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AdS/CFT CORRESPONDENCE, VISCOUS HYDRODYNAMICS AND TIME-DEPENDENT D7-BRANE EMBEDDING*

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AdS/CFT correspondence is used to model the dynamics of one-dimensional expansion of $\mathcal{N}=4$ SYM plasma. Criterium of nonsingularity of the dual geometry is shown to fix both, large proper time dynamics (to be of the perfect fluid type) and subleading corrections to it (viscosity and relaxation time). Time-dependent D7-brane embedding is shown as a first step towards adding a fundamental matter into the expanding plasma.

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1. Introduction

Quark–gluon plasma created in RHIC turned out to be a strongly coupled phase of QCD [1]. This raises a question of its theoretical understanding originating from the first principles. Recently the famous AdS/CFT correspondence [6] has been proposed as a tool to carry out analytical investigations in a strongly coupled gauge theory at a finite temperature [10]. In this paper gravity dual is used to model one-dimensional expanding plasma of $\mathcal{N}=4$ super Yang–Mills theory (SYM) with a very large number of colors in the boost-invariant setting [11]. In the first part of this paper, second order relativistic hydrodynamics [3–5] is used to extract viscosity and relaxation time from holographically reproduced energy-momentum tensor of the gauge theory [15]. These two quantities are of great importance because they can be compared to the ones describing QCD plasma.

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The second part is devoted to introducing the fundamental matter to $\mathcal{N} = 4$ SYM expanding plasma system (this reduces supersymmetry by the factor of 2, but at a finite temperature supersymmetry is broken anyway). On the dual side this corresponds to adding $N_f \ll N_c$ D7-branes [19]. A time-dependent D7-brane embedding is the first step towards the calculation of the meson spectra in the expanding plasma [20].

The paper is organized as follows: in Section 2 Bjorken hydrodynamics [2] is reviewed, next in Section 3 second order relativistic hydrodynamics [4] is introduced, Section 4 contains holographic renormalization method [7], in Section 5 systematic way of solving Einstein equations for large proper times is explained together with formulas for metric coefficients, Section 6 provides the details of viscosity and relaxation time calculations within the framework of second order relativistic hydrodynamics [15], Sections 7 and 8 are devoted to the embedding of the D7-brane in a time-dependent background [20], conclusions and summary of open problems are located in Section 9.

2. Boost invariant energy-momentum tensors

The operator of particular interest on the $\mathcal{N} = 4$ SYM side is the energy-momentum tensor. The gauge gravity duality will be used to model its dynamics in the QCD setting corresponding to one-dimensional expansion of quark–gluon plasma [11]. If x^0 is time-like coordinate and x^1 denotes direction of expansion, then the coordinates naturally adopted to probe one-dimensional expansion in boost-invariant setting are proper time $\tau = \sqrt{(x^0)^2 - (x^1)^2}$ and rapidity $y = \operatorname{arctanh}(x^1/x^0)$. Directions perpendicular to the expansion will be denoted $x_\perp$. Boost-invariance forces all physical quantities to depend on proper-time τ only [11].

After imposing all the symmetries on energy-momentum tensor $T^{\mu\nu}$ (conservation, tracelessness and rotational symmetry in the transverse plane) the resulting outcome is an energy-momentum tensor fully expressed in terms of energy density $f(\tau) = T^{00}$ [11].

The aim of applying the AdS/CFT correspondence in this context is to reproduce the dynamics of energy-momentum tensor of $\mathcal{N} = 4$ SYM from gravitational description. Having established the physical vacuum expectation value of energy-momentum tensor, second order relativistic hydrodynamics is used to calculate viscosity η and relaxation time τ_π.

