Vol. 38 (2007) ACTA PHYSICA POLONICA B No 12

# FIELD THEORY AT FINITE TEMPERATURE AND PHASE TRANSITIONS*

Mariano Quirós

Institució Catalana de Recerca i Estudis Avançats (ICREA)
Institut de Fisica d'Altes Energies (IFAE), 08193 Barcelona, Spain

*(Received October 29, 2007)*

We review different aspects of field theory at finite temperature, related to the theory of phase transitions. Finite temperature field theory is discussed in the real and imaginary time formalisms, showing their equivalence in simple examples. Bubble nucleation by thermal tunneling, and the subsequent development of the phase transition is described in some detail. Finally the application to baryogenesis at the electroweak phase transition is done in the Standard Model. We have translated the condition of not washing out any previously generated baryon asymmetry by upper bounds on the Higgs mass.

PACS numbers: 11.10.Wx, 05.70.Fh, 03.65.Xp, 64.60.Bd

## 1. Introduction

The formalism used in conventional quantum field theory is suitable to describe observables (*e.g.* cross-sections) measured in empty space-time, as particle interactions in an accelerator. However, in the early stages of the Universe, at high temperature, the environment had a non-negligible matter and radiation density, making the hypothesis of conventional field theories impracticable. For that reason, under those circumstances, the methods of conventional field theories are no longer in use, and should be replaced by others, closer to thermodynamics, where the background state is a thermal bath. This area has been called field theory at finite temperature and it is extremely useful to study all phenomena which happened in the early Universe: phase transitions, inflationary cosmology, .... Excellent articles [1, 2], review articles [3–5] and textbooks [6] exist which discuss different aspects of these issues. We will now review the main methods which will be useful for the theory of phase transitions at finite temperature.

* Presented at the XLVII Cracow School of Theoretical Physics, Zakopane, Poland, June 14–22, 2007.

In this section we shall give some definitions borrowed from thermodynamics and statistical mechanics. The **microcanonical ensemble** is used to describe an isolated system with fixed energy $E$, particle number $N$ and volume $V$. The **canonical ensemble** describes a system in contact with a heat reservoir at temperature $T$: the energy can be exchanged between them and $T$, $N$ and $V$ are fixed. Finally, in the **grand canonical ensemble** the system can exchange energy and particles with the reservoir: $T$, $V$ and the chemical potentials are fixed.

Consider now a dynamical system characterized by a Hamiltonian[1] $H$ and a set of conserved (mutually commuting) charges $Q_A$. The equilibrium state of the system at rest in the large volume $V$ is described by the **grand-canonical density operator**

$$\rho = \exp(-\Phi) = \exp\left\{-\sum_A \alpha_A Q_A - \beta H\right\}, \qquad (1.1)$$

where $\Phi \equiv \log \mathrm{Tr}\, \exp\{-\sum_A \alpha_A Q_A - \beta H\}$ is called the Massieu function (Legendre transform of the entropy), $\alpha_A$ and $\beta$ are Lagrange multipliers given by $\beta = T^{-1}$, $\alpha_A = -\beta\mu_A$, $T$ is the temperature and $\mu_A$ are the chemical potentials.

Using (1.1) one defines the **grand canonical average** of an arbitrary operator $\mathcal{O}$, as

$$\langle \mathcal{O} \rangle \equiv \mathrm{Tr}\,(\mathcal{O}\rho) \qquad (1.2)$$

satisfying the property $\langle \mathbf{1} \rangle = 1$. Some conventional definitions which follow are:

$$\begin{aligned} q_A &= \frac{1}{V}\langle Q_A \rangle = -\frac{1}{V}\frac{\partial \Phi}{\partial \alpha_A}\,, & (1.3)\\ E &= \frac{1}{V}\langle H \rangle = -\frac{1}{V}\frac{\partial \Phi}{\partial \beta}\,, & (1.4)\\ F &= -P = -\frac{1}{\beta V}\Phi\,, & (1.5)\\ S &= -\frac{1}{V}\langle \log \rho \rangle = \beta\left[E - F - \sum_A \mu_A q_A\right]. & (1.6) \end{aligned}$$

In the rest of these lectures we will always consider the case of zero chemical potential. It will be re-introduced when necessary.

[1] All operators will be considered in the Heisenberg picture.

## 2. Generating functionals

We will start by considering the case of a real scalar field $\phi(x)$, carrying no charges ($\mu_A = 0$), with Hamiltonian $H$, *i.e.*

$$\phi(x) = e^{itH}\phi(0,\vec{x})e^{-itH}\,, \tag{2.1}$$

where the time $x^0 = t$ is analytically continued to the complex plane.

We define the thermal Green function as the grand canonical average of the ordered product of the $n$ field operators

$$G^{(C)}(x_1,\ldots,x_n) \equiv \langle T_C\phi(x_1),\ldots,\phi(x_n)\rangle\,, \tag{2.2}$$

where the $T_C$ ordering means that fields should be ordered along the path $C$ in the complex $t$-plane. For instance the product of two fields is defined as,

$$T_C\phi(x)\phi(y) = \theta_C(x^0-y^0)\phi(x)\phi(y) + \theta_C(y^0-x^0)\phi(y)\phi(x)\,. \tag{2.3}$$

If we parameterize $C$ as $t = z(\tau)$, where $\tau$ is a real parameter, $T_C$ ordering means standard ordering along $\tau$. Therefore the step and delta functions can be given as $\theta_C(t) = \theta(\tau)$, $\delta_C(t) = (\partial z/\partial\tau)^{-1}\,\delta(\tau)$.

The rules of the functional formalism can be applied as usual, with the prescription $\delta j(y)/\delta j(x) = \delta_C(x^0-y^0)\delta^{(3)}(\vec{x}-\vec{y})$, and the generating functional $Z^\beta[j]$ for the full Green functions,

$$Z^\beta[j] = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int\limits_C d^4x_1\ldots d^4x_n j(x_1)\ldots j(x_n)G^{(C)}(x_1,\ldots,x_n) \tag{2.4}$$

can also be written as,

$$Z^\beta[j] = \left\langle T_C\exp\left\{i\int\limits_C d^4x j(x)\phi(x)\right\}\right\rangle \tag{2.5}$$

which is normalized to $Z^\beta[0] = \langle\mathbf{1}\rangle = 1$, as in (1.2), and where the integral along $t$ is supposed to follow the path $C$ in the complex plane.

Similarly, the generating functional for connected Green functions $W^\beta[j]$ is defined as $Z^\beta[j] \equiv \exp\{iW^\beta[j]\}$, and the generating functional for 1PI Green functions $\mathit{\Gamma}^\beta[\overline{\phi}]$, by the Legendre transformation,

$$\mathit{\Gamma}^\beta[\overline{\phi}] = W^\beta[j] - \int\limits_C d^4x\frac{\delta W^\beta[j]}{\delta j(x)}j(x)\,, \tag{2.6}$$

where the current $j(x)$ is eliminated in favor of the classical field $\overline{\phi}(x)$ as $\overline{\phi}(x) = \delta W^\beta[j]/\delta j(x)$. It follows that $\delta \Gamma^\beta[\overline{\phi}]/\delta\overline{\phi}(x) = -j(x)$, and $\overline{\phi}(x) = \langle\phi(x)\rangle$ is the grand canonical average of the field $\phi(x)$.

Symmetry violation is signaled by

$$\left.\frac{\delta \Gamma^\beta[\overline{\phi}]}{\delta\overline{\phi}}\right|_{j=0} = 0 \tag{2.7}$$

for a value of the field different from zero.

As in field theory at zero temperature, in a translationally invariant theory $\overline{\phi}(x) = \phi_{\rm c}$ is a constant. In this case, by removing the overall factor of space-time volume arising in each term of $\Gamma^\beta[\phi_{\rm c}]$, we can define the effective potential at finite temperature as,

$$\Gamma^\beta[\phi_{\rm c}] = -\int d^4x V^\beta_{\rm eff}(\phi_{\rm c}) \tag{2.8}$$

and symmetry breaking occurs when

$$\frac{\partial V^\beta_{\rm eff}(\phi_{\rm c})}{\partial\phi_{\rm c}} = 0 \tag{2.9}$$

for $\phi_{\rm c} \neq 0$.

## 3. Green functions

Not all the contours are allowed if we require Green functions to be analytic with respect to $t$. In this section we will compute Green functions for an arbitrary allowed contour.

### *3.1. Scalar fields*

For scalar fields, using (2.3) we can write the two-point Green function as,

$$G^{(C)}(x-y) = \theta_C(x^0-y^0)G_+(x-y) + \theta_C(y^0-x^0)G_-(x-y)\,, \tag{3.1}$$

where

$$G_+(x-y) = \langle\phi(x)\phi(y)\rangle\,, \quad G_-(x-y) = G_+(y-x)\,. \tag{3.2}$$

Now take the complete set of states $|n\rangle$ with eigenvalues $E_n$: $H|n\rangle = E_n|n\rangle$. One can readily compute (3.2) at the point $\vec{x} = \vec{y} = 0$ as

$$G_+(x^0-y^0) = e^{-\Phi}\sum_{m,n}|\langle m|\phi(0)|n\rangle|^2\, e^{-iE_n(x^0-y^0)}e^{iE_m(x^0-y^0+i\beta)} \tag{3.3}$$

so that the convergence of the sum implies that $-\beta \le \mathrm{Im}\,(x^0 - y^0) \le 0$ which requires $\theta_C(x^0 - y^0) = 0$ for $\mathrm{Im}\,(x^0 - y^0) > 0$. From (3.2) it follows that the similar property for the convergence of $G_-(x^0 - y^0)$ is that $0 \le \mathrm{Im}\,(x^0 - y^0) \le \beta$, which requires $\theta_C(y^0 - x^0) = 0$ for $\mathrm{Im}\,(x^0 - y^0) < 0$, and the final condition for the convergence of the complete Green function on the strip

$$-\beta \le \mathrm{Im}\,(x^0 - y^0) \le \beta \tag{3.4}$$

is that we define the function $\theta_C(t)$ such that $\theta_C(t) = 0 \quad \text{for} \quad \mathrm{Im}\,(t) > 0\,.$ The latter condition implies that $C$ must be such that *a point moving along it has a monotonically decreasing or constant imaginary part.*

A very important periodicity relation affecting Green functions can be easily deduced from the very definition of $G_+(x)$ and $G_-(x)$, Eq. (3.2). By using the definition of the grand canonical average and the cyclic permutation property of the trace of a product of operators, it can be easily deduced,

$$G_+(t - i\beta, \vec{x}) = G_-(t, \vec{x}) \tag{3.5}$$

which is known as the Kubo–Martin–Schwinger relation [7].

We can now compute the two-point Green function (3.1) for a free scalar field,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2\omega_p)^{1/2}} \left[a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right]\,, \tag{3.6}$$

where $\omega_p = \sqrt{\vec{p}^{\,2} + m^2}$, which satisfies the equation

$$\left[\partial^\mu \partial_\mu + m^2\right] G^{(C)}(x-y) = -i\delta_C(x-y) \equiv -i\delta_C(x^0 - y^0)\delta^{(3)}(\vec{x} - \vec{y})\,. \tag{3.7}$$

Using the time derivative of (3.6), and the equal time commutation relation,

$$\left[\phi(t,\vec{x}), \dot{\phi}(t,\vec{y})\right] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{3.8}$$

one easily obtains the commutation relation for creation and annihilation operators,

$$\left[a(p), a^\dagger(k)\right] = \delta^{(3)}(\vec{p} - \vec{k}) \tag{3.9}$$

and defining the Hamiltonian of the field as,

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p a^\dagger(p) a(p) \tag{3.10}$$

one can obtain, using (3.9) the thermodynamical averages,

$$\begin{aligned} \langle a^\dagger(p) a(k)\rangle &= n_{\mathrm{B}}(\omega_p)\delta^{(3)}(\vec{p} - \vec{k})\,, \\ \langle a(p) a^\dagger(k)\rangle &= [1 + n_{\mathrm{B}}(\omega_p)]\delta^{(3)}(\vec{p} - \vec{k})\,, \end{aligned} \tag{3.11}$$

where $n_{\rm B}(\omega)$ is the Bose distribution function,

$$n_{\rm B}(\omega) = \frac{1}{e^{\beta\omega}-1}\,. \tag{3.12}$$

We will give here a simplified derivation of expression (3.11). Consider the simpler example of a quantum mechanical state occupied by bosons of the **same** energy $\omega$. There may be any number of bosons in that state and no interaction between the particles: we will denote that state by $|n\rangle$. The set $\{|n\rangle\}$ is complete. Creation and annihilation operators are denoted by $a^\dagger$ and $a$, respectively. They act on the states $|n\rangle$ as, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a|n\rangle = \sqrt{n}|n-1\rangle$ , and satisfy the commutation relation, $\left[a, a^\dagger\right] = 1$. The Hamiltonian and number operators are defined as $H = \omega N$ and $N = a^\dagger a$, with eigenvalues $\omega n$ and $n$, respectively.

It is very easy to compute now $\langle a^\dagger a\rangle$ and $\langle a a^\dagger\rangle$ as in (3.11) using the completeness of $\{|n\rangle\}$. In particular,

$$\mathrm{Tr}\,(e^{-\beta H}) = \sum_{n=0}^{\infty}\langle n|e^{-\beta H}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta\omega n} = \frac{1}{1-e^{-\beta\omega}}$$

and

$$\mathrm{Tr}\,(e^{-\beta H}a^\dagger a) = \sum_{n=0}^{\infty} n e^{-\beta\omega n} = \frac{e^{-\beta\omega}}{(1-e^{-\beta\omega})^2}$$

from where $\langle a^\dagger a\rangle = n_{\rm B}(\omega)$, and $\langle a a^\dagger\rangle = 1 + n_{\rm B}(\omega)$, as we wanted to prove.

