Vol. 38 (2007) *ACTA PHYSICA POLONICA B* No 12

BIFURCATION IN THE SHAKURA MODEL*

Krzysztof Roszkowski

Department of General Relativity and Astrophysics
M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland

(Received October 30, 2007)

We investigate a Newtonian description of accretion of polytropic perfect fluids onto a luminous compact object possessing a hard surface. The selfgravitation of the fluid and its interaction with luminosity is included in the model. Using appropriate boundary conditions we find stationary, spherically symmetric solutions. For a given luminosity, asymptotic mass of the system and its asymptotic temperature there exist two sub-critical solutions. They differ by the ratio of fluid mass to the total mass.

PACS numbers: 97.10.Gz, 95.30.Lz

1. Introduction

In the paper we study a Newtonian, spherically symmetric accretion of selfgravitating gas onto a central compact body with a hard surface. We want to give an answer to the following question (it may be regarded as a kind of *an inverse problem*): assume that one knows (from the astronomical observations) total mass and luminosity of the system, asymptotic temperature and the equation of state of the accreting gas. Let us also assume the gravitational potential on the surface of the central body. Can we determine the mass of the central body and the abundance of the gas in the system? And is this solution unique?

The initial motivation to study the problem of such an accretion was an idea of gravastars, developed by Mazur and Mottola [1]. Gravastars are extremely compact objects (more compact than neutron stars), their radius is only slightly bigger than the radius of a black hole of the same mass but they do have a hard surface. To build such a hypothetical object one has to violate energy conditions [2] to avoid the Buchdahl theorem [3]. Have gravastars existed, they would be perfect laboratories to test quantum gravity effects.

* Presented at the XLVII Cracow School of Theoretical Physics, Zakopane, Poland, June 14–22, 2007.

The next question then arises: can one observationally detect gravastars and distinguish them from black holes and neutron stars? There exist contradictory opinions with regard to this subject [4, 5].

To address the above questions we chose the simplest model that can be regarded as a radiating system. Even though we investigated only the problem of Newtonian accretion we found that for a given set of observational data one can find two different solutions, having different ratio of the mass of infalling gas to the total mass of the system. This can be understood intuitively: for weakly luminous objects one can think of two configurations, the first, with a small amount of the accreting gas $m_f/M \ll 1$, and the second, with a light central object that does not accrete the matter effectively $m_f/M \approx 1$.

This result is not surprising. A recent general-relativistic analysis [6], including self-gravitation of a cloud, but neglecting luminosity, gives similar results. In this case, the mass accretion rate behaves like $y^2(1-y)$, where $y = M_{\mathrm{core}}/M$ is the ratio of the central object mass to the mass of the system. Thus the accretion is the most effective when the central object contains 2/3 of the total mass.

More detailed analysis of the problem presented in this paper can be found in [7].

2. The Shakura model

We will study the stationary accretion of spherically symmetric fluids in the extended Shakura model [9]. In the following text we will use comoving (Lagrangian) coordinates (r,t) and areal radius R. The velocity of a particle is given by $U(r,t) = \partial_t R$, p denotes pressure, L_{E} is the Eddington luminosity, and $L(R)$ is the local one. By ρ we denote the baryonic mass density of the infalling gas, so the quasilocal mass is govern by $\partial_R m(R) = 4\pi R^2 \rho$. The total mass, measured at the boundary R_∞ is $M = m(R_\infty)$. The mass accretion rate is

$$\dot{M} = -4\pi R^2 \rho U \,. \tag{1}$$

We assume the gas is guided by the polytropic equation of state $p = K\rho^{\Gamma}$, where Γ is a constant $(1 < \Gamma \leq 5/3)$. The speed of sound is $a = \sqrt{\partial_\rho p}$.

We study the stationary accretion, which means that all values measured at a fixed areal radius are constant. It is also obvious that any accretion leads to the growth of the central object's mass. We will then assume that the time scale is short and the accretion rate is small enough to ensure the quasilocal mass $m(R)$ is approximately constant. We also assume that at the outer boundary of the cloud the following condition holds true:

$$U_\infty^2 \ll \frac{m(R_\infty)}{R_\infty} \ll a_\infty^2 \,. \tag{2}$$

The full system is governed by a set of differential equations. It consists of the Euler momentum conservation equation

$$U\partial_R U = -\frac{Gm(R)}{R^2} - \frac{\partial_R p}{\rho} + \alpha\frac{L(R)}{R^2}\,, \tag{3}$$

the mass conservation

$$\partial_R \dot{M} = 0\,, \tag{4}$$

and the energy conservation

$$L_0 - L(R) = \dot{M}\left(\frac{a_\infty^2}{\Gamma-1} - \frac{a^2}{\Gamma-1} - \frac{U^2}{2} - \phi(R)\right)\,. \tag{5}$$

α is a dimensional constant $\alpha = \sigma_T/4\pi m_p c$, L_0 is the luminosity measured at the outer boundary of the fluid ball and

$$\phi(R) = -\frac{m(R)}{R} - 4\pi\int\limits_R^{R_\infty} r\rho(r)\mathrm{d}r \tag{6}$$

is the Newtonian gravitational potential. Actually, in (5) we used (2) to keep only the biggest term $\frac{a_\infty^2}{\Gamma-1}$.

