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SELFGRAVITATION AND STABILITY IN SPHERICAL ACCRETION*

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We review a steady spherical accretion of perfect fluids and show the effects of selfgravitation of the gas. The mass accretion rate is small either when most of the mass of the system is located in the center and in the contrasting case when the mass of the center is small compared to the mass of the accreting fluid. The maximal value of the accretion rate corresponds to the mass of the central black hole being equal $2/3$ of the total mass. The main focus is on the stability of selfgravitating flows. While the stability of the accretion in the test fluid case can be proven analytically, the regime of the massive, selfgravitating fluid requires a numerical analysis. Results obtained in the Newtonian approximation show stability against small and large amplitude perturbations.

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1. Introduction

The analysis of spherical accretion starts with the Newtonian investigation of Bondi [2]. This model has been promoted to the relativistic context by Michel [10] within the so-called test-fluid approximation — the motion of the fluid was considered in the fixed Schwarzschild background. The first investigation of the effects of selfgravitation of the accreting fluid was presented in the paper of Malec [9]; the more detailed analysis accompanied with direct numerical solutions has been published later by Karkowski *et al.* [4].

It turns out that selfgravity of accreting fluids can play an important role in the entire picture. Strictly speaking, there exists an upper bound on the mass accretion rate due to selfgravity. For a given accretion rate

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two quasistationary solutions appear: one corresponding to the case with a massive black hole and small amount of the accreting fluid, and the second describing an opposite case, where the mass of the black hole can be small compared to the mass of the fluid. This picture extends to systems emitting radiation. As expected, there appear two accreting regimes in the thin gas approximation [5].

Apart from astrophysical applications one can think of selfgravitating, steadily accreting solutions as providing tests for complex, general-relativistic, hydrodynamical codes. These solutions can be obtained by solving relatively simple sets of ordinary differential equations and many of their properties can be proven using analytical means.

In this talk we will discuss briefly these analytical results and then concentrate on the stability of selfgravitating, spherical accretion flows. This issue has been investigated numerically in the Newtonian approximation showing stability even in the regime characterised by a very large mass of the accreting fluid. Work is in progress to extend these simulations to the general-relativistic case.

2. Basic equations and definitions

A general, spherically symmetric space-time can be described by a line element

$$ds^2 = -N^2 dt^2 + \alpha dr^2 + R^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where lapse N, α and aerial radius R are functions of a coordinate radius r and an asymptotic time variable t.

Let us foliate this spacetime by the slices of a fixed time t. The non-zero elements of the second fundamental form of such a slice read $K_r^r = \partial_t \alpha / (2N\alpha)$, $K_\theta^\theta = K_\phi^\phi = \partial_t R/(NR)$ and thus $\mathrm{Tr} K = N^{-1} \partial_t \ln \left(\sqrt{\alpha} R^2 \right)$. Similarly the trace of the second fundamental form of a two-sphere of a constant radius r embedded in a slice of constant time t is given by $k = 2\partial_r R/(R\sqrt{\alpha})$.

We will deal with the perfect fluid, *i.e.*, one governed by the energy-momentum tensor of the form

$$T^{\mu\nu} = (p + \varrho) u^\mu u^\nu + p g^{\mu\nu},$$

where u^μ denotes the four-velocity of the fluid, p is the pressure and ϱ the energy density.

It turns out that the best choice of coordinates for the description of spherically accreting fluid is that of comoving ones, *i.e.*, such, where $u^r = u^\theta = u^\phi = 0$ (for more formal, geometric way of imposing comoving gauge consult [9]).

We will consider a ball of fluid that is falling onto a central black hole. Since we take into account the selfgravitation of the fluid, this ball has to be compact and have finite total mass (asymptotic mass for the entire spacetime). In the Bondi's or Michel's model of test-fluid accretion this cloud can formally extend to infinity. In our case we have to consider some gluing of the outer cloud boundary with the exterior Schwarzschild solution. For traditional reasons the quantities referring to the outer boundary, like aerial radius or energy density, will be still denoted with the ∞ subscript (*e.g.* R_∞, ϱ_∞).

Let us now define the following functions: $U = \partial_t R/N = R(\mathrm{Tr}K - K_r^r)/2$ playing the role of velocity and $m(R) = m_{\mathrm{tot}} - 4\pi \int_R^{R_\infty} dR' R'^2 \varrho$ being the quasilocal mass. Here m_{tot} stands for the total (asymptotic) mass of the system. The mass accretion rate is defined as $\dot{m} = (\partial_t - (\partial_t R)\partial_R) m(R)$. This can be understood as the time derivative of the mass function computed in the coordinate system obtained by the transformation $(t, r) \mapsto (t', r')$, where $t' = t$, $r' = R(t, r)$.