Using now (3.11) we can cast the two-point Green function as,

$$G^{(C)}(x-y) = \int \frac{d^4p}{(2\pi)^4}\rho(p)e^{-ip(x-y)}\left[\theta_C(x^0-y^0) + n_{\rm B}(p^0)\right]\,, \tag{3.13}$$

where the function $\rho(p)$ is defined by $\rho(p) = 2\pi[\theta(p^0)-\theta(-p^0)]\delta(p^2-m^2)$. Now the particular value of the Green function (3.13) depends on the chosen contour $C$. We will show later on two particular contours giving rise to the so-called imaginary and real time formalisms. Before coming to them we will describe how the previous formulae apply to the case of fermion fields.

### *3.2. Fermion fields*

We will replace here (3.1) and (3.2) by,

$$S^{(C)}_{\alpha\beta}(x-y) \equiv \langle T_C\psi_\alpha(x)\overline{\psi}_\beta(y)\rangle = \theta_C(x^0-y^0)S^+_{\alpha\beta} - \theta_C(y^0-x^0)S^-_{\alpha\beta}\,, \tag{3.14}$$

where $\alpha$ and $\beta$ are spinor indices, and

$$S^+_{\alpha\beta}(x-y) = \langle\psi_\alpha(x)\overline{\psi}_\beta(y)\rangle \tag{3.15}$$

is the reduced Green function, which satisfy the Kubo–Martin–Schwinger relation,

$$S^{+}_{\alpha\beta}(t-i\beta,\vec{x}) = -S^{-}_{\alpha\beta}(t,\vec{x})\,. \tag{3.16}$$

The calculation of the two-point Green function for a free fermion field, satisfying the equation

$$(i\gamma\cdot\partial - m)_{\alpha\sigma}\,S^{(C)}_{\sigma\beta}(x-y) = i\delta_C(x-y)\delta_{\alpha\beta} \tag{3.17}$$

follows lines similar to Eqs. (3.6) to (3.13). In particular, one can define a Green function $S^{(C)}$ as

$$S^{(C)}_{\alpha\beta}(x-y) \equiv (i\gamma\cdot\partial + m)_{\alpha\beta}S^{(C)}(x-y)\,, \tag{3.18}$$

where $S^{(C)}(x-y)$ satisfies the Klein–Gordon propagator equation (3.7). One can obtain for $S^{(C)}$ the expression,

$$S^{(C)}(x-y) = \int \frac{d^4p}{(2\pi)^4}\rho(p)e^{-ip(x-y)}\left[\theta_C(x^0-y^0) - n_{\mathrm{F}}(p^0)\right]\,, \tag{3.19}$$

where $n_{\mathrm{F}}(\omega)$ is the Fermi distribution function

$$n_{\mathrm{F}}(\omega) = \frac{1}{e^{\beta\omega}+1}\,. \tag{3.20}$$

Eq. (3.20) can be derived similarly to (3.12) as the mean number of fermions for a Fermi gas. This time the Pauli exclusion principle forbids more than one fermion occupying a single state, so that only the states $|0\rangle$ and $|1\rangle$ exist. They are acted on by creation and annihilation operators $b^\dagger$ and $b$, respectively as: $b^\dagger|0\rangle = |1\rangle$, $b^\dagger|1\rangle = 0$, $b|0\rangle = 0$, $b|1\rangle = |0\rangle$, and satisfy anticommutation rules, $\left\{b, b^\dagger\right\} = 1$. Defining the Hamiltonian and number operators as $H = \omega N$ and $N = b^\dagger b$, we can compute now the statistical averages of $\langle b^\dagger b\rangle$ and $\langle bb^\dagger\rangle$ using the completeness of $\{|n\rangle\}$.

$$\mathrm{Tr}\,(e^{-\beta H}) = \sum_{n=0}^{1}\langle n|e^{-\beta H}|n\rangle = \sum_{n=0}^{1} e^{-\beta\omega n} = 1 + e^{-\beta\omega}$$

and

$$\mathrm{Tr}\,(e^{-\beta H}b^\dagger b) = \sum_{n=0}^{1} n e^{-\beta\omega n} = e^{-\beta\omega}$$

from where $\langle b^\dagger b\rangle = n_{\mathrm{F}}(\omega)$, and $\langle bb^\dagger\rangle = 1 - n_{\mathrm{F}}(\omega)$, as we wanted to prove.

## 4. Imaginary time formalism

The calculation of the propagators in the previous sections depends on the chosen path $C$ going from an initial arbitrary time $t$ to $t-i\beta$, provided by the Kubo–Martin–Schwinger periodicity properties (3.5) and (3.16) of Green functions. The simplest path is to take a straight line along the imaginary axis $t=-i\tau$. It is called Matsubara contour, since Matsubara [8] was the first to set up a perturbation theory based upon this contour. In that case $\delta_C(t)=i\delta(\tau)$.

The two-point Green functions for scalar (3.13) and fermion (3.19) fields can be written as,

$$G(\tau,\vec{x})=\int\frac{d^4p}{(2\pi)^4}\rho(p)e^{i\vec{p}\vec{x}}e^{-\tau p^0}\left[\theta(\tau)+\eta n(p^0)\right]\,, \tag{4.1}$$

where the symbol $\eta$ stands as: $\eta_B=1$ ($\eta_{\rm F}=-1$) for bosons (for fermions). Analogously $n(p^0)$ stands either for $n_{\rm B}(p^0)$, as given by (3.12) for bosons, or $n_{\rm F}(p^0)$, as given by (3.20) for fermions. It can be defined as a function of $\eta$ as,

$$n(\omega)=\frac{1}{e^{\beta\omega}-\eta}\,. \tag{4.2}$$

The Green function (4.1) can be decomposed as in (3.1)

$$G(\tau,\vec{x})=G_+(\tau,\vec{x})\theta(\tau)+G_-(\tau,\vec{x})\theta(-\tau)\,. \tag{4.3}$$

Using now the Kubo–Martin–Schwinger relations, Eqs. (3.5) and (3.16), we can write $G(\tau+\beta)=\eta G(\tau)$ for $-\beta\leq\tau\leq 0$, $G(\tau-\beta)=\eta G(\tau)$ for $0\leq\tau\leq\beta$, which means that the propagator for bosons (fermions) is periodic (antiperiodic) in the *time* variable $\tau$, with period $\beta$.

It follows that the Fourier transform of (4.1)

$$\widetilde{G}(\omega_n,\vec{p})=\int\limits_{\alpha-\beta}^{\alpha}d\tau\int d^3x e^{i\omega_n\tau-i\vec{x}\vec{p}}G(\tau,\vec{x}) \tag{4.4}$$

(where $0\leq\alpha\leq\beta$) is independent of $\alpha$ and the discrete frequencies satisfy the relation $\eta e^{i\omega_n\beta}=1$, *i.e.* $\omega_n=2n\pi\beta^{-1}$ for bosons, and $\omega_n=(2n+1)\pi\beta^{-1}$ for fermions.

Inserting now (4.1) into (4.4) we can obtain the propagator in momentum space $\widetilde{G}$

$$\widetilde{G}(\omega_n,\vec{p})=\frac{1}{\vec{p}^{\,2}+m^2+\omega_n^2}\,. \tag{4.5}$$

We can now define the euclidean propagator, $\Delta(-i\tau,\vec{x})$, by

$$G(\tau,\vec{x}) = i\Delta(-i\tau,\vec{x})\,, \tag{4.6}$$

where $G(\tau,\vec{x})$ is the propagator defined in (4.1). Therefore, using (4.5), we can write the inverse Fourier transformation,

$$\Delta(x) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}e^{-i\omega_n\tau+i\vec{p}\vec{x}}\frac{-i}{\vec{p}^{\,2}+m^2+\omega_n^2}\,, \tag{4.7}$$

where the Matsubara frequencies $\omega_n$ are either for bosons or for fermions.

From (4.7) one can deduce the Feynman rules for the different fields in the imaginary time formalism. We can summarize them in the following way:

$$\begin{aligned}
&\text{Boson propagator}: && \frac{i}{p^2-m^2}\,;\ p^\mu = [2ni\pi\beta^{-1},\vec{p}\,]\,;\\
&\text{Fermion propagator}: && \frac{i}{\gamma\cdot p-m}\,;\ p^\mu = [(2n+1)i\pi\beta^{-1},\vec{p}\,]\,;\\
&\text{Loop integral}: && \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\,;\\
&\text{Vertex function}: && -i\beta(2\pi)^3\delta_{\sum\omega_i}\delta^{(3)}\left(\sum_i\vec{p}_i\right).
\end{aligned} \tag{4.8}$$

There is a standard trick to perform infinite summations as in (4.8). For the case of bosons we can have frequency sums as,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}f(p^0=i\omega_n) \tag{4.9}$$

with $\omega_n = 2n\pi\beta^{-1}$. Since the function $\frac{1}{2}\beta\coth(\frac{1}{2}\beta z)$ has poles at $z=i\omega_n$ and is analytic and bounded everywhere else, we can write (4.9) as,

$$\frac{1}{2\pi i\beta}\int_\gamma dz f(z)\frac{\beta}{2}\coth\left(\frac{1}{2}\beta z\right)\,,$$

where the contour $\gamma$ encircles anticlockwise all the previous poles of the imaginary axis. We are assuming that $f(z)$ does not have singularities along the imaginary axis (otherwise the previous expression is obviously not correct). The contour $\gamma$ can be deformed to a new contour consisting in two straight lines: the first one starting at $-i\infty+\epsilon$ and going to $i\infty+\epsilon$, and the second one starting at $i\infty-\epsilon$ and ending at $-i\infty-\epsilon$. Rearranging the exponentials in the hyperbolic cotangent one can write the previous expression as,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\frac{1}{2}[f(z)+f(-z)]+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dz[f(z)+f(-z)]\frac{1}{e^{\beta z}-1}$$

and the contour of the second integral can be deformed to a contour $C$ which encircles clockwise all singularities of the functions $f(z)$ and $f(-z)$ in the right half plane. Therefore we can write (4.9) as

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(p^0=i\omega_n)=\int_{-i\infty}^{i\infty}\frac{dz}{4\pi i}[f(z)+f(-z)]+\int_C\frac{dz}{2\pi i}n_{\rm B}(z)[f(z)+f(-z)]\,, \tag{4.10}$$

where $n_{\rm B}(z)$ is the Bose distribution function (3.12).

Eq. (4.10) can be generalized for both bosons and fermions as,

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty} f(p^0=i\omega_n)=\int_{-i\infty}^{i\infty}\frac{dz}{4\pi i}[f(z)+f(-z)]+\eta\int_C\frac{dz}{2\pi i}n(z)[f(z)+f(-z)]\,, \tag{4.11}$$

where the distribution functions $n(z)$ are defined in (4.2). Eq. (4.11) shows that the frequency sum naturally separates into a $T$ independent piece, which should coincide with the similar quantity computed in the field theory at zero temperature, and a $T$ dependent piece which vanishes in the limit $T\to 0$, *i.e.* $\beta\to\infty$.

## 5. Real time formalism

The obvious disadvantage of the imaginary time formalism is to compute Green functions along imaginary time, so that going to the real time has to be done through a process of analytic continuation. However, a direct evaluation of Green function in the real time is possible by a judicious choice of the contour $C$ in (2.2). The family of such real time contours is depicted in Fig. 1 where the contour $C$ is $C=C_1\bigcup C_2\bigcup C_3\bigcup C_4$, where $C_1$ goes from the initial time $t_i$ to the final time $t_f$, $C_3$ from $t_f$ to $t_f-i\sigma$, with $0\leq\sigma\leq\beta$, $C_2$ from $t_f-i\sigma$ to $t_i-i\sigma$, and $C_4$ from $t_i-i\sigma$ to $t_i-i\beta$. Different choices of $\sigma$ lead to an equivalence class of quantum field theories at finite temperature [9]. For instance the choice $\sigma=0$ leads to the Keldysh perturbation expansion [10], while the choice $\sigma=\beta/2$ is the preferred one to compute Green functions.

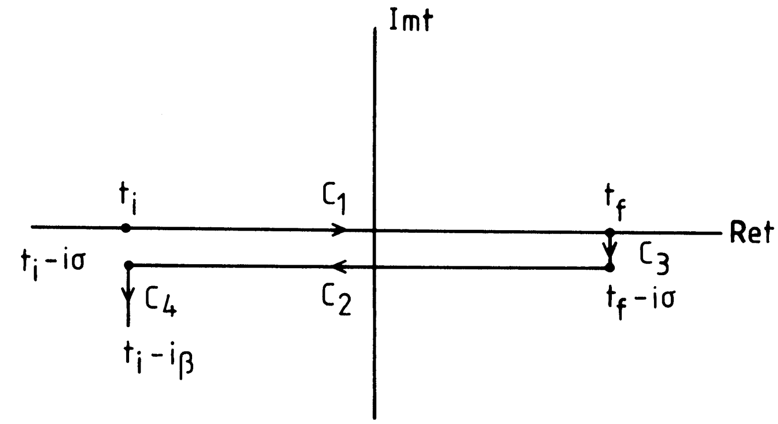


Fig. 1. Contour used in the real time formalism.