3. Second order viscous hydrodynamics

Viscous corrections to perfect fluid energy-momentum tensor are of great importance because viscosity carries the information whether the system is weakly or strongly coupled. Second order hydrodynamics is a framework in which thermodynamical quantities are expanded up to the second order in gradient expansion [4, 5]. Introduction of relaxation time, parameter corresponding to characteristic scale of equilibration, results in a causal theory. Evolution equation takes the form

$$\frac{d\rho}{d\tau} = -\frac{4}{3}\frac{\rho}{\tau} + \frac{\Phi}{\tau}\,, \tag{1}$$

where

$$\Phi = \frac{4}{3}\frac{\eta}{\tau} \qquad \text{(first order formalism)}\,, \tag{2}$$

$$\tau_\pi \frac{d\Phi}{d\tau} = -\Phi + \frac{4}{3}\frac{\eta}{\tau} \qquad \text{(second order formalism)}\,. \tag{3}$$

Limit $\tau_\pi \to 0$ reduces equations to parabolic type which violates causality. In order to extract relaxation time, it is assumed on dimensional reasons that $t_\pi = r t_\pi^{(\mathrm{B})}$ where relaxation time for Boltzmann free gas assumes the form $t_\pi^{(\mathrm{B})} = \frac{3}{2}\frac{\eta}{p}$ [5]. This reduces the problem of finding dimensionless constant r [15].

4. Selected aspects of AdS/CFT correspondence

The AdS/CFT correspondence postulates an equivalence between string theory on $\mathrm{AdS}_5 \times \mathrm{S}_5$ and $\mathcal{N} = 4$ SYM theory in four-dimensional Minkowski space [6]. In the limit of large number of colors $N_c \to \infty$ and infinite 't Hooft coupling $\lambda = g_{\mathrm{YM}} N_c^2 \to \infty$ string theory reduces to supergravity defined in asymptotic AdS_5 space, which corresponds to strongly coupled gauge theory in the planar limit.

The dictionary of gauge/gravity duality connects each gauge-invariant local operator in $\mathcal{N} = 4$ SYM with a field in the bulk. There is a systematic prescription to extract vacuum expectation values of these operators from near-boundary behavior of fields in asymptotically AdS_5 space. The operator of particular interest is energy-momentum tensor of the gauge theory and the field dual to it — the metric. The line element of asymptotically AdS_5 space in Fefferman–Graham coordinates [8] takes the form

$$ds^2 = \frac{\widetilde{g}_{\mu\nu} dx^\mu dx^\nu + dz^2}{z^2}\,. \tag{4}$$

Near-boundary ($z = 0$) behavior of the metric following from five-dimensional Einstein equations $R_{ab} - \frac{1}{2} R g_{ab} - 6 g_{ab} = 0$ is

$$\widetilde{g}_{\mu,\nu} = \widetilde{g}^{(0)}_{\mu\nu} + \widetilde{g}^{(2)}_{\mu\nu} z^2 + \widetilde{g}^{(4)}_{\mu\nu} z^4 + \widetilde{g}^{(6)}_{\mu\nu} z^6 + \dots\,, \tag{5}$$

where $\widetilde{g}^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ (four-dimensional Minkowski metric), $\widetilde{g}^{(2)}_{\mu\nu} = 0$ (consistency condition) and $\widetilde{g}^{(4)}_{\mu\nu} = \frac{N_c^2}{2\pi^2}\langle T_{\mu\nu}\rangle$. All higher order terms depend only on the form of energy-momentum tensor [11].

It appears that holography works both ways, near boundary behavior sets energy-momentum tensor and any traceless and conserved energy-momentum tensor reproduces five-dimensional space-time. The energy-momentum tensor describing boost-invariant expansion of plasma is fully expressed in terms of energy density, which is fixed by the dynamics of boundary theory. This means that an additional criterium on the gravity side is needed so that holographically reproduced data on gauge theory are physical. In [11] the regularity of $R_{\mu\nu\rho\delta}R^{\mu\nu\rho\delta}$ was shown to be the correct constraint.

