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HIGHER YM THEORIES AND THE COMPACTIFICATION IN STRING THEORY*

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The Higher YM theories generalize these of the standard model of particles. From the string theory side, the geometrical constructions of gerbes appear, when describing non-vanishing B-fields on branes. Both, higher YM and gerbes are proved to be the same mathematical object. Thus, a natural candidate for the intermediate stage of the compactification in string theory appears. Moreover, replacing smooth 2-spaces by some categories of smooth topoi gives rise to the generalized spacetime based on topoi.

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1. Introduction

There exists a substantial effort recently toward getting the Standard Model (SM) of particles from superstring theory. One way to this end is to perform the compactification of extra dimensions and leave uncompactified 4 spacetime dimensions. This process should lead to the known 4-dimensional physics and SM in particular. Even though the compactification leads to the known 4-dimensional physics this process is highly non unique. The set of possible solutions (vacua) of string theory totals about 10^{500} possibilities. The predictive power of string theory is dubious. String theory leads to the highly non unique set of predictions relating our world. This situation enforced some authors to apply statistical methods to the landscape of possible solutions of string theory and to make statistical predictions. In this paper we present some mathematical tools of gerbes, higher Yang–Mills theories and topoi as possibly relevant to the compactification of string theory in the sense that the result of the compactification should lead more naturally to the higher YM theories rather than just Yang–Mills theories (YM) of SM.

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2. The categorification, higher YM theories and gerbes

The program of categorification in mathematics was proposed by Baez and Dolan [4] and currently is under rapid development. It consists in replacing the category SET of sets and maps, by category of categories and functors between these and replacing categories by higher categories. This is performed such that diagrams defining some structures in a category are weakened and they hold only up to higher isomorphisms from the higher category. When the isomorphisms are replaced by identities one gets back 1-category structures.

2-category is the category which consists of a class of objects, a class of 1-morphisms, a class of 2- morphisms (cells) with suitable set of coherence diagrams which are supposed to hold true [3, 4]. One way of showing the importance of categorification for physics is to categorify the mathematical theory of fibre bundles and connections and watch for the physically valid ingredients which are not usually present in traditional YM theories. Such an enterprise was performed indeed by Baez *et al.* [3] giving rise to the higher gauge theories. A gauge theory describes the parallel transport of point particles using connections on bundles. A higher gauge theory describes the parallel transport of 1-dimensional, string-like, objects using 2-connections on 2-bundles. A 2-bundle is a categorified version of a bundle: that is, one where the fiber is not a manifold but a category with a suitable smooth structure. The essential role in categorification is played by the process of internalization of one construction or category in some other categories. Thus, a smooth manifolds and smooth maps between these are enhanced to 2-smooth spaces and 2-smooth maps in a suitable sense [3]. 2-Lie groups correspond to groups internally in the 2-category of 2-spaces. On such a categorified 2-bundles one can define 2-connections. These are defined in terms of 2-holonomies. Following [3], for each path in the path's groupoid $P_1(M)$ we assign a holonomy taking values in some Lie group G. The composition of paths corresponds to multiplication in G. Also for each 1-parameter family of paths with fixed endpoints we have a holonomy taking values in some other Lie group H. The vertical composition corresponds to multiplication in H [3].

We can parallel transport an element $g \in G$ along a 1-parameter family of paths and get an element $\hat{g} \in G$. There emerges a category $\mathcal{K}_G$ where the set of objects is G, while the set of morphisms consists of ordered pairs $f = (g, h) \in G \times H$ [3]. We write $f : g \to \hat{g}$ for a morphism between the morphisms.

Moreover, we can vertically compose $f : g \to \hat{g}$ and $\hat{f} : \hat{g} \to \hat{\hat{g}}$ to get $f\hat{f} : g \to \hat{\hat{f}}$. This is just the composition of morphisms in the category $\mathcal{K}_G$. We can also compose horizontally $f_1 : g_1 \to \hat{g}_1$ and $f_2 : g_2 \to \hat{g}_2$ to get $f_1 \circ f_2 : g_1 g_2 \to \hat{g}_1 \hat{g}_2$.