Computing the Green function for scalar (3.13) and fermion (3.19) fields taking path $C$ is a matter of calculation, as we did for the imaginary time formalism in (4.1)–(4.5). One can prove that the contribution from the contours $C_3$ and $C_4$ can be neglected [4, 11]. Therefore, for the propagator between $x^0$ and $y^0$ there are four possibilities depending on whether they are on $C_1$ or $C_2$. Correspondingly, there are four propagators which are labeled by (11), (12), (21) and (22).

Making the choice $\sigma = \beta/2$, the propagators for scalar fields (3.13) can be written, in momentum space, as

$$G(p) \equiv \begin{pmatrix} G^{(11)}(p) & G^{(12)}(p) \\ G^{(21)}(p) & G^{(22)}(p) \end{pmatrix} = M_{\rm B}(\beta,p) \begin{pmatrix} \Delta(p) & 0 \\ 0 & \Delta^*(p) \end{pmatrix} M_{\rm B}(\beta,p)\,, \tag{5.1}$$

where $\Delta(p)$ is the boson propagator at zero temperature, and the matrix $M_{\rm B}(\beta,p)$ is given by,

$$M_{\rm B}(\beta,p) = \begin{pmatrix} \cosh\theta(p) & \sinh\theta(p) \\ \sinh\theta(p) & \cosh\theta(p) \end{pmatrix}, \tag{5.2}$$

where

$$\begin{aligned} \sinh\theta(p) &= e^{-\beta\omega_p/2}\left(1 - e^{-\beta\omega_p}\right)^{-1/2}, \\ \cosh\theta(p) &= \left(1 - e^{-\beta\omega_p}\right)^{-1/2}. \end{aligned} \tag{5.3}$$

Using now (5.1), (5.2), (5.3), one can easily write the expression for the four bosonic propagators, as

$$G^{(11)}(p) = \Delta(p) + 2\pi n_{\rm B}(\omega_p)\delta(p^2 - m^2)\,,$$

$$
\begin{aligned}
G^{(22)}(p) &= G^{(11)*}\,, \\
G^{(12)} &= 2\pi e^{\beta\omega_p/2} n_{\rm B}(\omega_p)\delta(p^2-m^2)\,, \\
G^{(21)} &= G^{(12)}\,.
\end{aligned} \tag{5.4}
$$

Similarly, the propagators for fermion fields can be written as

$$
\begin{aligned}
S(p)_{\alpha\beta} &\equiv \begin{pmatrix} S^{(11)}_{\alpha\beta}(p) & S^{(12)}_{\alpha\beta}(p) \\ S^{(21)}_{\alpha\beta}(p) & S^{(22)}_{\alpha\beta}(p) \end{pmatrix} \\
&= M_{\rm F}(\beta,p)\begin{pmatrix} (\gamma\cdot p+m)_{\alpha\beta}\Delta(p) & 0 \\ 0 & (\gamma\cdot p+m)_{\alpha\beta}\Delta^*(p) \end{pmatrix} M_{\rm F}(\beta,p)\,,
\end{aligned} \tag{5.5}
$$

where the matrix $M_{\rm F}(\beta,p)$ is,

$$
M_{\rm F}(\beta,p) = \begin{pmatrix} \cos\theta(p) & \sin\theta(p) \\ \sin\theta(p) & \cos\theta(p) \end{pmatrix} \tag{5.6}
$$

with

$$
\begin{aligned}
\sin\theta(p) &= e^{-\beta\omega_p/2}\left(1+e^{-\beta\omega_p}\right)^{-1/2}\,, \\
\cos\theta(p) &= [\theta(p^0)-\theta(-p^0)]\left(1+e^{-\beta\omega_p}\right)^{-1/2}\,.
\end{aligned} \tag{5.7}
$$

In the same way, using now (5.5), (5.6), and (5.7) one can easily write the expression for the four fermionic propagators, as

$$
\begin{aligned}
S^{(11)}(p) &= (\gamma\cdot p+m)\left(\Delta(p)-2\pi n_{\rm F}(\omega_p)\delta(p^2-m^2)\right)\,, \\
S^{(22)}(p) &= S^{(11)*}\,, \\
S^{(12)} &= -2\pi(\gamma\cdot p+m)[\theta(p^0)-\theta(-p^0)]e^{\beta\omega_p/2}n_{\rm F}(\omega_p)\delta(p^2-m^2)\,, \\
S^{(21)} &= -S^{(12)}\,.
\end{aligned} \tag{5.8}
$$

As one can see from (5.4) and (5.8), the main feature of the real time formalism is that the propagators come in two terms: one which is the same as in the zero temperature field theory, and a second one where all the temperature dependence is contained. This is welcome. However the propagators (12), (21) and (22) are unphysical since one of their time arguments has an imaginary component. They are required for the consistency of the theory. The only physical propagator is the (11) component in (5.4) and (5.8).

Now the Feynman rules in the real time formalism are very similar to those in the zero temperature field theory. In fact all diagrams have the same topology as in the zero temperature field theory and the same symmetry factors. However, associated to every field there are two possible vertices, 1 and 2, and four possible propagators, (11), (12), (21) and (22) connecting them. All of them have to be considered for the consistency of the theory. In the Feynman rules, type 2 vertices are hermitian conjugate with respect to type 1 vertices. The golden rule is that: *physical legs must always be attached to type 1 vertices*. Apart from the previous prescription, one must sum over all the configurations of type 1 and type 2 vertices, and use the propagator $G^{(ab)}$ or $S^{(ab)}$ to connect vertex $a$ with vertex $b$.

There is now a general agreement in the sense that the imaginary time formalism and the real time formalism should give the same physical answer [12]. Using one or the other is sometimes a matter of taste, though in some cases the choice is dictated by calculational simplicity depending on the physical problem one is dealing with.

## 6. The effective potential at finite temperature

In this section we will construct the (one-loop) effective potential at finite temperature, using all the tools provided in the previous sections. As we will see, in particular, the effective potential at finite temperature contains the effective potential at zero temperature. The usefulness of this construction is addressed to the theory of phase transitions at finite temperature. The latter being essential for the understanding of phenomena as: inflation, baryon asymmetry generation, quark–gluon plasma transition in QCD,. . . . We will compare different methods leading to the same result, including the use of both the imaginary and the real time formalisms. This exercise can be useful mainly to face more complicated problems than those which will be developed in these lectures.

### *6.1. Scalar fields*

We will consider here the simplest model of one self-interacting scalar fields described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - V_0(\phi) \tag{6.1}$$

with a tree-level potential

$$V_0 = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \,. \tag{6.2}$$

We have to compute the diagrams contained in Fig. 2 using the Feynman rules described in (4.8), for the imaginary time formalism, or in (5.4) for the real time formalism. We will write the result as,

$$V_{\rm eff}^{\beta}(\phi_{\rm c}) = V_0(\phi_{\rm c}) + V_1^{\beta}(\phi_{\rm c})\,, \tag{6.3}$$

where $V_0(\phi_{\rm c})$ is the tree level potential.

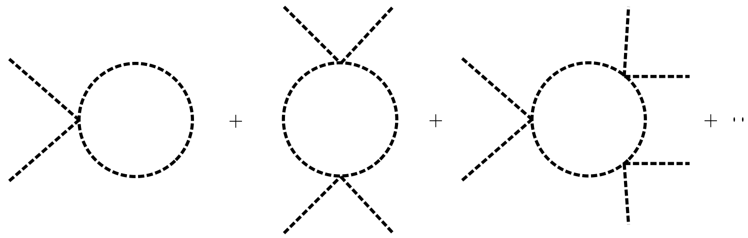

Fig. 2. 1PI diagrams contributing to the one-loop effective potential of (6.1).

#### 6.1.1. Imaginary time formalism

We will compute the diagrams in Fig. 2. Using the Feynman rules in Eq. (4.8), the expression

$$V_1(\phi_{\rm c}) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\log\left[p^2 + m^2(\phi_{\rm c})\right] \tag{6.4}$$

translates into

$$V_1^{\beta}(\phi_{\rm c}) = \frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^3p}{(2\pi)^3}\log(\omega_n^2 + \omega^2)\,, \tag{6.5}$$

where $\omega_n$ are the bosonic Matsubara frequencies and

$$\omega^2 = \vec{p}^{\,2} + m^2(\phi_{\rm c}) \tag{6.6}$$

$m^2(\phi_{\rm c})$ being defined in

$$m^2(\phi_{\rm c}) = m^2 + \frac{1}{2}\lambda\phi_{\rm c}^2 = \frac{d^2V_0(\phi_{\rm c})}{d\phi_{\rm c}^2}\,. \tag{6.7}$$

The sum over $n$ in (6.5) diverges, but the infinite part does not depend on $\phi_{\rm c}$. The finite part, which contains the $\phi_{\rm c}$ dependence, can be computed by the following method [1]. Let us define,

$$v(\omega) = \sum_{n=-\infty}^{\infty}\log(\omega_n^2 + \omega^2) \tag{6.8}$$

then,

$$\frac{\partial v}{\partial \omega} = \sum_{n=-\infty}^{\infty} \frac{2\omega}{\omega_n^2 + \omega^2} \,. \tag{6.9}$$

Using now the identity,

$$\begin{aligned} f(y) = \sum_{n=1}^{\infty} \frac{y}{y^2+n^2} &= -\frac{1}{2y} + \frac{1}{2}\pi \coth \pi y \\ &= -\frac{1}{2y} + \frac{\pi}{2} + \pi \frac{e^{-2\pi y}}{1-e^{-2\pi y}} \end{aligned} \tag{6.10}$$

with $y = \beta\omega/2\pi$ we obtain,

$$\frac{\partial v}{\partial \omega} = 2\beta \left[ \frac{1}{2} + \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \right] \tag{6.11}$$

and

$$v(\omega) = 2\beta \left[ \frac{w}{2} + \frac{1}{\beta} \log\left(1-e^{-\beta\omega}\right) \right] + \omega - \text{independent terms} \,. \tag{6.12}$$

Substituting finally (6.12) into (6.5) one gets,

$$V_1^\beta(\phi_c) = \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\omega}{2} + \frac{1}{\beta} \log\left(1-e^{-\beta\omega}\right) \right] \,. \tag{6.13}$$

One can easily prove that the first integral in (6.13) is the one-loop effective potential at zero temperature. For that we have to prove the identity,

$$-\frac{i}{2} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \log(-x^2 + \omega^2 - i\epsilon) \;=\; \frac{\omega}{2} + \text{const} \tag{6.14}$$

*i.e.*

$$\omega \int_{-\infty}^{\infty} \frac{dx}{2\pi i} \frac{1}{-x^2+\omega^2-i\epsilon} \;=\; \frac{1}{2} \,. \tag{6.15}$$

The integral (6.15) can be performed closing the integration interval $(-\infty, \infty)$ in the complex $x$ plane along a contour going anticlockwise and picking the pole of the integrand at $x = -\sqrt{\omega^2 - i\epsilon}$ with a residue $1/2\omega$. Using the residues theorem Eq. (6.15) can be easily checked. Now we can use identity (6.14) to write the temperature independent part of (6.13) as

$$\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega = -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \log(-p_o^2 + \omega^2 - i\epsilon) \tag{6.16}$$

and, after making the Wick rotation $p^0 = ip_E$ in (6.16) we obtain,

$$\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\log[p^2 + m^2(\phi_c)] \tag{6.17}$$

which is the same result we obtained in zero temperature field theory.

Now the temperature dependent part in (6.13) can be easily written as,

$$\frac{1}{\beta}\int \frac{d^3p}{(2\pi)^3}\log\left(1 - e^{-\beta\omega}\right) = \frac{1}{2\pi^2\beta^4}J_\mathrm{B}[m^2(\phi_c)\beta^2]\,, \tag{6.18}$$

where the thermal bosonic function $J_\mathrm{B}$ is defined as,

$$J_\mathrm{B}[m^2\beta^2] = \int_0^\infty dx\ x^2 \log\left[1 - e^{-\sqrt{x^2+\beta^2 m^2}}\right]\,. \tag{6.19}$$

The integral (6.19) and therefore the thermal bosonic effective potential admits a high-temperature expansion which will be very useful for practical applications. It is given by

$$\begin{aligned} J_\mathrm{B}(m^2/T^2) \ =\ & -\frac{\pi^4}{45} + \frac{\pi^2}{12}\frac{m^2}{T^2} - \frac{\pi}{6}\left(\frac{m^2}{T^2}\right)^{3/2} - \frac{1}{32}\frac{m^4}{T^4}\log\frac{m^2}{a_b T^2} \\ & -2\pi^{7/2}\sum_{\ell=1}^{\infty}(-1)^\ell \frac{\zeta(2\ell+1)}{(\ell+1)!}\Gamma\left(\ell+\frac{1}{2}\right)\left(\frac{m^2}{4\pi^2 T^2}\right)^{\ell+2}, \end{aligned} \tag{6.20}$$

where $a_b = 16\pi^2\exp(3/2 - 2\gamma_E)$ ($\log a_b = 5.4076$) and $\zeta$ is the Riemann $\zeta$-function.