5. Perturbative solution to Einstein equations

Five-dimensional metric respecting all the symmetries of energy-momentum tensor takes the form

$$ds^2 = \frac{-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx_\perp^2 + dz^2}{z^2}\,. \tag{6}$$

For an energy-density of $\mathcal{N} = 4$ SYM plasma asymptotically behaving like $f(\tau) = \frac{1}{\tau^s}$ the condition of nonsingularity selects both, $s = \frac{4}{3}$ and the form of large proper-time expansion of metric functions to be

$$a(\tau,z) = a_0\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{2/3}}a_1\left(\frac{z}{\tau^{1/3}}\right) + \frac{1}{\tau^{4/3}}a_2\left(\frac{z}{\tau^{1/3}}\right) + \ldots \tag{7}$$

Equivalent expression holds for $b(\tau,z)$, $c(\tau,z)$ and $R_{\alpha\beta\rho\delta}(\tau,z)R^{\alpha\beta\rho\delta}(\tau,z)$. Metric functions in the lowest two orders take the form

$$\begin{aligned}
a(v) &= \frac{\left(3-v^4\right)^2}{3\left(3+v^4\right)} + \frac{1}{\tau^{2/3}}2\,\eta_0\frac{v^4\left(9+v^4\right)}{9-v^8}\,,\\
b(v) &= 3+v^4+\frac{1}{\tau^{2/3}}\left(-2\,\eta_0\frac{v^4}{3+v^4}+2\,\eta_0\log\frac{3-v^4}{3+v^4}\right),\\
c(v) &= 3+v^4+\frac{1}{\tau^{2/3}}\left(-2\,\eta_0\frac{v^4}{3+v^4}-\eta_0\log\frac{3-v^4}{3+v^4}\right),
\end{aligned} \tag{8}$$

where $v = \frac{z}{\tau^{1/3}}$. Second and third order functions are more complicated and can be found in [15]. Regularity of metric in the third order requires passing to the string frame, where the Einstein frame metric $g^{(\mathrm{E})}$ is multiplied by the exponent of dilaton field $\phi(\tau, z)$ so that $g^{(s)} = e^{\frac{1}{2}\phi} g^{(\mathrm{E})}$. Nontrivial dilaton profile in the bulk corresponds to nonzero $\mathrm{tr}F^2$ in the dual field theory — plasma's color electric and magnetic fields do not equilibrate.

Recently it was proposed to investigate early time dynamics of energy-momentum tensor in the Bjorken setting using condition of analyticity of the metric in the leading order [16]. The features of particular importance are poles and cuts in the perturbative expansion. It turns out that certain resummation removes poles from the first order functions (at $\frac{1}{\tau^{2/3}}$).

6. Viscosity and relaxation time of $\mathcal{N} = 4$ SYM plasma

Energy density extracted from five-dimensional metric in the second order takes the form

$$f(\tau) = \frac{1}{\tau^{4/3}} - \frac{\sqrt{2}}{3^{3/4}\tau^2} + \frac{1 + 2\log 2}{12\sqrt{3}\,\tau^{8/3}}\,. \tag{9}$$

Viscosity and constant r fixing relaxation time can be extracted using second order hydrodynamics equations. The results are

$$\begin{aligned} r &= \frac{1 - \log 2}{9}\,, \\ \eta &= \frac{1}{\sqrt{2}\,3^{3/4}\frac{1}{\tau}}\,. \end{aligned} \tag{10}$$

This gives relaxation time almost 30 times smaller than in the weak coupling limit and viscosity saturating the bound $\frac{\eta}{s} = \frac{1}{4\pi}$ [15].