This operation guarantees that the morphisms of $\mathcal{K}_G$ consist a group. The interchange law, as presented below, enables one to have well-defined a surface holonomy (for G and H also nonabelian):

$$\begin{tikzcd}[column sep=large]
\bullet \arrow[r, bend left=60, "g_1"{name=U1}] \arrow[r, "g_2"{name=M1}] \arrow[r, bend right=60, "g_3"'{name=D1}] & \bullet \arrow[r, bend left=60, "\hat{g}_1"{name=U2}] \arrow[r, "\hat{g}_2"{name=M2}] \arrow[r, bend right=60, "\hat{g}_3"'{name=D2}] & \bullet
\arrow[Rightarrow, from=U1, to=M1, "f_1"]
\arrow[Rightarrow, from=M1, to=D1, "\hat{f}_1"]
\arrow[Rightarrow, from=U2, to=M2, "f_2"]
\arrow[Rightarrow, from=M2, to=D2, "\hat{f}_2"]
\end{tikzcd}$$

$$(f_1\hat{f}_1) \circ (f_2\hat{f}_2) = (f_1 \circ f_2)(\hat{f}_1 \circ \hat{f}_2)\,.$$

A structure of Lie 2-group emerges. This is an internal category in the category of Lie groups, meaning it is a category where the set of objects is a Lie group, the set of morphisms is a Lie group, and all the usual category operations are Lie group homomorphisms.

A 2-category with all morphisms and 2-morphisms invertible is a 2-groupoid. A 2-groupoid with only one object is a 2-group. Let C be a cartesian closed category containing among its objects smooth manifolds and among its morphisms smooth maps. This category possesses all limits and colimits [3]. Thus, according to the internalization procedure:

1. A smooth group is a group in C^∞.
2. A smooth groupoid is a groupoid in C^∞.
3. A smooth category is a category in C^∞.
4. A smooth 2-group is a 2-group in C^∞.
5. A smooth 2-groupoid is a 2-groupoid in C^∞.
6. A smooth 2-category is a 2-category in C^∞.

The categorification of a fibre bundle with a connection gives the crucial correspondence [3]:

Theorem 1 *There is a one-to-one correspondence between smooth 2-functors* $\mathrm{hol} : \mathrm{P}_2(\mathrm{M}) \to \mathcal{K}_G$

$$\mathrm{hol}_i \quad : \mathrm{P}_2(U_i) \quad \to \quad \mathcal{K}_G$$

$$\begin{tikzcd}
x \arrow[r, bend left=50, "\gamma"{name=A}] \arrow[r, bend right=50, "\eta"'{name=B}] & y \arrow[r, phantom, "\mapsto"] & \bullet \arrow[r, bend left=50, "\mathrm{hol_i}(\gamma)"{name=C}] \arrow[r, bend right=50, "\mathrm{hol_i}(\eta)"'{name=D}] & \bullet
\arrow[Rightarrow, from=A, to=B, "\Sigma"]
\arrow[Rightarrow, from=C, to=D, "\mathrm{hol_i}(\Sigma)"]
\end{tikzcd}$$

and pairs $(A; B)$ *consisting of a* $\mathfrak{g}$*-valued 1-form* A *and an* $\mathfrak{h}$*-valued 2-form* B *on* M *with vanishing fake curvature* $dA + A \wedge A + dt(B) = 0$.

Here $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of the groups G and H, respectively, and $P_2(M)$ is a 2-grupoid of the thin homotopy classes of surfaces in M. The fact that any 2-group is equivalent to the crossed module, where dt is the differential form of the morphism t defining the module with respect to the groups G and H, is used here [3].

When $\mathcal{K}_G = \mathrm{AUT(H)}$ for some Lie group H, the pair $(A;B)$ is a connection on a trivial nonabelian H-gerbe. In general, the pair (A,B) is exactly a connection on nonabelian gerbe [3]. Gerbes are generalizations of vector bundles and are defined by specifying the following data [1, 3, 9]:

1. A base space manifold M;
2. A good cover U of M;
3. A crossed module (G,H,α,t) with differential crossed module $(\mathfrak{g},\mathfrak{h},d\alpha,dt)$;
4. Transition functions (here ($U^{[n]}$ is the n-fold intersection of local patches of a good cover U): $U^{[2]} \to \Omega^0(M,G)$ such that $(x,i,j) \mapsto g_{ij}(x)$;
5. Connection 1-forms: $U^{[1]} \to \Omega^1(M,\mathfrak{g})$ such that $(x,i) \mapsto A_i(x)$;
6. Curving 2-forms: $U^{[1]} \to \Omega^2(M,\mathfrak{h})$ such that $(x,i) \mapsto B_i(x)$;
7. Transition transformation 0-forms: $U^{[3]} \to \Omega^0(M,H)$ such that $(x,i,j,k) \mapsto f_{ijk}(x)$
8. Connection transformation 1-forms: $U^{[2]} \to \Omega^1(M,\mathfrak{h})$ such that $(x,i,j) \mapsto a_{ij}(x)$;
9. Curving transformation 2-forms $U^{[2]} \to \Omega^2(M,\mathfrak{h})$ such that $(x,i,j) \mapsto d_{ij}(x)$;
10. Twist p-forms are also defined [3] such that the following transition laws are satisfied:
    (a) transition law for the transition functions
    $\phi_{ij}(x)\phi_{jk}(x) = t(f_{ijk}(x))\phi_{ik}(x)\,, \quad \forall (x,i,j,k) \in U^{[3]}$
    (b) transition law for the connection 1-forms
    $A_i(x) + dt(a_{ij(x)}) = \phi_{ij}(x)A_j(x)\phi_{ij}^{-1}(x) + \phi_{ij}(x)(d\phi_{ij}^{-1})(x)\,,$
    $\forall (x,i,j) \in U^{[2]}\,.$
    (c) transition law for the curving 2-forms
    $B_i(x) = \alpha(\phi_{ij}(x))(B_j(x)) + k_{ij}(x) - d_{ij}(x) - \beta_{ij}(x)\,,$
    $\forall (x,i,j) \in U^{[2]}\,.$
    (d) transition law for the curving transformation 2-forms
    $d_{ij} + \phi_{ij}(d_{jk}) = f_{ijk}\, d_{ik}\, f_{ijk}^{-1} + f_{ijk} d\alpha(dt(B_i) + F_{A_i})\, f_{ijk}^{-1}\,,$
    $\forall (x,i,j) \in U^{[2]}\,.$

3. The modified compactification to 2-YM

The spectrum of IIB superstring theory contains, among others, B-field, which is the collection of real 2-forms B_i on each U_i, satisfying:

On double overlaps U_{ij} we have the relation: $B_j - B_i = dA_{ij}$, where A_{ij} are real 1-forms. These 1-forms satisfy a consistency condition on triple overlaps U_{ijk}: $A_{ij}+A_{jk}+A_{ki} = -id\log\alpha_{ijk}$ where α_{ijk} are $U(1)$-valued 0-forms. α_{ijk} have to satisfy U_{ijkl}: $\alpha_{ijk}\alpha^{-1}_{jkl}\alpha_{ikl}\alpha^{-1}_{ijl} = 1$ on quadruple overlaps.

The above equations indicate that B-field is a connection on a gerbe. This is the generalization of the correspondence of the brane charges and K-theory classes $K(Q)$ for branes wraping a cycle $Q \in M$.

The idea of modified compactification relies on considering the intermediate stage in between string theory and 4-dimensional YM theories of standard model of particle physics. The stage is not a MSSM. From the perspective of 4-YM it is the higher, in fact 2-YM, version of YM in 4-dimensions. From the perspective of sustring theory this stage should contain gerbes constructions, such that the conection on the gerbe is the pair of string's field *i.e.* (A, B). These fields can appear as well in 4-dimensional compactified versions of the theory. However, Theorem 1 shows the unique correspondence between the pairs (A, B) and 2-bundles defined by cross module (G, H, t, α). Thus, given 2-bundle corresponding to 4-YM it contains automatically (A, B)-fields of string theory. The point is that now we can consider 4-dimensional YM which contains B and H fields. This is more close to sustring than 4-YM of SM, which does not contain these „stringy" fields. From the other side, higher YM need not be supersymmetric, hence these are also more close to the YM of SM than traditional compactifications of sustring theory.

4. Discussion and topoi models of spacetime

When one takes a smooth topos, say Basel $\mathcal{B}$ [10], rather than the category C^∞, one can formulate internally in $\mathcal{B}$ higher categorical ingredients of the 2-bundle with a 2-connection. This internalization in smooth topoi can be performed in some local patches of spacetime manifold M. This leads to the recently proposed models of spacetime [6–8] where manifold M is covered by internal in topoi objects R^4's rather than external $\mathbb{R}^4$.

From the point of view of coherent formulation of the theory of quantum gravity with YM theories of SM, it seems unavoidable to consider as fundamental symmetry of physics some kind of gauge group which is rather higher 2-group than ordinary Lorentz 1-group. This kind of generalization leads to the natural appearance of B and H string fields as the connections of these extended 2-YM. Moreover, the internalization of the construction in the smooth space shows the connection with the spacetimes locally described by the internal spaces of some smooth topoi. This is also the indication for the possible role assigned to the exotic smooth 4-spaces in the regime of QG [2, 5].

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