There is a very simple way of computing the effective potential: it consists in *computing its derivative* **in the shifted theory** *and then integrating*! In fact the derivative of the effective potential

$$\frac{dV_1^\beta}{d\phi_c}$$

is described diagrammatically by the tadpole diagram. Using the Feynman rules in (4.8) one can easily write for the tadpole the expression,

$$\frac{dV_1^\beta}{d\phi_c} = \frac{\lambda\phi_c}{2}\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_n^2+\omega^2} \tag{6.21}$$

or, using the expression (6.7) for $m^2(\phi_c)$,

$$\frac{dV_1^\beta}{dm^2(\phi_c)} = \frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_n^2+\omega^2}\,. \tag{6.22}$$

Now we can perform the infinite sum in (6.22) using the result in Eq. (4.10) with a function $f$ defined as,

$$f(z) = \frac{1}{\omega^2 - z^2} \tag{6.23}$$

and obtain for the tadpole (6.22) the result

$$\frac{dV_1^\beta}{dm^2(\phi_c)} = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{1}{\omega^2 - z^2} + \int_C \frac{dz}{2\pi i} \frac{1}{e^{\beta z} - 1} \frac{1}{\omega^2 - z^2} \right\}. \tag{6.24}$$

The first term in (6.24) gives the $\beta$-independent part of the tadpole contribution as,

$$\frac{1}{2} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{1}{\omega^2 - z^2} . \tag{6.25}$$

We can now close the integration contour of (6.25) anticlockwise and pick the pole of (6.23) at $z = -\omega$ with a residue $1/2\omega$. The result of (6.25) is

$$\frac{1}{4\omega} . \tag{6.26}$$

The second term in (6.24) gives the $\beta$-dependent part of the tadpole contribution. Here the integration contour encircles the pole at $z = \omega$ with a residue

$$-\frac{1}{2\omega} \frac{1}{e^{\beta\omega} - 1} . \tag{6.27}$$

Adding (6.26) and (6.27) we obtain for the tadpole the final expression,

$$\frac{dV_1(\phi_c)}{dm^2(\phi_c)} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{1}{2\omega} + \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1} \right] . \tag{6.28}$$

Now, integration of (6.28) with respect to $m^2(\phi_c)$ leads to the expression (6.13) for the thermal effective potential and, therefore, to the final expression given by (6.17) and (6.18).

#### 6.1.2. Real time formalism

As we will see in this section, the final result for the effective potential (6.13) can be also obtained using the real time formalism. Let us compute the tadpole diagram. Since physical legs must be attached to type 1 vertices, the vertex in the tadpole must be considered of type 1, and the propagator circulating around the loop has to be considered as a (11) propagator.

Application of the Feynman rules (5.4) to the tadpole diagram leads to the expression[2]

$$\frac{dV_1^\beta}{d\phi_c} = \frac{\lambda\phi_c}{2}\int\frac{d^4p}{(2\pi)^4}\left[\frac{i}{p^2-m^2(\phi_c)+i\epsilon}+2\pi n_B(\omega)\delta(p^2-m^2(\phi_c))\right] \tag{6.29}$$

or, using as before the expression (6.7) for $m^2(\phi_c)$,

$$\frac{dV_1^\beta}{dm^2(\phi_c)} = \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\left[\frac{-i}{-p^2+m^2(\phi_c)-i\epsilon}+2\pi n_B(\omega)\delta(p^2-m^2(\phi_c))\right]. \tag{6.30}$$

Now the $\beta$-independent part of (6.30), after integration on $m^2(\phi_c)$ contributes to the effective potential as

$$-\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\log(-p^2+m^2(\phi_c)-i\epsilon)\,. \tag{6.31}$$

Finally using Eq. (6.14) to perform the $p^0$ integral, we can cast Eq. (6.31) as

$$\int\frac{d^3p}{(2\pi)^3}\frac{\omega}{2} \tag{6.32}$$

which coincides with the first term in (6.13).

Integration over $p^0$ in the $\beta$-dependent part of (6.30) can be easily performed with the help of the identity

$$\delta(p^2-m^2) = \frac{1}{2\omega_p}\left[\delta(p^0+\omega_p)+\delta(p^0-\omega_p)\right] \tag{6.33}$$

leading to,

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}n_B(\omega) \tag{6.34}$$

which, upon integration over $m^2(\phi_c)$ leads to the second term of Eq. (6.13).

We have thus checked trivially that the real time and imaginary time formalisms lead to the same expression of the thermal effective potential, in the one loop approximation.

[2] We are replacing in (5.4) the value of $\omega_p$ by the corresponding value $\omega$ given by (6.6) in the shifted theory.

### *6.2. Fermion fields*

We will consider here a theory with fermion fields described by the Lagrangian

$$\mathcal{L} = i\overline{\psi}_a \gamma \cdot \partial \psi^a - \overline{\psi}_a (M_f)^a_b \psi^b \,. \tag{6.35}$$

As in the scalar case, we have to compute the diagrams contained in Fig. 3 using the Feynman rules either for the imaginary or for the real time formalism, and decompose the thermal effective potential as in (6.3).

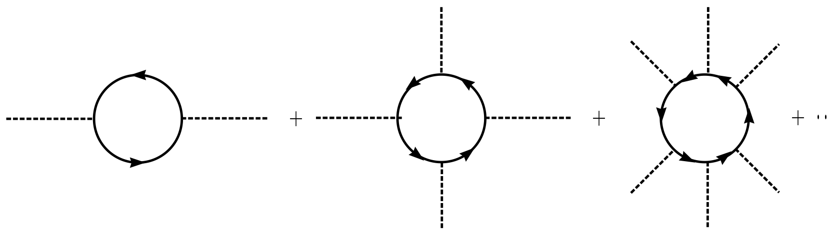

Fig. 3. 1PI diagrams contributing to the one-loop effective potential of (6.35).

#### **6.2.1. Imaginary time formalism**

The calculation of the diagrams in Fig. 3, using the Feynman rules (4.8), yields,

$$V_1^\beta(\phi_{\rm c}) = -\frac{2\lambda}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \log(\omega_n^2 + \omega^2) \,, \tag{6.36}$$

where $\omega_n$ are the fermionic Matsubara frequencies and

$$\omega^2 = \vec{p}^{\,2} + M_f^2 \,. \tag{6.37}$$

The sum over $n$ is done with the help of the same trick employed in (6.8)–(6.12). Let $f(y)$ be given by (6.10), then,

$$\begin{aligned} \sum_{m=2,4,\ldots} \frac{y}{y^2+m^2} &= \sum_{n=1}^{\infty} \frac{y}{y^2+4n^2} = \frac{1}{2} f\left(\frac{y}{2}\right), \\ \sum_{m=1,3,\ldots} \frac{y}{y^2+m^2} &= f(y) - \frac{1}{2} f\left(\frac{y}{2}\right) \end{aligned} \tag{6.38}$$

and using (6.10) we get,

$$\sum_{m=1,3,\ldots} \frac{y}{y^2+m^2} = \frac{\pi}{4} - \frac{\pi}{2} \frac{1}{e^{\pi y}+1} \,. \tag{6.39}$$

The function $v(\omega)$ in this case can be written as,

$$v(\omega) = 2 \sum_{n=1,3,\ldots} \log\left[\frac{\pi^2 n^2}{\beta^2} + \omega^2\right] \tag{6.40}$$

and its derivative,

$$\frac{\partial v}{\partial \omega} = \frac{4\beta}{\pi} \sum_{1,3,\ldots} \frac{y}{y^2+n^2}\,, \tag{6.41}$$

where $y = \beta\omega/\pi$. Then using (6.39) we get

$$\frac{\partial v}{\partial \omega} = 2\beta\left[\frac{1}{2} - \frac{1}{1+e^{\beta\omega}}\right] \tag{6.42}$$

and, after integration with respect to $\omega$,

$$v(\omega) = 2\beta\left[\frac{w}{2} + \frac{1}{\beta}\log\left(1+e^{-\beta\omega}\right)\right] + \omega - \text{independent terms}\,. \tag{6.43}$$

Replacing finally (6.43) into (6.36) one gets,

$$V_1^\beta(\phi_c) = -2\lambda \int \frac{d^3p}{(2\pi)^3}\left[\frac{\omega}{2} + \frac{1}{\beta}\log\left(1+e^{-\beta\omega}\right)\right]. \tag{6.44}$$

The first integral in (6.44) can be proven, as in (6.14)–(6.17), to lead to the one-loop effective potential at zero temperature

$$V_1 = -2\lambda\frac{1}{2}\mathrm{Tr}\int \frac{d^4p}{(2\pi)^4}\log\left[p^2 + M_f^2(\phi_c)\right]. \tag{6.45}$$

The second integral, which contains all the temperature dependent part, can be written as,

$$-2\lambda\frac{1}{\beta}\int \frac{d^3p}{(2\pi)^3}\log\left(1+e^{-\beta\omega}\right) = -2\lambda\frac{1}{2\pi^2\beta^4}J_F[M_f^2(\phi_c)\beta^2]\,, \tag{6.46}$$

where the thermal fermionic function $J_F$ is defined as,

$$J_F[m^2\beta^2] = \int_0^\infty dx\, x^2 \log\left[1 + e^{-\sqrt{x^2+\beta^2 m^2}}\right]. \tag{6.47}$$

As in the case of scalar fields, the integral (6.47) and therefore the thermal fermionic effective potential admits a high-temperature expansion which will be very useful for practical applications. It is given by

$$J_{\rm F}\left(\frac{m^2}{T^2}\right) = \frac{7\pi^4}{360} - \frac{\pi^2}{24}\frac{m^2}{T^2} - \frac{1}{32}\frac{m^4}{T^4}\log\frac{m^2}{a_f T^2}$$
$$-\frac{\pi^{7/2}}{4}\sum_{\ell=1}^{\infty}(-1)^{\ell}\frac{\zeta(2\ell+1)}{(\ell+1)!}\left(1-2^{-2\ell-1}\right)\Gamma\left(\ell+\frac{1}{2}\right)\left(\frac{m^2}{\pi^2 T^2}\right)^{\ell+2}, \quad (6.48)$$

where $a_f = \pi^2\exp(3/2 - 2\gamma_E)$ ($\log a_f = 2.6351$) and $\zeta$ is the Riemann $\zeta$-function.

As we did in the case of the scalar field, there is a very simple way of obtaining the effective potential, computing the tadpole in the shifted theory, and integrating over $\phi_{\rm c}$. Using for the fermion propagator (4.8)

$$i\frac{\gamma\cdot p + M_f}{p^2 - M_f^2}$$

and the trace formula,

$$\mathrm{Tr}\ (\gamma\cdot p + M_f) = 2\lambda M_f$$

we can write for the tadpole the expression,

$$\frac{dV_1^\beta}{d\phi_{\rm c}} = -2\lambda\Gamma M_f\frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_n^2+\omega^2} \quad (6.49)$$

or, using the expression $M_f(\phi_{\rm c}) = \Gamma\phi_{\rm c}$, where $\Gamma$ is the Yukawa coupling,

$$\frac{dV_1^\beta}{dM_f^2(\phi_{\rm c})} = -2\lambda\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_n^2+\omega^2}\,. \quad (6.50)$$

Now the infinite sum in (6.50) can be done with the help of (4.11), with $f(z)$ given by (6.23), as

$$\frac{dV_1^\beta}{dM_f^2(\phi_{\rm c})} = -2\lambda\int\frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2}\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{1}{\omega^2-z^2} - \int_C\frac{dz}{2\pi i}\frac{1}{e^{\beta z}+1}\frac{1}{\omega^2-z^2}\right\}. \quad (6.51)$$

The first term of (6.51) reproduces the zero temperature result (6.45), after $M_f^2$ integration, by closing the integration contour of (6.25) anticlockwise and picking the pole at $z=-\omega$ with a residue $1/2\omega$. The second term in (6.51) gives the $\beta$-dependent part of the tadpole contribution. Here the integration contour $C$ encircles the pole at $z = \omega$ with a residue

$$(-2\lambda)\frac{1}{2\omega}\,\frac{1}{e^{\beta\omega}+1}\,. \tag{6.52}$$

Adding all of them together, we obtain for the tadpole the final expression

$$\frac{dV_1(\phi_c)}{dM_f^2(\phi_c)} = -\lambda\int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{2\omega}-\frac{1}{\omega}\frac{1}{e^{\beta\omega}+1}\right] \tag{6.53}$$

and, upon integration with respect to $M_f^2$ we obtain the result previously presented in Eq. (6.44).

**6.2.2. Real time formalism**

As for the case of scalar fields, the thermal effective potential for fermions (6.44) can also be very easily obtained using the real time formalism. We compute again the tadpole diagram, where the vertex between the two fermions and the scalar is of type 1 and the fermion propagator circulating along the loop is a (11) propagator. Application of the Feynman rules (5.8) leads to the expression

$$\frac{dV_1^\beta}{d\phi_c} = -\mathit{\Gamma}\,\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}(\gamma\cdot p+M_f)\left[\frac{i}{p^2-M_f^2+i\epsilon}-2\pi n_F(\omega)\delta(p^2-M_f^2)\right] \tag{6.54}$$

or, using as before the expression for $M_f^2$,

$$\frac{dV_1^\beta}{dM_f^2(\phi_c)} = -\frac{\mathrm{Tr}\,\mathbf{1}}{2}\int\frac{d^4p}{(2\pi)^4}\left[\frac{-i}{-p^2+M_f^2-i\epsilon}-2\pi n_F(\omega)\delta(p^2-M_f^2)\right]. \tag{6.55}$$

Now the $\beta$-independent part of (6.55), after integration on $M_f^2$, contributes to the effective potential,

$$-\mathrm{Tr}\,\mathbf{1}\int\frac{d^3p}{(2\pi)^3}\frac{\omega}{2} \tag{6.56}$$

which coincides with the first term in (6.44).