7. Time-dependent D7-brane embedding

Having constructed a perturbative dual geometry in the time-dependent framework, we may proceed to introduce fundamental matter or more precisely $\mathcal{N} = 2$ hypermultiplet in the fundamental representation [18]. This is accomplished by placing a probe D7-brane into that geometry

$$\begin{aligned} ds_{10}^2 &= G_{MN}dx^\mu dx^\nu = \frac{g_{ij}(x,z)dx^i dx^j}{z^2} + \frac{dz^2}{z^2} + d\Omega_5^2 \\ &= \frac{g_{ij}(x,z)dx^i dx^j}{z^2} + \frac{dz^2}{z^2} + \rho^2 d\Omega_3^2 + (dz_8)^2 + (dz_9)^2\,, \end{aligned} \tag{11}$$

where g_{ij} is given by Eq. (6). The SO(2) symmetry in the 8,9-plane can be used to rotate the embedding of the D7 to

$$z_8 = 0, \qquad z_9 = \phi(\tau, \rho)\,. \tag{12}$$

The scalar function $\phi(\tau,\rho)$ completely describes the profile of the embedded D7-brane. In order to determine this function we have to consider the action

$$S_{\rm D7}=\mu_7\int d^8\xi\, e^{-\varphi}\sqrt{\det P[g]_{ab}+F_{ab}}\,, \tag{13}$$

where $P[...]$ denotes the pull-back to the world-volume. The resulting equations of motion are as follows

$$\Box\phi(\tau,\rho)+3\tan[\phi(\tau,\rho)]-\frac{1}{2}\frac{G^{MN}\partial_M\phi(\tau,\rho)\partial_N[G^{KL}\partial_K\phi(\tau,\rho)\partial_L\phi(\tau,\rho)]}{1+G^{AB}\partial_A\phi(\tau,\rho)\partial_B\phi(\tau,\rho)}=0. \tag{14}$$

We attempt to extract the quark mass and the quark condensate expectation value which are given by the UV asymptotic behavior of the solutions to the supergravity equations of motion. We expect to find the time dependence of the latter. Being motivated by adiabatic approximation results, we consider the following ansatz

$$\phi(\tau,z)=m+\frac{f(z)}{\tau^{8/3}}+\frac{h(z,\eta_0)}{\tau^{10/3}}+\dots \tag{15}$$

From the equations of motion we get two differential equations for $f(z)$ and $h(z,\eta_0)$. The solution reads

$$f(z)=-\tfrac{m^2z^2+m^8z^8}{27(2m^7-2m^9z^2)}\,, \tag{16}$$

$$h(z,\eta_0)=\tfrac{2\eta_0z^2(1-m^2z^2+m^4z^4)}{27m^5}\,. \tag{17}$$

Close to the boundary the embedding behaves as

$$\phi(\tau,z)=m+c(\tau)z^2+c_1(z,\eta_0)z^2+\dots=m+\tilde{c}z^2+\dots \tag{18}$$

$\tilde{c}$ is

$$\tilde{c}=-\frac{1}{54m^5\tau^{8/3}}\Big(1-\frac{4\eta_0}{\tau^{2/3}}+\dots\Big)\,. \tag{19}$$

m and $\tilde{c}$ are related to the bare quark mass m_q and chiral condensate $\langle\mathcal{O}\rangle$ by

$$m_q=\frac{m}{2\pi\alpha'}\,, \tag{20}$$

$$\langle\mathcal{O}\rangle=-\frac{N_{\rm f}N_c}{(2\pi\ell_s^2)^3\,\lambda}\,\tilde{c}\,, \tag{21}$$

where $\lambda=g^2_{\rm YM}N_c=2\pi g_sN_c=R^4/(2\ell_s^4)$.

8. Regularized D7 action

Calculating the D7 action (14) we face IR divergences in the bulk and UV divergences on the SYM side. The standard procedure to remove these divergences was presented in [19]. It regularizes the action by introducing a cut-off and then adding covariant counterterms that cancel divergences when cut-off is removed. The regularized action takes the form

$$S_{\text{reg}} = \int\limits_{z_{\min}}^{z_{\max}} \mathcal{L}_{\text{D7}}\, dz + \sum L_i(z_{\min})\,, \tag{22}$$