The Eddington luminosity can be easily calculated if one knows the mass of the system: $L_{\mathrm{E}} = GM/\alpha$. The total luminosity L_0 can be found from $L_0 = \dot{M}\phi_0$. By ϕ_0 we understand the potential measured at the surface of the compact object $\phi_0 \equiv |\phi(R_0)|$. Knowing ϕ_0 we define another measure of the central body size, a modified radius $\tilde{R}_0 = GM/\phi_0$. Shakura [9] found an expression describing the luminosity at a given radius, for test fluids. Its generalization,

$$L(R) = L_0 \exp\left(-\frac{L_0\tilde{R}_0}{L_{\mathrm{E}}R}\right) \tag{7}$$

holds in our case [7,8].

3. Sonic point

To simplify our calculations we will assume the configuration possesses a sonic point, the same analysis leads to non unique results also when there are no such points.

The sonic point (more precisely: sonic horizon, it is not a point but a sphere of a given radius) is a point where the infalling velocity becomes equal to the speed of sound $|U| = a$. In the following we will denote by an asterisk all the values measured at the sonic point. From the Eqs. (3)–(5) one finds:

$$a_*^2 = U_*^2 = \frac{GM_*}{2R_*}\left(1 - \frac{L_*\alpha}{GM_*}\right) = \left(1 - \frac{L_* M}{L_{\rm E} M_*}\right) . \tag{8}$$

From that follows:

$$\frac{L_* M}{L_{\rm E} M_*} < 1 . \tag{9}$$

Let us introduce auxiliary variables: relative luminosity $x = L_0/L_{\rm E}$, relative mass at the sonic point $y = M_*/M$ and a kind of "compactness" measure $\gamma = \tilde{R}_0/R_*$. Condition (9) becomes now:

$$x \exp(-x\gamma) < y . \tag{10}$$

We assume that $\gamma < 1$ and $x\gamma \ll 1$. After simple, but tedious algebra [7], one can calculate the mass accretion rate as a function of sonic point characteristics:

$$\dot{M} = G^2 \pi^2 M^2 \frac{\rho_\infty}{a_\infty^3} (y - x\exp(-x\gamma))^2 \left(\frac{a_*^2}{a_\infty^2}\right)^{\frac{5-3\Gamma}{2(\Gamma-1)}} . \tag{11}$$

This in turn allows us to express the relative luminosity using values measured during the astronomical observations:

$$L_0 = \phi_0 G^2 \pi^2 \chi_\infty \frac{M^3}{a_\infty^3} (1-y)(y - x\exp(-x\gamma))^2 \left(\frac{2}{5-3\Gamma}\right)^{\frac{5-3\Gamma}{2(\Gamma-1)}} , \tag{12}$$

where χ_∞ is roughly the inverse of the volume of the gas outside of the sonic sphere.

4. Bifurcation

The above equation may be rewritten in the following form:

$$x = \beta(1-y)(y - x\exp(-x\gamma))^2 , \tag{13}$$

where β is a numerical constant whose value is known from the observations of the specific configuration. We will now restrict our attention to the analysis of this equation. Below we show that for any β, $0 \leq \gamma < 1$ and x smaller than certain "critical" value a there exist two solutions $y(x)$.

Theorem: Let us define a function

$$F(x,y) = x - \beta(1-y)(y - x\exp(-x\gamma)) . \tag{14}$$

Then, for any β, $0 \leq \gamma < 1$ and x smaller than a certain critical value:

1. There exists a critical point $x = a, y = b$ of F ($F(a,b) = 0$ and $\partial_y F(x,y)|_{(a,b)} = 0$). The parameters a and b satisfy $0 < a, b < 1$ and $3b = 2 + a\exp(-a\gamma)$.
2. For any $0 < x < a$ there exist two solutions $y(x)^{\pm}$, bifurcating from (a,b). They are locally approximated by:

$$y^{\pm} = b \pm \frac{\sqrt{(a-x)(b + a\exp(-a\gamma)(1-2a\gamma))}}{\sqrt{\beta(b - a\exp(-a\gamma))(1 - a\exp(-a\gamma))}}\,. \tag{15}$$

3. The relative luminosity is extremized at the critical point (a,b).

Sketch of a proof: Two criticality conditions are:

$$a - \beta(1-b)(b - a\exp(-a\gamma))^2 = 0\,, \tag{16}$$

$$b - a\exp(-a\gamma) = 2(1-b)\,. \tag{17}$$

A straightforward calculation leads to the part 1 of the theorem.