Integration over $p^0$ in the $\beta$-dependent part of (6.55) can be easily performed with the help of the identity (6.33) leading to,

$$-\mathrm{Tr}\,\mathbf{1}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}[-n_F(\omega)] \tag{6.57}$$

which, upon integration over $M_f^2$ leads to the second term of Eq. (6.44).

### *6.3. Gauge bosons*

The thermal effective potential for gauge bosons in a theory described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\,(D_\mu\phi_a)^\dagger D^\mu\phi^a + \cdots \tag{6.58}$$

is computed in the same way as for previous fields. The simplest thing is to compute the tadpole diagram using the shifted mass for the gauge boson. In the Landau gauge, the gauge boson propagator reads as,

$$\Pi^\mu{}_\nu(p)^{(\alpha\beta)} = \Delta^\mu{}_\nu G^{(\alpha\beta)}(p)\,, \tag{6.59}$$

where $\Delta$ is the projector defined in

$$\Delta^\mu{}_\nu = g^\mu{}_\nu - \frac{p^\mu p_\nu}{p^2} \tag{6.60}$$

with a trace equal to

$$\mathrm{Tr}\,(\Delta) = 3\,. \tag{6.61}$$

Therefore the final expression for the thermal effective potential is computed as,

$$V_1^\beta(\phi_\mathrm{c}) = \mathrm{Tr}\,(\Delta)\left\{\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\log[p^2 + M_{gb}^2(\phi_\mathrm{c})] + \frac{1}{2\pi^2\beta^4}J_\mathrm{B}[M_{gb}^2(\phi_\mathrm{c})\beta^2]\right\}, \tag{6.62}$$

where the thermal bosonic function $J_\mathrm{B}$ in (6.19). The first term of (6.62) agrees with the zero temperature effective potential

$$V_1 = \mathrm{Tr}\,(\Delta)\frac{1}{2}\mathrm{Tr}\int\frac{d^4p}{(2\pi)^4}\log\left[p^2 + (M_{gb})^2(\phi_\mathrm{c})\right] \tag{6.63}$$

and the second one just counts that of a scalar field theory a number of times equal to the number of degrees of freedom (3) of the gauge boson.

## 7. The Standard Model

In this subsection we will apply the above ideas to compute the one loop effective potential for the Standard Model of electroweak interactions. The spin-zero fields of the Standard Model are described by the SU(2) doublet,

$$\Phi = \begin{pmatrix} \chi_1 + i\chi_2 \\ \dfrac{\phi_\mathrm{c} + h + i\chi_3}{\sqrt{2}} \end{pmatrix}, \tag{7.1}$$

where $\phi_{\rm c}$ is the real constant background, $h$ the Higgs field, and $\chi_a$ ($a$=1,2,3) are the three Goldstone bosons. The tree level potential reads, in terms of the background field, as

$$V_0(\phi_{\rm c}) = -\frac{m^2}{2}\phi_{\rm c}^2 + \frac{\lambda}{4}\phi_{\rm c}^4 \tag{7.2}$$

with positive $\lambda$ and $m^2$, and the tree level minimum corresponding to

$$v^2 = \frac{m^2}{\lambda}\,.$$

The spin-zero field dependent masses are

$$\begin{aligned} m_h^2(\phi_{\rm c}) \;&=\; 3\lambda\phi_{\rm c}^2 - m^2\,, \\ m_\chi^2(\phi_{\rm c}) \;&=\; \lambda\phi_{\rm c}^2 - m^2\,, \end{aligned} \tag{7.3}$$

so that $m_h^2(v) = 2\lambda v^2 = 2m^2$ and $m_\chi^2(v) = 0$. The gauge bosons contributing to the one-loop effective potential are $W^\pm$ and $Z$, with tree level field dependent masses,

$$\begin{aligned} m_W^2(\phi_{\rm c}) \;&=\; \frac{g^2}{4}\phi_{\rm c}^2\,, \\ m_Z^2(\phi_{\rm c}) \;&=\; \frac{g^2+g'^2}{4}\phi_{\rm c}^2\,. \end{aligned} \tag{7.4}$$

Finally, the only fermion which can give a significant contribution to the one loop effective potential is the top quark, with a field-dependent mass

$$m_t^2(\phi_{\rm c}) = \frac{h_t^2}{2}\phi_{\rm c}^2\,, \tag{7.5}$$

where $h_t$ is the top quark Yukawa coupling.

The Standard Model one-loop effective potential at zero temperature can be computed using various renormalization schemes and the contribution of gauge and Higgs bosons and the top quark fermion to radiative corrections. Here we will compute the corresponding one-loop effective potential at finite temperature which contains, as we already observed, the zero-temperature potential as its first term. We will use the renormalization scheme provided by imposing that the minimum, at $v = 246.22$ GeV, and the Higgs mass do not change with respect to their tree level values, *i.e.*,

$$\begin{aligned} \left.\frac{d(V_1+V_1^{c.t.})}{d\phi_{\rm c}}\right|_{\phi_{\rm c}=v} \;&=\; 0\,, \\ \left.\frac{d^2(V_1+V_1^{c.t.})}{d\phi_{\rm c}^2}\right|_{\phi_{\rm c}=v} \;&=\; 0\,. \end{aligned} \tag{7.6}$$

We will further consider only the contribution of $W$ and $Z$ bosons, and the top quark to radiative corrections. This is expected to be a good enough approximation for Higgs masses lighter than the $W$ mass.

Using Eqs. (6.46) and (6.62) one can easily see that the finite-temperature part of the one-loop effective potential can be written as,

$$\Delta V^{(1)}(\phi_{\rm c},T)=\frac{T^4}{2\pi^2}\left[\sum_{i=W,Z}n_iJ_{\rm B}[m_i^2(\phi_{\rm c})/T^2]+n_tJ_{\rm F}[m_t^2(\phi_{\rm c})/T^2]\right]\,, \quad (7.7)$$

where the functions $J_{\rm B}$ and $J_{\rm F}$ were defined in Eqs. (6.19) and (6.47), respectively, and $n_W=2,\ n_Z=6,\ n_t=-12$.

Using now the high temperature expansions (6.20) and (6.48), and the one loop effective potential at zero temperature, one can write the total potential as,

$$V(\phi_{\rm c},T)=D(T^2-T_0^2)\phi_{\rm c}^2-ET\phi_{\rm c}^3+\frac{\lambda(T)}{4}\phi_{\rm c}^4\,, \quad (7.8)$$

where the coefficients are given by

$$D = \frac{2m_W^2+m_Z^2+2m_t^2}{8v^2}\,, \quad (7.9)$$

$$E = \frac{2m_W^3+m_Z^3}{4\pi v^3}\,, \quad (7.10)$$

$$T_0^2 = \frac{m_h^2-8Bv^2}{4D}\,, \quad (7.11)$$

$$B = \frac{3}{64\pi^2v^4}\left(2m_W^4+m_Z^4-4m_t^4\right)\,, \quad (7.12)$$

$$\lambda(T) = \lambda-\frac{3}{16\pi^2v^4}\left(2m_W^4\log\frac{m_W^2}{A_{\rm B}T^2}+m_Z^4\log\frac{m_Z^2}{A_{\rm B}T^2}-4m_t^4\log\frac{m_t^2}{A_{\rm F}T^2}\right)\,, \quad (7.13)$$

where $\log A_{\rm B}=\log a_b-3/2$ and $\log A_{\rm F}=\log a_{\rm F}-3/2$, and $a_{\rm B}$, $a_{\rm F}$ are given in (6.20) and (6.48). All the masses which appear in the definition of the coefficients, Eqs. (7.9) to (7.13), are the physical masses, *i.e.* the masses at the zero temperature minimum. The peculiar form of the potential, as given by Eq. (7.8), will be useful to study the associated phase transition, as we will see in subsequent sections.

## 8. Cosmological phase transitions

All cosmological applications of field theories are based on the theory of phase transitions at finite temperature, that we will briefly describe throughout this section. The main point here is that at finite temperature, the equilibrium value of the scalar field $\phi$, $\langle\phi(T)\rangle$, does not correspond to the minimum of the effective potential $V_{\rm eff}^{T=0}(\phi)$, but to the minimum of the finite temperature effective potential $V_{\rm eff}^{\beta}(\phi)$. Thus, even if the minimum of $V_{\rm eff}^{T=0}(\phi)$ occurs at $\langle\phi\rangle = \sigma \neq 0$, very often, for sufficiently large temperatures, the minimum of $V_{\rm eff}^{\beta}(\phi)$ occurs at $\langle\phi(T)\rangle = 0$: this phenomenon is known as **symmetry restoration** at high temperature, and gives rise to the phase transition from $\phi(T) = 0$ to $\phi = \sigma$. It was discovered by Kirzhnits [13] in the context of the electroweak theory (symmetry breaking between weak and electromagnetic interactions occurs when the Universe cools down to a critical temperature $T_{\rm c} \sim 10^2$ GeV) and subsequently confirmed and developed by other authors [1, 2, 14, 15].

The cosmological scenario can be drawn as follows: In the theory of the hot big bang, the Universe is initially at very high temperature and, depending on the function $V_{\rm eff}^{\beta}(\phi)$, it can be in the **symmetric phase** $\langle\phi(T)\rangle = 0$, *i.e.* $\phi = 0$ can be the stable absolute minimum. At some critical temperature $T_{\rm c}$ the minimum at $\phi = 0$ becomes metastable and the phase transition may proceed. The phase transition may be **first** or **second** order. First-order phase transitions have supercooled (out of equilibrium) symmetric states when the temperature decreases and are of use for baryogenesis purposes. Second-order phase transitions are used in the so-called new inflationary models [16]. We will illustrate these kinds of phase transitions with very simple examples.

### *8.1. First and second order phase transitions*

We will illustrate the difference between **first** and **second** order phase transitions by considering first the simple example of a potential[3] described by the function,

$$V(\phi, T) = D(T^2 - T_0^2)\phi^2 + \frac{\lambda(T)}{4}\phi^4\,, \tag{8.1}$$

where $D$ and $T_0^2$ are constant terms and $\lambda$ is a slowly varying function of $T$[4]. A quick glance at (6.20) and (6.48) shows that the potential (8.1) can be part of the one-loop finite temperature effective potential in field theories.

At zero temperature, the potential has a negative mass-squared term, which indicates that the state $\phi = 0$ is unstable, and the energetically favored state corresponds to the minimum at $\phi(0) = \pm\sqrt{\frac{2D}{\lambda}}T_0$, where the symmetry $\phi \leftrightarrow -\phi$ of the original theory is spontaneously broken.

[3] The $\phi$ independent terms in (8.1), *i.e.* $V(0, T)$, are not explicitly considered.
[4] The $T$ dependence of $\lambda$ will often be neglected in this section.

The curvature of the finite temperature potential (8.1) is now $T$-dependent,

$$m^2(\phi, T) = 3\lambda\phi^2 + 2D(T^2 - T_0^2) \tag{8.2}$$

and its stationary points, *i.e.* solutions to $dV(\phi, T)/d\phi = 0$, given by,

$$\phi(T) \;=\; 0$$

and

$$\phi(T) \;=\; \sqrt{\frac{2D(T_0^2 - T^2)}{\lambda(T)}}\,. \tag{8.3}$$

Therefore the critical temperature is given by $T_0$. At $T > T_0$, $m^2(0, T) > 0$ and the origin $\phi = 0$ is a minimum. At the same time only the solution $\phi = 0$ in (8.3) does exist. At $T = T_0$, $m^2(0, T_0) = 0$ and both solutions in (8.3) collapse at $\phi = 0$. The potential (8.1) becomes,

$$V(\phi, T_0) = \frac{\lambda(T_0)}{4}\phi^4\,. \tag{8.4}$$

At $T < T_0$, $m^2(0, T) < 0$ and the origin becomes a maximum. Simultaneously, the solution $\phi(T) \neq 0$ does appear in (8.3). This phase transition is called of **second order**, because there is no barrier between the symmetric and broken phases. Actually, when the broken phase is formed, the origin (symmetric phase) becomes a maximum. The phase transition may be achieved by a thermal fluctuation for a field located at the origin.