We will further assume that the flow of fluid is steady — all hydrodynamic quantities and the mass accretion rate should be constant at a fixed aerial radius, *i.e.*, $(\partial_t - (\partial_t R)\partial_R)X = 0$, $X = \varrho, U \ldots$. One should point here that the accretion must cause a slight growth of the central mass and, consequently, a change of some geometric quantities. Thus the notion of the steady accretion can only be approximate. In order not to change the quasilocal mass $m(R)$ too significantly the mass accretion rate should be small and the time scale short. It turns out that these conditions can indeed be satisfied.

Under the above assumptions the whole system can be described by the following set of equations: The lapse equation

$$N = \frac{kR}{k_\infty R_\infty} \exp\left(-16\pi \int_R^{R_\infty} \frac{(p + \varrho) dR'}{k^2 R'} \right),$$

the Hamiltonian constraint equation

$$Rk = \sqrt{1 - \frac{2m(R)}{R} + U^2}$$

and two equations expressing the conservation of the baryonic current nu^μ and the energy-momentum tensor $T^{\mu\nu}$ respectively

$$U = \frac{A}{R^2 n}, \qquad N = \frac{Bn}{\varrho + p}.$$

Here n denotes the baryonic density, A and B stand for integration constants.

To close the above system an equation of state should be added. We have been working with polytropic equations of state of the form $p = Kn^\Gamma$, where K and Γ are constants as well as with polytropes given by $p = K\varrho^\Gamma$. These are essentially distinct equations of state, both present in astrophysical and relativistic literature. Many of the following results can be also generalised to barotropic equation of state, where p can be an almost arbitrary function of density [6].

3. Sonic point

The sonic point is defined as a location, where $|U| = kRa/2$. Here a denotes the local speed of sound given by the formula $a^2 = dp/d\varrho$. In the Newtonian limit this definition coincides with the standard condition $|U| = a$; in both theories (Newtonian and relativistic) precise information about parameters of the sonic point can provide important characteristics of the accretion flow. To give an example, let us restrict for the moment to the test-fluid approximation (*i.e.* Michel's model). The condition for that case can be written as

$$4\pi \int\limits_{R>2m_{\mathrm{tot}}} dR' R'^2 \varrho \ll m_{\mathrm{tot}} \,.$$

If we additionally assume polytropic equation of state of the form $p = Kn^\Gamma$, then precisely one sonic point exists and the value of the local speed of sound at this point reads[1]

$$\begin{aligned} a_*^2 \;=\; & \frac{1}{9}\Bigg\{6\Gamma - 7 + 2(3\Gamma - 2)\cos\left[\frac{\pi}{3} + \frac{1}{3}\arccos\Bigg\{\frac{1}{2(3\Gamma-2)^3}\right. \\ & \left.\times\Big(54\Gamma^3 - 351\Gamma^2 - 558\Gamma + 486(\Gamma-1)a_\infty^2 - 243a_\infty^4 - 259\Big)\Bigg\}\right]\Bigg\}. \end{aligned}$$

The symbol with an asterisk has been used here to denote a value referring to the sonic point (this convention will be followed in the rest of the paper). Now, accretion rate can be generally written as

$$\dot{m} = -4\pi N R^2 U(p+\varrho) = -4\pi AB \,.$$

Evaluating this expression at the sonic point we can finally arrive at the formula

$$\dot{m} = \pi n_\infty m_{\mathrm{tot}}^2 \frac{\Gamma - 1}{\Gamma - 1 - a_\infty^2}\left(\frac{1+3a_*^2}{a_*^2}\right)^{\frac{3}{2}}\left(\frac{a_*^2}{a_\infty^2}\frac{\Gamma - 1 - a_\infty^2}{\Gamma - 1 - a_*^2}\right)^{\frac{1}{\Gamma-1}} \,,$$

[1] It is a curious fact connected with the test-fluid accretion model and the equation of state $p = Kn^\Gamma$, that the speed of sound at the sonic point can be expressed by an exact, analytical formula. It is not so in the $p = K\varrho^\Gamma$ case.

which, together with the expression for a_*^2, gives the accretion rate solely in terms of the asymptotic density n_∞ and the asymptotic speed of sound a_∞. This result can be also used to estimate $\dot{m}$ for a more general class of barotropic equations of state.

4. Selfgravitating *versus* test-fluid flow

The effects of selfgravitation of the accreting fluid are best seen when contrasted with the simplified test-fluid model. Concerning the quantities characterising the sonic point, one can show that some of them remain the same in both models but some differ. This result can be formulated as follows [4, 7].

