We look for configurations of the form

$$\bar{g}_{MN} = \mathrm{diag}\,(-A, A, A, A, 1)\,, \quad A = e^{-2\sigma(y)}, \quad \phi = \bar{\phi}(y)\,, \tag{7}$$

which maintain Lorentz invariance in the non-compact directions; we require $\bar{\phi}(y)$ and $\sigma(y)$ periodic. In this case the Einstein equations imply

$$\frac{3}{4\pi k}\left(\sigma'\right)^2 \;=\; \tfrac{1}{2}\left(\bar{\phi}'\right)^2 - \bar{V} + 6\xi\left(\bar{\phi}\sigma'\right)^2 - 4\xi\sigma'\left(\bar{\phi}^2\right)'\,, \tag{8}$$

$$\frac{3}{8\pi k}\sigma'' \;=\; \left(\bar{\phi}'\right)^2 - \xi\left(\bar{\phi}^2\right)'' - \xi\sigma'\left(\bar{\phi}^2\right)' + 3\xi\sigma''\bar{\phi}^2\,. \tag{9}$$

One can attempt a perturbative solution around the zero-field configuration: $\bar{\phi} = \varepsilon\bar{\phi}_1 + \varepsilon^2\bar{\phi}_2 + \ldots$, $\sigma = \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots$; to lowest order we find $\phi_1 = P\,\cos(\omega y + \alpha)$, when $V(\phi = 0) = V'(\phi = 0) = 0$, $V''(0) = -\omega^2$. In this case, however, we find that σ_k contains non-periodic terms $\propto (\omega y)^k$. Of course, these might add up to a periodic function but to determine that the whole series must be summed: this perturbative expansion is not useful in obtaining periodic solutions.

A different approach that does produce periodic solutions and which is motivated through numerical simulations assumes $\bar{\phi}(0) = \varphi \to \infty$ and $V(\phi) = -v(\phi - \varphi)$ with the fields having the following expansion

$$\bar{\phi} = \varphi + \bar{\phi}^{(0)} + \frac{1}{\varphi}\bar{\phi}^{(1)} + \ldots\,, \qquad \sigma = \frac{1}{\varphi}\sigma^{(1)} + \ldots\,. \tag{10}$$

This also implies that the metric is almost flat: $g_{MN} = \mathrm{diag}(-1, 1, 1, 1, 1) + \mathcal{O}(1/\varphi)$. Substituting in the Einstein equations gives, to lowest order,

$$\tfrac{1}{2}\zeta\left[\bar{\phi}^{(0)\prime}\right]^2 + v\left(\bar{\phi}^{(0)}\right) = 0\,, \qquad \zeta = 1 - \frac{16}{3}\xi \tag{11}$$

that corresponds to zero-energy Newtonian motion in a potential v/ζ for which it is easy to choose potentials v that lead to periodic solutions. The higher order terms can then be expressed in terms of $\bar\phi^{(0)}$ (so that they are also periodic with the same period):

$$\begin{aligned} \sigma^{(1)} &= (2/3)\bar\phi^{(0)}\,, \qquad \bar\phi^{(1)} = 0 \\ \sigma^{(2)} &= [2/(3\xi)]\mathcal{F}\left(\bar\phi^{(0)}\right) - \bar\phi^{(0)2}/3\,, \qquad \bar\phi^{(2)} = \mathcal{G}\left(\bar\phi^{(0)}\right)\,, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \mathcal{F}(\phi) &= -\frac{\zeta}{2}\int_0^\phi d\lambda \int_0^\lambda d\gamma \sqrt{\frac{v(\gamma)}{v(\lambda)}}\,, \\ \mathcal{G}(\phi) &= \frac{1}{3\zeta}\int_0^\phi d\gamma \sqrt{\frac{v(\phi)}{v(\gamma)}}\left\{\frac{1}{\pi k} - \frac{8}{\xi}\left[\mathcal{F}'(\gamma)\right]^2\right\}\,. \end{aligned} \tag{13}$$

For example, if $v(\phi)/\zeta = V_1 - (V_0+V_1)\theta(A-|\phi|),\ V_{0,1}>0$ then

$$\mathcal{F} = -\frac{\zeta}{4}\phi^2\,, \qquad \mathcal{G} = \frac{2\zeta}{9\xi}\left(\frac{3\xi}{2\pi k\zeta^2} - \phi^2\right)\phi\,, \qquad \text{for} |\phi| < A \tag{14}$$

and the solution matches the configuration used in the Randall–Sundrum (RS) model [1] with bulk cosmological constant $\Lambda = -2V_0/(3\pi k\varphi^2)$ and brane tension $\lambda = 2(V_1+V_0)/(8\pi k\sqrt{2V_0}\varphi)$.