However, in many interesting theories there is a **barrier** between the symmetric and broken phases. This is characteristic of **first order** phase transitions. A typical example is provided by the potential[5],

$$V(\phi, T) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{\lambda(T)}{4}\phi^4\,, \tag{8.5}$$

where, as before, $D$, $T_0$ and $E$ are $T$ independent coefficients, and $\lambda$ is a slowly varying $T$-dependent function. Notice that the difference between (8.5) and (8.1) is the cubic term with coefficient $E$. This term can be provided by the contribution to the effective potential of bosonic fields (6.20). The behavior of (8.5) for the different temperatures is reviewed in Refs. [17, 18]. At $T > T_1$ the only minimum is at $\phi = 0$. At $T = T_1$

$$T_1^2 = \frac{8\lambda(T_1)DT_0^2}{8\lambda(T_1)D - 9E^2} \tag{8.6}$$

[5] See, *e.g.* the one-loop effective potential for the Standard Model, Eq. (7.8).

a local minimum at $\phi(T) \neq 0$ appears as an inflection point. The value of the field $\phi$ at $T = T_1$ is,

$$\langle \phi(T_1) \rangle = \frac{3ET_1}{2\lambda(T_1)} \,. \tag{8.7}$$

A barrier between the latter and the minimum at $\phi = 0$ starts to develop at lower temperatures. Then the point (8.7) splits into a maximum

$$\phi_M(T) = \frac{3ET}{2\lambda(T)} - \frac{1}{2\lambda(T)} \sqrt{9E^2T^2 - 8\lambda(T)D(T^2 - T_0^2)} \tag{8.8}$$

and a local minimum

$$\phi_m(T) = \frac{3ET}{2\lambda(T)} + \frac{1}{2\lambda(T)} \sqrt{9E^2T^2 - 8\lambda(T)D(T^2 - T_0^2)} \,. \tag{8.9}$$

At a given temperature $T = T_c$

$$T_c^2 = \frac{\lambda(T_c)DT_0^2}{\lambda(T_c)D - E^2} \tag{8.10}$$

the origin and the minimum (8.9) become degenerate. From (8.8) and (8.9) we find that

$$\phi_M(T_c) = \frac{ET_c}{\lambda(T_c)} \tag{8.11}$$

and

$$\phi_m(T_c) = \frac{2ET_c}{\lambda(T_c)} \,. \tag{8.12}$$

For $T < T_c$ the minimum at $\phi = 0$ becomes metastable and the minimum at $\phi_m(T) \neq 0$ becomes the global one. At $T = T_0$ the barrier disappears, the origin becomes a maximum

$$\phi_M(T_0) = 0 \tag{8.13}$$

and the second minimum becomes equal to

$$\phi_m(T_0) = \frac{3ET_0}{\lambda(T_0)} \,. \tag{8.14}$$

The phase transition starts at $T = T_c$ by tunneling. However, if the barrier is high enough the tunneling effect is very small and the phase transition does effectively start at a temperature $T_c > T_t > T_0$. In some models $T_0$ can be equal to zero. The details of the phase transition depend therefore on the process of tunneling from the false to the global minimum. These details will be studied in the rest of this section.

### *8.2. Thermal tunneling*

The transition from the false to the true vacuum proceeds via **thermal tunneling** at finite temperature. It can be understood in terms of formation of bubbles of the broken phase in the sea of the symmetric phase. Once this has happened, the bubble spreads throughout the Universe converting false vacuum into true one.

The tunneling rate [19–21] is computed by using the rules of field theory at finite temperature [22]. In the previous section we defined the critical temperature $T_c$ as the temperature at which the two minima of the potential $V(\phi, T)$ have the same depth. However, tunneling with formation of bubbles of the field $\phi$ corresponding to the second minimum starts somewhat later, and goes sufficiently fast to fill the Universe with bubbles of the new phase only at some lower temperature $T_t$ when the corresponding euclidean action $S_{\rm E} = S_3/T$ suppressing the tunneling becomes $\mathcal{O}(130–140)$ [17,23,24], as we will see in the next section.

We will use as prototype the potential of Eq. (8.5) which can trigger, as we showed in this section, a first order phase transition. In this case the false minimum is $\phi = 0$, and the value of the potential at the origin is zero, $V(0, T) = 0$. The tunneling probability per unit time per unit volume is given by [22]

$$\frac{\Gamma}{\nu} \sim A(T)e^{-S_3/T} . \tag{8.15}$$

In (8.15) the prefactor $A(T)$ is roughly of $\mathcal{O}(T^4)$ while $S_3$ is the three-dimensional euclidean action defined as

$$S_3 = \int d^3x \left[\frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + V(\phi, T)\right] . \tag{8.16}$$

At very high temperature the **bounce** solution has O(3) symmetry [22] and the euclidean action is then simplified to,

$$S_3 = 4\pi \int_0^\infty r^2 dr \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + V(\phi(r), T)\right] , \tag{8.17}$$

where $r^2 = \vec{x}^{\,2}$, and the euclidean equation of motion yields,

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = V'(\phi, T) \tag{8.18}$$

with the boundary conditions

$$\lim_{r\to\infty} \phi(r) \;=\; 0 \, , \tag{8.19}$$

$$\left.\frac{d\phi}{dr}\right|_{r=0} \;=\; 0 \, . \tag{8.20}$$

From here on we will follow the discussion in Ref. [17]. Let us take $\phi = 0$ outside a bubble. Then (8.17), which is also the surplus free energy of a true vacuum bubble, can be written as

$$S_3 = 4\pi \int_0^R r^2 dr \left[ \frac{1}{2} \left( \frac{d\phi}{dr} \right)^2 + V(\phi(r), T) \right] , \tag{8.21}$$

where $R$ is the bubble radius. There are two contributions to (8.21): a surface term $F_{\rm S}$, coming from the derivative term in (8.21), and a volume term $F_{\rm V}$, coming from the second term in (8.21). They scale like,

$$S_3 \sim 2\pi R^2 \left( \frac{\delta\phi}{\delta R} \right)^2 \delta R + \frac{4\pi R^3 \langle V \rangle}{3} , \tag{8.22}$$

where $\delta R$ is the thickness of the bubble wall, $\delta\phi = \phi_m$ and $\langle V \rangle$ is the average of the potential inside the bubble.

For temperatures just below $T_{\rm c}$, the height of the barrier $V(\phi_M, T)$ is large compared to the depth of the potential at the minimum, $-V(\phi_m, T)$. In that case, the solution of minimal action corresponds to minimizing the contribution to $F_V$ coming from the region $\phi = \phi_M$. This amounts to a very small bubble wall $\delta R/R \ll 1$ and so a very quick change of the field from $\phi = 0$ outside the bubble to $\phi = \phi_m$ inside the bubble. Therefore, the first formed bubbles after $T_{\rm c}$ are **thin wall** bubbles.

Subsequently, when the temperature drops towards $T_0$ the height of the barrier $V(\phi_M, T)$ becomes small as compared with the depth of the potential at the minimum $-V(\phi_m, T)$. In that case the contribution to $F_{\rm V}$ from the region $\phi = \phi_M$ is negligible, and the minimal action corresponds to minimizing the surface term $F_{\rm S}$. This amounts to a configuration where $\delta R$ is as large as possible, *i.e.* $\delta R/R = \mathcal{O}(1)$: **thick wall** bubbles. So whether the phase transition proceeds through thin or thick wall bubbles depends on how large the bubble nucleation rate (8.15) is, or how small $S_3$ is, before thick bubbles are energetically favored.

For the potential (8.5) an analytic formula has been obtained in Ref. [18] without assuming the thin wall approximation. It is given by,

$$\frac{S_3}{T} = \frac{13.72}{E^2} \left[ D \left( 1 - \frac{T_0^2}{T^2} \right) \right]^{3/2} f \left[ \frac{\lambda(T) D}{E^2} \left( 1 - \frac{T_0^2}{T^2} \right) \right] , \tag{8.23}$$

$$f(x) = 1 + \frac{x}{4} \left[ 1 + \frac{2.4}{1-x} + \frac{0.26}{(1-x)^2} \right] . \tag{8.24}$$

The case of two fields is extremely more complicated. In particular the two-Higgs situation in the supersymmetric standard theory has been recently solved in Ref. [25]. The connection between zero temperature and finite temperature tunneling is manifest. In particular at temperatures much less than the inverse radius the ($T = 0$) O(4) solution has the least action. This can happen for theories with a supercooled symmetric phase: for instance in the presence of a barrier that does not disappear when the temperature drops to zero. At temperatures much larger than the inverse radius, the O(3) solution has the least action.

### *8.3. Bubble nucleation*

In the previous subsection we have established the free energy and the critical radius of a bubble large enough to grow after formation. The subsequent progress of the phase transition depends on the ratio of the rate of production of bubbles of true vacuum, as given by (8.15), over the expansion rate of the Universe. For example if the former remains always smaller than the latter, then the state will be trapped in the supercooled false vacuum. Otherwise the phase transition will start at some temperature $T_t$ by **bubble nucleation**. The probability of bubble formation per unit time per unit volume is given by (8.15) where $B(T) = S_3(T)/T$, $A(T) = \omega T^4$, where the parameter $\omega$ will be taken of $\mathcal{O}(1)$.

Since the progress of the phase transition should depend on the expansion rate of the Universe, we have to describe the Universe at temperatures close to the electroweak phase transition. A homogeneous and isotropic (flat) Universe is described by a Robertson–Walker metric which, in comoving coordinates, is given by $ds^2 = dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2\right)$, where $a(t)$ is the scale factor of the Universe. The Universe expansion is governed by the equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{\rm Pl}^2}\rho\,, \tag{8.25}$$

where $M_{\rm Pl}$ is the Planck mass, and $\rho$ is the energy density. For temperatures $T \sim 10^2$ GeV the Universe is radiation dominated, and its energy density is given by,

$$\rho = \frac{\pi^2}{30} g(T) T^4\,, \tag{8.26}$$

where $g(T) = g_{\rm B}(T) + \frac{7}{8} g_{\rm F}(T)$, and $g_{\rm B}(T)$ ($g_{\rm F}(T)$) is the effective number of bosonic (fermionic) degrees of freedom at the temperature $T$. For the standard model we have $g^{\rm SM} = 106.75$ which can be considered as temperature independent.

The equation of motion (8.25) can be solved, and assuming an adiabatic expansion of the Universe, $a(T_1)T_1 = a(T_2)T_2$, one obtains the following relationship,

$$t = \zeta \frac{M_{\rm Pl}}{T^2}\,, \tag{8.27}$$

where $\zeta = \frac{1}{4\pi}\sqrt{\frac{45}{\pi g}} \sim 3 \times 10^{-2}$. Using (8.27) the horizon volume is given by

$$V_H(t) = 8\zeta^3 \frac{M_{\rm Pl}^3}{T^6}\,. \tag{8.28}$$

The onset of nucleation happens at a temperature $T_t$ such that **the probability for a single bubble to be nucleated within one horizon volume is $\sim$ 1**, *i.e.* [25]

$$\int_{T_t}^{\infty} \frac{dT}{T}\left(\frac{2\zeta M_{\rm Pl}}{T}\right)^4 \exp\{-S_3(T)/T\} = \mathcal{O}(1) \tag{8.29}$$

which implies numerically,

$$B(T_t) \sim 137 + \log\frac{10^2 E^2}{\lambda D} + 4\log\frac{100\ \text{GeV}}{T_t}\,, \tag{8.30}$$

where we have normalized $T_t \sim 100$ GeV and $E^2/(\lambda D) \sim 10^{-2}$ which are typical values which will be obtained in the standard model of electroweak interactions.

## 9. Baryogenesis at phase transitions

There are two essential problems to be understood related with the baryon number of the Universe:

(*i*) There is no evidence of antimatter in the Universe. In fact, there is no antimatter in the solar system, and only $\overline{p}$ in cosmic rays. However antiprotons can be produced as secondaries in collisions ($pp \to 3p + \overline{p}$) at a rate similar to the observed one. Numerically, $\frac{n_{\overline{p}}}{n_p} \sim 3 \times 10^{-4}$, and $\frac{n_{^4\rm He}}{n_{^4\overline{\rm He}}} \sim 10^{-5}$. We can conclude that $n_B \gg n_{\overline{B}}$, so $n_{\Delta B} \equiv n_{\rm B} - n_{\overline{B}} \sim n_{\rm B}$.

(*ii*) The second problem is to understand the origin of

$$\eta \equiv \frac{n_{\rm B}}{n_\gamma} \sim (4.7 - 6.5) \times 10^{-10} \tag{9.1}$$

today. This parameter is essential for primordial nucleosynthesis [26]. $\eta$ may not have changed since nucleosynthesis. At these energy scales

($\sim 1$ MeV) baryon number is conserved if there are no processes which would have produced entropy to change the photon number. We can easily estimate from $\eta$ the baryon to entropy ratio by using

$$s = \frac{\pi^4}{45\zeta(3)} 3.91\ n_\gamma = 7.04\ n_\gamma \tag{9.2}$$

and the range (9.1).

In the standard cosmological model there is no explanation for the smallness of the ratio (9.1) if we start from $n_{\Delta B} = 0$. An initial asymmetry has to be imposed by hand as an initial condition (which violates any naturalness principle) or has to be dynamically generated at phase transitions, which is the way we will explore all along this section.

### *9.1. Conditions for baryogenesis*

As we have seen in the previous subsection the Universe was initially baryon symmetric ($n_{\rm B} \simeq n_{\overline{B}}$) although the matter–antimatter asymmetry appears to be large today ($n_{\Delta B} \simeq n_{\rm B} \gg n_{\overline{B}}$). In the standard cosmological model there is no explanation for the value of $\eta$ consistent with nucleosynthesis, Eq. (9.1), and it has to be imposed by hand as an initial condition. However, it was suggested by Sakharov long ago [27] that a tiny $n_{\Delta B}$ might have been produced in the early Universe leading, after $p\overline{p}$ annihilations, to (9.1). The three ingredients necessary for baryogenesis are:

#### 9.1.1. *B*-nonconserving interactions

This condition is obvious since we want to start with a baryon symmetric universe ($\Delta B = 0$) and evolve it to a universe where $\Delta B \neq 0$. $B$-nonconserving interactions might mediate proton decay; in that case the phenomenological constraints are provided by the proton lifetime measurements [28] $\tau_p \gtrsim 10^{31-33}$ yr.