Using the relations between a and b and Eq. (17) we get:

$$b = \frac{2}{3} + \frac{4\beta}{3}\,(1-b)^3 \exp\left(-4\beta\,(1-b)^3\,\gamma\right)\,. \tag{18}$$

Both sides are continuous functions of b and for $b = 0$ the right hand side is bigger than 0, while for $b = 1$ it is smaller than 1. Therefore there must exist a solution. It can also be shown that there is just a single solution. Inserting $x = a + \varepsilon$ and $y = b + y(\varepsilon)$ into $F(x,y) = 0$, and expanding it we finally come to the relation (15).

To prove the last part of the theorem we calculate the derivative of x along a non-critical solution $F(x(y), y) = 0$:

$$\frac{dx}{dy} = \beta\frac{\left(y - x\mathrm{e}^{-x\gamma}\right)\left(3y - 2 - x\mathrm{e}^{-x\gamma}\right)}{1 + 2\beta(1-y)\left(y - x\mathrm{e}^{-x\gamma}\right)\left(\mathrm{e}^{-x\gamma} - \gamma x\mathrm{e}^{-x\gamma}\right)}\,. \tag{19}$$

The nominator is equal to 0 only at the critical point, while the denominator is always positive.

In Fig. 1 we show a plot of the dependence of the mass of the gas on the relative luminosity. For luminosities smaller than the critical one there exist two solutions with different ratio of the gas mass to the whole system mass. Different values of β lead to plots which are qualitatively the same.

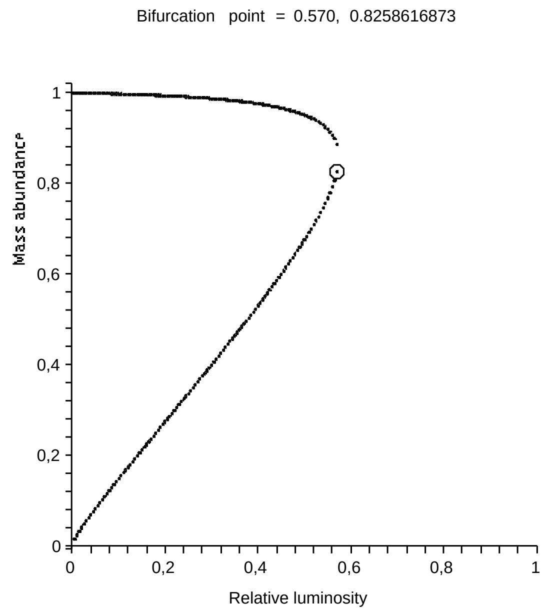

Fig. 1. Mass abundance y *versus* relative luminosity x. The critical point is enclosed in a circle. $\beta = 50$.

5. Numerical examples

In the previous sections we described the behavior of a system consisting of a cloud of gas accreting on a central compact object. To achieve the relation (13) between relative luminosity and the amount of gas in the system, we made a few simplifying assumptions. To test the final results a series of numerical calculations was performed. It turned out that the relative error of the assumptions was of the order of 10^{-3}.

To show the bifurcation of the system we chose the following observational data:

- Total mass is given in the units of solar mass $M_\odot = 1.989 \times 10^{33}$ g, $M = M_\odot(M/M_\odot)$.
- Eddington luminosity $L_{\mathrm{E}} = 1.3 \times 10^{38}(M/M_\odot)$ erg/s and for $x = 0.1$ we have $L_0 = 1.3 \times 10^{37}(M/M_\odot)$ erg/s.
- The asymptotic speed of sound $a_\infty = c/50 = 6 \times 10^8$ cm/s, the radius of the sphere enclosing the gas $R_\infty = 1.5 \times 10^{11}(M/M_\odot)$ cm and the surface potential $\phi(R_0) = -0.25c^2 = -2.25 \times 10^{20}$ cm^2/s^2.
- The modified size measure $\tilde{R}_0 = 6 \times 10^5(M/M_\odot)$ cm.

Now we have two systems which provide the above data:

Solution I:

- (sonic point parameters) $R_* = 8.35 \times 10^7 (M/M_\odot)$ cm, $a_* = |U_*| = 8.46 \times 10^8$ cm/s;
- (size and mass of the hard core) $R_0 = 5.93 \times 10^5 (M/M_\odot)$ cm, $M_{\rm core} = 1.98 \times 10^3 3(M/M_\odot)$ g;
- (asymptotic mass density) $\rho_\infty \approx 6 \times 10^{-4} (M_\odot/M)^2$ g/cm^3.