where the counterterms L_i are given by

$$\begin{aligned}
L_1 &= -\frac{1}{4}\sqrt{\gamma}\,,\\
L_2 &= -\frac{1}{48}\sqrt{\gamma}R_\gamma\,,\\
L_3 &= -\ln(z_{\min})\sqrt{\gamma}\frac{1}{32}\left(R_{ij}R^{ij} - \frac{1}{3}R_\gamma\right)\,,\\
L_4 &= \frac{1}{2}\sqrt{\gamma}\Psi^2\,,\\
L_5 &= -\frac{1}{2}\ln(z_{\min})\Psi\left(\partial_\tau\gamma^{\tau\tau}\sqrt{\gamma}\partial_\tau + \frac{1}{6}\sqrt{\gamma}R_\gamma\right)\Psi\,,\\
L_{\text{f}} &= -\frac{5}{12}\gamma\Psi^4\,.
\end{aligned} \tag{23}$$

L_{f} is a finite counterterm with a constant parameter that corresponds to different renormalization schemes. We fix it by the requirement that in the supersymmetric setting the on-shell action vanishes. γ is the induced metric on the $z = z_{\min}$ slice and

$$\Psi = \arcsin(z\Phi) \tag{24}$$

is the embedding coordinate of the D7 expressed as an angle on the internal S^5. Now we construct the regularized action

$$S_{\text{ren}} = \frac{1}{2}\frac{N_c N_f}{(2\pi\ell_s^2)^4\lambda}\int dy\, d^2x_\perp d\tau\, \tau\left[-\frac{1}{108m^4\tau^{8/3}} + \frac{\eta_0}{27m^4\tau^{10/3}}\right]\,. \tag{25}$$

We see that for very late times the configuration indeed relaxes to the supersymmetric setting.

9. Summary and open problems

In this talk, solution to fully nonlinear Einstein equations corresponding to the one-dimensional expansion of $\mathcal{N} = 4$ SYM plasma in the regime of large proper time is discussed together with the viscosity and relaxation time calculations. Introduction of the scaling variable is shown to be a valid way of reproducing an asymptotic behavior of plasma's energy-momentum tensor. Condition of nonsingularity of the dual geometry (regularity of $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$) is shown to fix this behavior to be of the perfect fluid type together with the form of subasymptotic corrections. A systematic method of seeking for subasymptotic terms is discussed up to the third order in perturbative expansion around large proper-time solution. Passing to the string frame with nontrivial profile of the dilaton field is proposed as a resolution of logarithmic divergence in the third order of Riemann squared. Turning on the dilaton field contributes to the nonvanishing expectation value of $\mathrm{tr}F^2$ on the dual field theory side, which means that for subasymptotic times the color magnetic and electric field are not in equilibrium. Viscosity η and relaxation time r are calculated within the framework of the second order viscous hydrodynamics with the resulting η satisfying famous viscosity to entropy density bound and relaxation time being 30 times smaller than the weak coupling approximation. It would be very interesting to reproduce this result with the use of other methods.

Finally, it is shown how to introduce the fundamental matter by putting a probe D7-brane into the time-dependent geometry. Using equations of motion the D7-brane embedding is calculated. The D7 action is shown to suffer from divergences, so in the last step this action is regularized using the holographic renormalization method.

Further investigations into the late proper time dynamics of the energy-momentum tensor should include the analysis of evolving black hole using the framework of dynamical horizons [17], turning on other SUGRA fields and more-dimensional expansion.

Another interesting issue is the small proper time dynamics in case of one-dimensional expansion using the nonsingularity argument, which recently was argued to result in asymptotically constant energy-momentum tensor with possible logarithmic corrections [16].

Having found the embedding, one can study small fluctuations of the brane and calculate the meson spectrum [20]. It would be also interesting to investigate the thermodynamics of such time-dependent system but up till now there is no clear definition of thermodynamic quantities.

The authors are sure, that further applications of the AdS/CFT correspondence to the finite temperature QFT will produce many interesting, if not surprising, results.

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