#### 9.1.2. C and CP violation

The action of C (charge conjugation) and CP (combined action of charge conjugation and parity) interchanges particles with antiparticles, changing therefore the sign of $B$. For instance if we describe spin-$\frac{1}{2}$ fermions by two-component fields of definite chirality (left-handed fields $\psi_{\rm L}$ and right-handed fields $\psi_{\rm R}$) the action of C and CP over them is given by

$$\begin{aligned}
\mathrm{P}\ &:\ \psi_{\rm L} \longrightarrow \psi_{\rm R}, \quad \psi_{\rm R} \longrightarrow \psi_{\rm L}\,, \\
\mathrm{C}\ &:\ \psi_{\rm L} \longrightarrow \psi_{\rm L}^C \equiv \sigma_2 \psi_{\rm R}^*, \quad \psi_{\rm R} \longrightarrow \psi_{\rm R}^C \equiv -\sigma_2 \psi_{\rm L}^*\,, \\
\mathrm{CP}\ &:\ \psi_{\rm L} \longrightarrow \psi_{\rm R}^C, \quad \psi_{\rm R} \longrightarrow \psi_{\rm L}^C\,.
\end{aligned} \tag{9.3}$$

If the Universe is initially matter–antimatter symmetric, and without a preferred direction of time as in the standard cosmological model, it is represented by a C and CP invariant state, $|\phi_0\rangle$, with $B=0$. If C and CP were conserved, *i.e.* $[\mathrm{C},H]=[\mathrm{CP},H]=0$, $H$ being the Hamiltonian, then the state of the Universe at a later time $t$, $|\phi(t)\rangle = e^{iHt}|\phi_o\rangle$ would be C and CP invariant and, therefore, baryon number conserving, $\Delta B=0$. The only way to generate a net $\Delta B \neq 0$ is to have $C$ and $CP$ violating interactions.

#### 9.1.3. Departure from thermal equilibrium

If all particles in the Universe remained in thermal equilibrium, then no direction for time would be defined and CPT invariance would prevent the appearance of any baryon excess, rendering CP violating interactions irrelevant [29].

A particle species is in thermal equilibrium if all its reaction rates, $\Gamma$, are much faster than the expansion rate of the Universe, $H$. On the other hand a departure from thermal equilibrium is expected whenever a rate crucial for maintaining it is less than the expansion rate ($\Gamma < H$). Deviation from thermal equilibrium cannot occur in a homogeneous isotropic universe containing only massless species: massive species are needed in general for such deviations to occur.

### *9.2. Baryogenesis at the electroweak phase transitions*

It has been recently realized [30,31] that the three Sakharov's conditions for baryogenesis can be fulfilled at the electroweak phase transition:

- Baryonic charge non-conservation was discovered by 't Hooft [32]. In fact baryon and lepton number are conserved anomalous global symmetries in the Standard Model. They are violated by non-perturbative effects.

- CP violation can be generated in the Standard Model from phases in the fermion mass matrix, Cabibbo, Kobayashi, Maskawa (CKM) phases [33]. This effect is much too small to explain the observed baryon to entropy ratio. However, in extensions of the Standard Model as the minimal supersymmetric standard model (MSSM), a sizeable CP violation can happen through an extended Higgs sector.

- The out of equilibrium condition can be achieved, if the phase transition is strong enough first order, in the bubble walls. In that case the $B$-violating interactions are out of equilibrium in the bubble walls and a net $B$-number can be generated during the phase transition [31].

**9.2.1. Baryon and lepton number violation in the electroweak theory**

Violation of baryon and lepton number in the electroweak theory is a very striking phenomenon. Classically, baryonic and leptonic currents are conserved in the electroweak theory. However, that conservation is spoiled by quantum corrections through the chiral anomaly associated with triangle fermionic loop in external gauge fields. The calculation gives,

$$\partial_\mu j^\mu_{\mathrm{B}} = \partial_\mu j^\mu_L = N_f \left( \frac{g^2}{32\pi^2} W\widetilde{W} - \frac{g'^2}{32\pi^2} Y\widetilde{Y} \right), \tag{9.4}$$

where $N_f$ is the number of fermion generations, $W_{\mu\nu}$ and $Y_{\mu\nu}$ are the gauge field strength tensors for SU(2) and $\mathrm{U}(1)_Y$, respectively, and the tilde means the dual tensor.

A very important feature of (9.4) is that the difference $B - L$ is strictly conserved, and so only the sum $B + L$ is anomalous and can be violated. Another feature is that fluctuations of the gauge field strengths can lead to fluctuations of the corresponding value of $B + L$. The product of gauge field strengths on the right hand side of Eq. (9.4) can be written as four-divergences, $W\widetilde{W} = \partial_\mu k^\mu_W$, $Y\widetilde{Y} = \partial_\mu k^\mu_Y$, where

$$\begin{aligned} k^\mu_Y &= \epsilon^{\mu\nu\alpha\beta} Y_{\nu\alpha} Y_\beta \,, \\ k^\mu_W &= \epsilon^{\mu\nu\alpha\beta} \left( W^a_{\nu\alpha} W^a_\beta - \frac{g}{3} \epsilon_{abc} W^a_\nu W^b_\alpha W^c_\beta \right) \end{aligned} \tag{9.5}$$

and $W_\mu$, $Y_\mu$ are the gauge fields of SU(2) and $\mathrm{U}(1)_Y$, respectively. In general total derivatives are unobservable because they can be integrated by parts and drop from the integrals. This is true for the terms in the four-vectors (9.5) proportional to the field strengths $W_{\mu\nu}$ and $Y_{\mu\nu}$. This means that for the abelian subgroup $U(1)_Y$ the current non conservation induced by quantum effects becomes non observable. However this is not mandatory for gauge fields, for which the integral can be nonzero. Hence only for non-abelian groups can the current non conservation induced by quantum effects become observable. In particular one can write $\varDelta B = \varDelta L = N_f \varDelta N_{\mathrm{CS}}$, where $N_{\mathrm{CS}}$ is the so-called Chern–Simons number characterizing the topology of the gauge field configuration,

$$N_{\mathrm{CS}} = \frac{g^2}{32\pi^2} \int d^3x \epsilon^{ijk} \left( W^a_{ij} W^a_k - \frac{g}{3} \epsilon_{abc} W^a_i W^b_j W^c_k \right). \tag{9.6}$$

Note that though $N_{\mathrm{CS}}$ is not gauge invariant, its variation $\varDelta N_{\mathrm{CS}}$ is so.

We want to compute now $\varDelta B$ between an initial and a final configuration of gauge fields. We are considering vacuum field strength tensors $W_{\mu\nu}$ which vanish. The corresponding potentials are not necessarily zero but can be represented by purely gauge fields,

$$W_\mu = -\frac{i}{g} U(x) \partial_\mu U^{-1}(x) \,. \tag{9.7}$$

There are two classes of gauge transformations keeping $W_{\mu\nu} = 0$:

- Continuous transformations of the potentials yielding $\varDelta N_{\rm CS} = 0$.
- If one tries to generate $\varDelta N_{\rm CS} \neq 0$ by a continuous variation of the potentials, then one has to enter a region where $W_{\mu\nu} \neq 0$. This means that vacuum states with different topological charges are separated by potential barriers.

The probability of barrier penetration can be calculated using the quasi-classical approximation [19]. In euclidean space time, the trajectory in field space configuration which connects two vacua differing by a unit of topological charge is called instanton. The euclidean action evaluated at this trajectory gives the probability for barrier penetration as $\varGamma \sim \exp\left(-\frac{4\pi}{\alpha_W}\right) \sim 10^{-162}$, where $\alpha_W = g^2/4\pi$. This number is so small that the calculation of the pre-exponential is unnecessary and the probability for barrier penetration is essentially zero.

#### 9.2.2. Baryon violation at finite temperature: sphalerons

However, in a system with non zero temperature a particle may classically go over the barrier with a probability determined by the Boltzmann exponent, as we have seen.

What we have is a potential which depends on the gauge field configuration $W_\mu$. This potential has an infinite number of degenerate minima, labeled as $\varOmega_n$. These minima are characterized by different values of the Chern–Simons number. The minimum $\varOmega_0$ corresponds to the configuration $W_\mu = 0$ and we can take conventionally the value of the potential at this point to be zero. Other minima have gauge fields given by (9.7). In the temporal gauge $W_0 = 0$, the gauge transformation $U$ must be time independent (since we are considering gauge configurations with $W_{\mu\nu} = 0$), *i.e.* $U = U(\vec{x})$, and so functions $U$ define maps,

$$U : S^3 \longrightarrow {\rm SU}(2) \,.$$

All the minima with $W_{\mu\nu} = 0$ have equally zero potential energy, but those defined by a map $U(\vec{x})$ with nonzero Chern–Simons number

$$n[U] = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \mathrm{Tr}\,(U\partial_i U^{-1} U\partial_j U^{-1} U\partial_k U^{-1}) \tag{9.8}$$

correspond to degenerate minima in the configuration space with non-zero baryon and lepton number.

Degenerate minima are separated by a potential barrier. The field configuration at the top of the barrier is called **sphaleron**, which is a *static unstable* solution to the classic equations of motion [34]. The sphaleron solution has been explicitly computed in Ref. [34] for the case of zero Weinberg angle, (*i.e.* neglecting terms $\mathcal{O}(g')$), and for an arbitrary value of $\sin^2\theta_W$ in Ref. [35].

An ansatz for the sphaleron solution for the case of zero Weinberg angle was given (for the zero temperature potential) in Ref. [34], for the Standard Model with a single Higgs doublet, as,

$$W_i^a \sigma^a dx^i = -\frac{2i}{g} f(\xi) dU\, U^{-1} \tag{9.9}$$

for the gauge field, and

$$\Phi = \frac{v}{\sqrt{2}} h(\xi) U \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.10}$$

for the Higgs field, where the gauge transformation $U$ is taken to be,

$$U = \frac{1}{r} \begin{pmatrix} z & x+iy \\ -x+iy & z \end{pmatrix} \tag{9.11}$$

and we have introduced the dimensionless radial distance $\xi = gvr$.

Using the ansatz (9.9), (9.10) and (9.11) the field equations reduce to,

$$\begin{aligned} \xi^2 \frac{d^2 f}{d\xi^2} &= 2f(1-f)(1-2f) - \frac{\xi^2}{4} h^2 (1-f)\,, \\ \frac{d}{d\xi}\left(\xi^2 \frac{dh}{d\xi}\right) &= 2h(1-f)^2 + \frac{\lambda}{g^2} \xi^2 (h^2-1) h \end{aligned} \tag{9.12}$$

with the boundary conditions, $f(0) = h(0) = 0$ and $f(\infty) = h(\infty) = 1$. The energy functional becomes then,

$$\begin{aligned} E \;=\; \frac{4\pi v}{g} \int_0^\infty \Bigg\{ 4\left(\frac{df}{d\xi}\right)^2 + \frac{8}{\xi^2}[f(1-f)]^2 + \frac{1}{2}\xi^2 \left(\frac{dh}{d\xi}\right)^2 \\ + [h(1-f)]^2 + \frac{1}{4}\left(\frac{\lambda}{g^2}\right) \xi^2 (h^2-1)^2 \Bigg\} d\xi\,. \end{aligned} \tag{9.13}$$

The solution to Eqs. (9.12) has to be found numerically. The solutions depend on the gauge and quartic couplings, $g$ and $\lambda$. Once replaced into the energy functional (9.13) they give the sphaleron energy which is the height of the barrier between different degenerate minima. It is customary to write the solution as,

$$E_{\rm sph} = \frac{2m_W}{\alpha_W} B\left(\frac{\lambda}{g^2}\right) , \tag{9.14}$$

where $B$ is the constant which requires numerical evaluation. For the standard model with a single Higgs doublet this parameter ranges from $B(0) = 1.5$ to $B(\infty) = 2.7$. A fit valid for values of the Higgs mass 25 GeV $\leq m_h \leq$ 250 GeV can be written as,

$$B(x) = 1.58 + 0.32x - 0.05x^2 , \tag{9.15}$$

where $x = m_h/m_W$.

The previous calculation of the sphaleron energy was performed at zero temperature. The sphaleron at finite temperature was computed in Ref. [36, 37]. Its energy follows the approximate scaling law,

$$E_{\rm sph}(T) = E_{\rm sph}\frac{\langle\phi(T)\rangle}{\langle\phi(0)\rangle}$$

which, using (9.14), can be written as,

$$E_{\rm sph}(T) = \frac{2m_W(T)}{\alpha_W} B(\lambda/g^2) , \tag{9.16}$$

where $m_W(T) = \frac{1}{2}g\langle\phi(T)\rangle$.

#### 9.2.3. Baryon violation rate at $T > T_{\rm c}$

The calculation of the baryon violation rate at $T > T_{\rm c}$, *i.e.* in the symmetric phase, is very different from that in the broken phase, that will be reviewed in the next section. In the symmetric phase, at $\phi = 0$, the perturbation theory is spoiled by infrared divergences, and so we cannot rely upon perturbative calculations to compute the baryon violation rate in this phase. In fact, the infrared divergences are cut off by the non-perturbative generation of a **magnetic mass**, $m_M \sim \alpha_W T$, *i.e.* a **magnetic screening length** $\xi_M \sim (\alpha_W T)^{-1}$. The rate of baryon violation per unit time and unit volume $\Gamma$ does not contain any exponential Boltzmann factor[6]. The pre-exponential can be computed from dimensional grounds [38] as

[6] It would disappear from (8.15) in the limit $T \to \infty$.

$$\Gamma = k(\alpha_W T)^4 \,, \tag{9.17}$$

where the coefficient $k$ has been evaluated numerically in Ref. [39] with the result [7] $0.1 \lesssim k \lesssim 1.0$.