Solution II:

- (sonic point parameters) $R_* = 9.15 \times 10^6 (M/M_\odot)$ cm, $a_* = |U_*| = 8.46 \times 10^8$ cm/s;
- (size and mass of the hard core) $R_0 = 1.18 \times 10^5 (M/M_\odot)$ cm, $M_{\rm core} = 3.92 \times 10^{32} (M/M_\odot)$ g;
- (asymptotic mass density) $\rho_\infty \approx 1.1 \times 10^{-1} (M_\odot/M)^2$ g/cm^3.

In Fig. 2 we present a plot of the rescaled infall speed profile for those two solutions. We plotted $\frac{U^2}{2m(R)/R}$, as in the case of free falling gas (*i.e.*, with selfgravitation and luminosity effects neglected) the rescaled speed should equal 1. As expected, when the system consists of a heavy center and a small amount of infalling matter, the speed is closer to the free-fall limit.

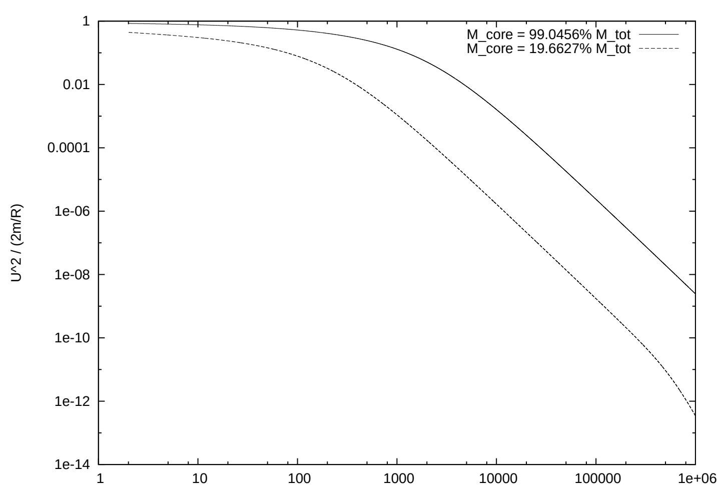

Fig. 2. The speed of the infalling gas in case of the two solutions. $\frac{U^2}{2m(R)/R} = 1$ for the freely falling gas.

We also used numerical calculations to test the assumptions regarding stationarity of the accretion. Stationary solutions were used as initial data for full time dependent equations. It turned out that for reasonably long times the exact solutions do not differ significantly from the stationary ones.

Another aspect, which is currently tested numerically, is the stability of the solutions. One can add a small initial perturbation to the stationary profiles and evolve them in time. The analysis performed so far confirms that the two branches of solution are stable [10]. This issue requires, however, deeper investigations, we also want to extend it into a two-dimensional problem.

6. Conclusions

In this short paper we showed that there may exist two radiating systems having the same mass, luminosity, asymptotic temperature and potential at the surface of the core. If the luminosity is much smaller than the critical one (the largest consistent with other observational data) those solutions correspond to a light central object surrounded by a heavy cloud of gas, and to a very heavy core on which a small amount of gas accretes. As our initial motivations was a question whether observations could distinguish between neutron stars and gravastars — the answer is: even within this simple framework those objects are observationally identical. We believe the same result will hold in case of more complicated models as well.

We tested numerically the analytical solutions and proved the validity of simplifying assumptions used in the derivation of the relative luminosity (12). It turns out that the stationary accretion is a good approximation for sufficiently long times. We also found that both of the bifurcating solutions are stable with regard to small perturbations of the initial data.

The author wishes to thank the Foundation for Polish Science for financial support. This paper has been partially supported by the Ministry of Science and Higher Education grant 1P03B 01229.

REFERENCES

[1] P.O. Mazur, E. Mottola, *Proc. Nat. Acad. Sci.* **101**, 9545 (2004).

[2] S.W. Hawking, G.F.R. Ellis, *The Large Scale Strucure of Space-Time*, Cambridge University Press, 1975.

[3] A.H. Buchdahl, *Phys. Rev.* **116**, 1027 (1957).

[4] A.E. Broderick, R. Narayan, *Astrophys. J.* **638**, L21 (2006).

[5] M. Abramowicz, W. Kluźniak, J.-P. Lassota, *Astron. Astrophys.* **396**, L31 (2002).

[6] J. Karkowski, B. Kinasiewicz, P. Mach, E. Malec, Z. Świerczyński, *Phys. Rev.* **D73**, 021503(R) (2006).

[7] J. Karkowski, E. Malec, K. Roszkowski, `astro-ph/0611393`, to appear in *Astron. Astrophys.*

[8] E. Malec, K. Roszkowski, *J. Phys.* **C66**, 012062 (2007).

[9] N.I. Shakura, *Astron. Zh.* **51**, 441 (1974).

[10] B. Rachwał, M.Sc. Thesis, Jagellonian University, 2007.