5. Stability

To determine the conditions under which the configurations described above can represent background states of the models considered we must study the stability of these periodic solutions under small perturbations. We restrict ourselves to linear perturbation theory and write perturbations

$$g_{\mu\nu} = \bar g_{\mu\nu} + e^{ix^\alpha p_\alpha}h_{\mu\nu}(y)\,, \quad g_{44} = \bar g_{44} + e^{ix^\alpha p_\alpha}\gamma(y)\,, \quad \phi = \bar\phi + e^{ix^\alpha p_\alpha}\chi(y)\,, \tag{15}$$

where the over-bar indicates the background solution and $p_{0,1,2,3}$ are y-independent numbers; γ is the so-called "dilaton" field. We assume that

$$h_{\mu\nu}(y) = H(y)p^2\eta_{\mu\nu} + X(y)p_\mu p_\nu\,, \qquad p^2 = p_\rho p_\tau \eta^{\rho\tau} \tag{16}$$

and take the $X(y)=0$ gauge. We also assume the perturbations have large φ expansions of the form

$$\chi = \sum \frac{1}{\varphi^n}\chi^{(n)}\,, \qquad \gamma = \sum \frac{1}{\varphi^n}\gamma^{(n)}\,, \qquad H = \sum \frac{1}{\varphi^n}H^{(n)}\,. \tag{17}$$

Substituting these expressions into the equation of motion ($u = 1 - 8\pi k\xi\phi^2$)

$$-\frac{1}{16\pi k}\Big(uG_{kl} - u_{;k;l} + g_{kl}\ g^{ij}u_{;i;j}\Big) + \tfrac{1}{2}\phi_{,k}\phi_{,l} - \tfrac{1}{2}\Big(\tfrac{1}{2}g^{ij}\phi_{,i}\phi_{,j} + V\Big)g_{kl} = 0 \quad (18)$$

yields, to the lowest orders, $\gamma^{(0)} = H^{(0)} = 0$, $\gamma^{(1)} = p^2 H^{(1)} = 2\chi^{(0)}$ and

$$-\chi^{(0)\prime\prime} + \Big[\boldsymbol{P}^2 + U\Big]\chi^{(0)} = \omega^2\chi^{(0)}\,, \quad (19)$$

where $\boldsymbol{P} = (p_1, p_2, p_3),\ \omega = p_0$ and

$$\boldsymbol{P}^2 = \sum_{a=1}^{3} p_a^2\,, \qquad p^2 = \boldsymbol{P}^2 - \omega^2\,, \qquad U(y) = -\frac{1}{\zeta}\,\frac{d^2 v}{d\phi^2}\bigg|_{\phi=\bar{\phi}^{(0)}(y)}\,. \quad (20)$$

So the solutions found will be stable (at least under small perturbations) provided the Schrödinger-like equation for $\chi^{(0)}$ has only positive eigenvalues that is $\omega^2 > 0$ for all $\boldsymbol{P}$. This can be examined in general without need to specify the potential v: a straightforward extension of the Bargmann–Schwinger [6] method to periodic potentials such as U shows that (19) has only positive eigenvalues only if $\boldsymbol{P}^2 + U$ is positive. Unfortunately *any* v that leads to periodic solutions to (11) leads to a potential U which is *not* positive definite so that the $\boldsymbol{P} = 0$ mode is necessarily unstable.

6. Outlook

The main point of this talk has been to demonstrate that a simple modification of the Lagrangian evades the constraints imposed by the sum rules of Ref. [4] to scalars minimally coupled to gravity. The modified coupling is quite natural (and, in fact, it is generated though loop effects) and leads to periodic solutions similar to the ones postulated in the RS model.

The solutions found are not stable under small perturbations but this is not necessarily fatal: one can modify the model by introducing a new (Goldberger–Wise) stabilizing scalar ψ with the following coupling:

$$\mathcal{L} \to \mathcal{L} - \tfrac{1}{2}g^{MN}\ \partial_M\psi\ \partial_N\psi - W(\psi) + \lambda(\psi)g^{MN}\partial_M\psi\ \partial_N\phi\,. \quad (21)$$

The last terms approximates $\phi''\times$ function of ψ that mimics the stabilizing term $\sum_b \delta(y - y_b)\lambda_b(\psi^2 - v_b^2)^2$ used in Ref. [5]. This work is currently in progress.

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