#### 9.2.4. Baryon violation rate at $T < T_c$

After the phase transition, the calculation of baryon violation rate can be done using the semiclassical approximations given by Eq. (8.15). The rate per unit time and unit volume for fluctuations between neighboring minima contains a Boltzmann suppression factor $\exp\left(-E_{\rm sph}(T)/T\right)$, where $E_{\rm sph}(T)$ is given by (9.16), and a pre-factor containing the determinant of all zero and non-zero modes. The prefactor was computed in Ref. [41] as

$$\Gamma \sim 2.8 \times 10^5\, T^4 \left(\frac{\alpha_W}{4\pi}\right)^4 \kappa \frac{\zeta^7}{B^7} e^{-\zeta} \,, \tag{9.18}$$

where we have defined $\zeta(T) = E_{\rm sph}(T)/T$, the coefficient $B$ is the function of $\lambda/g^2$ defined in (9.15) and $\kappa$ is the functional determinant associated with fluctuations about the sphaleron. It has been estimated [24] to be in the range, $10^{-4} \lesssim \kappa \lesssim 10^{-1}$.

The equation describing the dilution $S$ of the baryon asymmetry in the anomalous electroweak processes reads [42]

$$\frac{\partial S}{\partial t} = -V_{\rm B}(t) S \,, \tag{9.19}$$

where $V_{\rm B}(t)$ is the rate of the baryon number non-conserving processes. Assuming $T$ is constant during the phase transition the integration of (9.19) yields $S = e^{-X}$ and $X = \dfrac{13}{2} N_f \dfrac{\Gamma}{T^3} t$. Using now (9.18) and (8.27) we can write the exponent $X$ as, $X \sim 10^{10} \kappa \zeta^7 e^{-\zeta}$, where we have taken the values of the parameters, $B = 1.87$, $\alpha_W = 0.0336$, $N_f = 3$, $T_c \sim 10^2$ GeV. Imposing now the condition $S \gtrsim 10^{-5}$, or $X \lesssim 10$, leads to the condition on $\zeta(T_c)$,

$$\zeta(T_c) \gtrsim 7 \log \zeta(T_c) + 9 \log 10 + \log \kappa \,. \tag{9.20}$$

Now, taking $\kappa$ at its upper bound, $\kappa = 10^{-1}$, we obtain from (9.20) the bound [43]

$$\frac{E_{\rm sph}(T_c)}{T_c} \gtrsim 45 \,, \tag{9.21}$$

[7] In fact recent lattice computations [40] seem to provide an extra factor in (9.17) as $k' \alpha_W$ which is roughly speaking $\mathcal{O}(1)$.

and using the lower bound, $\kappa = 10^{-4}$ we obtain,

$$\frac{E_{\rm sph}(T_{\rm c})}{T_{\rm c}} \gtrsim 37\,. \tag{9.22}$$

Eq. (9.21) is the usual bound used to test different theories while Eq. (9.22) gives an idea on how much can one move away from the bound (9.21), *i.e.* the uncertainty on the bound (9.21).

The bounds (9.21) and (9.22) can be translated into bounds on $\phi(T_{\rm c})/T_{\rm c}$. Using the relation (9.16) we can write

$$\frac{\phi(T_{\rm c})}{T_{\rm c}} = \frac{g}{4\pi B}\frac{E_{\rm sph}(T_{\rm c})}{T_{\rm c}} \sim \frac{1}{36}\frac{E_{\rm sph}(T_{\rm c})}{T_{\rm c}}\,, \tag{9.23}$$

where we have used the previous values of the parameters. The bound (9.21) translates into

$$\frac{\phi(T_{\rm c})}{T_{\rm c}} \gtrsim 1.3 \tag{9.24}$$

while the bound (9.22) translates into

$$\frac{\phi(T_{\rm c})}{T_{\rm c}} \gtrsim 1.0\,. \tag{9.25}$$

These bounds, Eqs. (9.24) and (9.21), require that the phase transition is strong enough first order. In fact for a second order phase transition, $\phi(T_{\rm c}) \simeq 0$ and any previously generated baryon asymmetry would be washed out during the phase transition. For the case of the Standard Model the previous bounds translate into a bound on the Higgs mass.

For the Standard Model we can use Eq. (8.12) and $m_h^2 = 2\lambda v^2$ to write,

$$\frac{\phi(T_{\rm c})}{T_{\rm c}} \sim \frac{4Ev^2}{m_h^2}\,. \tag{9.26}$$

In this way the bound (9.24) translates into the bound on the Higgs mass,

$$m_h \lesssim \sqrt{\frac{4E}{1.3}} \sim 42 \text{ GeV}\,. \tag{9.27}$$

The bound (9.27) is excluded by LEP measurements [28], and so the Standard Model is unable to keep any previously generated baryon asymmetry. Including two-loop effects the bound is slightly increased to $\sim 45$ GeV [44]. In this way we see that the Standard Model is unable to explain the baryon number of the Universe which motivates the necessity of physics beyond the Standard Model. Some progress has been done in this direction although it is outside the scope of these lectures.

## 10. Conclusion

There is a number of missing topics in these lectures that we did not have time to cover. They can be summarized as follows:

- Field theory at finite temperature has an IR divergence: it has to be cured by improving the theory with resummations, *e.g.* hard thermal loops, . . . .
- Electroweak Baryogenesis requires large $CP$ violation and strong first-order phase transition: neither of them are provided by the Standard Model effective potential.
- They can possibly be provided in extensions of the Standard Model: *e.g.* its minimal supersymmettric extension (MSSM).
- The theory of phase transitions has wide applications in model building of inflation: old inflation, new inflation, extended inflation, hybrid inflation, . . . .

Work supported in part by the European Commission under the European Union through the Marie Curie Research and Training Networks "UniverseNet" (MRTN-CT-2006-035863); and by CICYT, Spain, under contracts FPA 2004-02012, FPA 2004-02015 and FPA 2005-02211.

## REFERENCES

[1] L. Dolan, R. Jackiw, *Phys. Rev.* **D9**, 3320 (1974).
[2] S. Weinberg, *Phys. Rev.* **D9**, 3357 (1974).
[3] R.H. Brandenberger, *Rev. Mod. Phys.* **57**, 1 (1985).
[4] N.P. Landsman, Ch.G. van Weert, *Phys. Rep.* **145**, 141 (1987).
[5] M. Quirós, *Helv. Phys. Acta* **67**, 451 (1994).
[6] J.I. Kapusta, *Finite-Temperature Field Theory*, Cambridge University Press, 1989.
[7] R. Kubo, *J. Phys. Soc. Japan* **12**, 570 (1957); P.C. Martin, J. Schwinger, *Phys. Rev.* **115**, 1342 (1959).
[8] T. Matsubara, *Prog. Theor. Phys.* **14**, 351 (1955).
[9] H. Matsumoto, Y. Nakano, H. Umezawa, F. Mancini, M. Marinaro, *Prog. Theor. Phys.* **70**, 599 (1983); H. Matsumoto, Y. Nakano, H. Umezawa, *J. Math. Phys.* **25**, 3076 (1984).
[10] L.V. Keldish, *Sov. Phys. JETP* **20**, 1018 (1964).

[11] T.S. Evans, *Phys. Rev.* **D47**, R4196 (1993); T. Altherr, CERN preprint, CERN-TH.6942/93.

[12] R. Kobes, *Phys. Rev.* **D42**, 562 (1990); *Phys. Rev. Lett.* **67**, 1384 (1991); T.S. Evans, *Phys. Lett.* **B249**, 286 (1990); *Phys. Lett.* **B252**, 108 (1990); *Nucl. Phys.* **B371**, 340 (1992); P. Aurenche, T. Becherrawy, *Nucl. Phys.* **B379**, 259 (1992); M.A. van Eijck, Ch.G. van Weert, *Phys. Lett.* **B278**, 305 (1992).

[13] D.A. Kirzhnits, *JETP Lett.* **15**, 529 (1972).

[14] D.A. Kirzhnits, A.D. Linde, *Phys. Lett.* **42B**, 471 (1972); D.A. Kirzhnits, A.D. Linde, *JETP* **40**, 628 (1974); D.A. Kirzhnits, A.D. Linde, *Ann. Phys.* **101**, 195 (1976).

[15] A.D. Linde, *Rep. Prog. Phys.* **42**, 389 (1979); A.D. Linde, *Phys. Lett.* **99B**, 391 (1981); A.D. Linde, *Particle Physics and Inflationary Cosmology*, Harwood, Chur, Switzerland, 1990.

[16] A.D. Linde, *Phys. Lett.* **B108**, 389 (1982); A. Albrecht, P.J. Steinhardt, *Phys. Rev. Lett.* **48**, 1226 (1082).

[17] G.W. Anderson, L.J. Hall, *Phys. Rev.* **D45**, 2685 (1992).

[18] M. Dine, R.G. Leigh, P. Huet, A. Linde, D. Linde, *Phys. Lett.* **B283**, 319 (1992); *Phys. Rev.* **D46**, 550 (1992).

[19] S. Coleman, *Phys. Rev.* **D15**, 2929 (1977).

[20] C.G. Callan, S. Coleman, *Phys. Rev.* **D16**, 1762 (1977).

[21] S. Coleman, F. De Luccia, *Phys. Rev.* **D21**, 3305 (1980).

[22] A.D. Linde, *Phys. Lett.* **70B**, 306 (1977); **100B**, 37 (1981); *Nucl. Phys.* **B216**, 421 (1983).

[23] L. McLerran, M. Shaposhnikov, N. Turok, M. Voloshin, *Phys. Lett.* **B256**, 451 (1991).

[24] M. Dine, P. Huet, R. Singleton Jr., *Nucl. Phys.* **B375**, 625 (1992).

[25] J.M. Moreno, M. Quirós, M. Seco, *Nucl. Phys.* **B526**, 489 (1998).

[26] E.W. Kolb, M.S. Turner, *The Early Universe*, Addison-Wesley, 1990.

[27] A.D. Sakharov, *Zh. Eksp. Teor. Fiz. Pis'ma* **5**, 32 (1967); *JETP Lett.* **91B**, 24 (1967).

[28] Particle Data Group, Review of Particle Properties, *J. Phys. G: Nucl. Part. Phys.* **33**, 1 (2006).

[29] E.W. Kolb, S. Wolfram, *Nucl. Phys.* **B172**, 224 (1980); *Phys. Lett.* **B91**, 217 (1980).

[30] V.A. Kuzmin, V.A. Rubakov, M.E. Shaposhnikov, *Phys. Lett.* **B155**, 36 (1985); *Phys. Lett.* **B191**, 171 (1987).

[31] A.G. Cohen, D.B. Kaplan, A.E. Nelson, *Annu. Rev. Nucl. Part. Sci.* **43**, 27 (1993).

[32] G. 't Hooft, *Phys. Rev. Lett.* **37**, 8 (1976); *Phys. Rev.* **D14**, (1976) 3432.

[33] M. Kobayashi, M. Maskawa, *Prog. Theor. Phys.* **49**, 652 (1973).

[34] N.S. Manton, *Phys. Rev.* **D28**, 2019 (1983); F.R. Klinkhamer, N.S. Manton, *Phys. Rev.* **D30**, 2212 (1984).

[35] J. Kunz, B. Kleihaus, Y. Brihaye, *Phys. Rev.* **D46**, 3587 (1992).

[36] Y. Brihaye, J. Kunz, *Phys. Rev.* **D48**, 3884 (1993).

[37] J.M. Moreno, D.H. Oaknin, M. Quirós, *Nucl. Phys.* **B483**, 267 (1997) [hep-ph/9605387].

[38] P. Arnold, L. McLerran, *Phys. Rev.* **D36**, 581 (1987); S.Yu. Khlebnikov, M.E. Shaposhnikov, *Nucl. Phys.* **B308**, 885 (1988).

[39] J. Ambjorn, M. Laursen, M. Shaposhnikov, *Phys. Lett.* **B197**, 49 (1989); J. Ambjorn, T. Askaard, H. Porter, M. Shaposhnikov, *Phys. Lett.* **B244**, 479 (1990); *Nucl. Phys.* **B353**, 346 (1991); J. Ambjorn, K. Farakos, *Phys. Lett.* **B294**, 248 (1992); J. Ambjorn, A. Krasnitz, *Phys. Lett.* **B362**, 97 (1995).

[40] P. Arnold, D. Son, L.G. Yaffe,*Phys. Rev.* **D55**, 6264 (1997); *Phys. Rev.* **D59**, 105020 (1999) [hep-ph/9810216]; hep-ph/9810217; G.D. Moore, *Phys. Rev.* **D59**, 014503 (1999); *Nucl. Phys.* **B568**, 367 (2000) [hep-ph/981031]; D. Bodeker, *Phys. Lett.* **B426**, 351 (1998); hep-ph/9810265.

[41] L. Carson, Xu Li, L. McLerran, R.-T. Wang, *Phys. Rev.* **D42**, 2127 (1990).

[42] M.E. Shaposhnikov, *Nucl. Phys.* **B287**, 757 (1987); *Nucl. Phys.* **B299**, 797 (1988); A.I. Bochkarev, M.E. Shaposhnikov, *Mod. Phys. Lett* **A2**, 417 (1987).

[43] A.I. Bochkarev, S.V. Kuzmin, M.E. Shaposhnikov, *Phys. Rev.* **D43**, 369 (1991).

[44] P. Arnold, O. Espinosa, *Phys. Rev.* **D47**, 3546 (1993).