Consider two models with the same polytropic equation of state, say $p = K\varrho^\Gamma$ and same asymptotic data R_∞, a_∞^2, $m_{\rm tot} = m(R_\infty)$. One of them should be computed taking into account selfgravity of the accreting fluid while for the second simplified formulae of the test-fluid approximation are used. For these two models the following sonic point parameters are the same: a_*^2, U_* and $m(R_*)/R_*$. The masses contained within the sonic point $m(R_*)$ differ.

It can also be shown that $m(R_*)$ changes linearly with ϱ_∞, *i.e.*, $m(R_*) = m_{\rm tot} - \gamma\varrho_\infty$ [4]. The constant γ appearing here is determined practically only by R_∞. The accretion rate can be written as

$$\dot{m} = -4\pi m(R_*)^2 \varrho_\infty \frac{R_*^2}{m(R_*)^2} U_* \left(\frac{a_*}{a_\infty}\right)^{\frac{2}{\Gamma-1}} \left(1 + \frac{a_*^2}{\Gamma}\right).$$

Here the whole dependence on ϱ_∞ is contained in the single term $m(R_*)^2\varrho_\infty$. One can show [4] that there exist a maximum of $\dot{m}$ at $m(R_*) = 2m_{\rm tot}/3$ and $\dot{m} \to 0$ for $\varrho_\infty \to 0$ and $m(R_*)/m_{\rm tot} \to 0$. The quantity $m(R_*)$ is almost equal to the mass of the central black hole $m_{\rm BH}$ — only a very small amount of mass is contained in the shell between apparent horizon of the black hole and the sonic point sphere.

A numerical example confirming these analytical results can be seen on Fig. 1. Clearly, for a given accretion rate and fixed asymptotic mass there exist two different models with different mass contained in the fluid zone $m_{\rm fluid} = m_{\rm tot} - m_{\rm BH}$. This also means that estimating the accretion rate basing on the test-fluid formulae may become highly inaccurate.

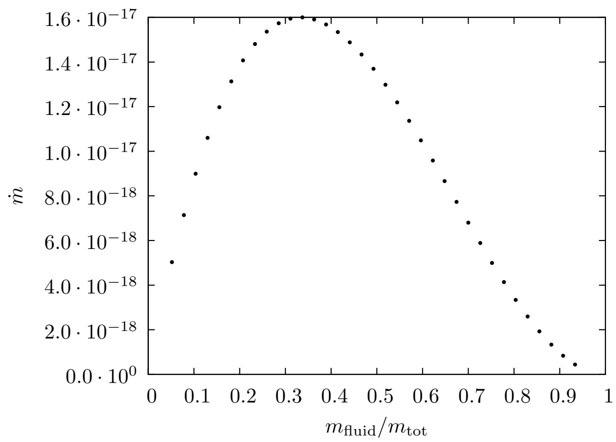

Fig. 1. The dependence of the accretion rate $\dot{m}$ on the ratio $m_{\rm fluid}/m_{\rm tot}$, where $m_{\rm fluid}$ denotes the mass contained in the fluid region.

5. Stability of the selfgravitating accretion flows

The stability of Bondi accretion had been first investigated by Balazs [1]. His Lagrangian approach was in principle correct but inconclusive, because of a too restrictive understanding of linearised stability. The first proof of stability of spherically symmetric accretion in the test-fluid approximation was given by Moncrief [11] both for Newtonian and relativistic case. The Eulerian version of method used by Moncrief was especially designed to work with test-fluid cases. It is, however, possible to reproduce his result using Lagrangian approach as well. To demonstrate this, let us restrict ourselves to technically simpler, Newtonian case.

The key equations governing the flow can be written as

$$\partial_t U + U\partial_R U = -\frac{\partial_R p}{\varrho} - \frac{m(R)}{R^2}, \tag{1}$$

$$\partial_t \varrho = -\frac{1}{R^2}\partial_R \left(R^2 \varrho U\right). \tag{2}$$

Here U stands for radial velocity component while all other quantities retain their hitherto meaning. We will also distinguish between Eulerian coordinate R and comoving radius r as before.

Let us introduce $\zeta(r,t) = \Delta R(r,t)$ being deviation from the particle position in the unperturbed flow. The perturbation of the velocity reads $\Delta U = \partial_t^L = (\partial_t + U\partial_R)\zeta$, where the symbol $\partial_t^L = \partial_t + U\partial_R$ denotes the derivative with respect to the Lagrangian time. Perturbation of the density

$$\Delta\varrho = -\varrho\left(\frac{2\zeta}{R} + \partial_R\zeta\right)$$

follows from the continuity equation (2) (it can be also obtained immediately from $\Delta m(R(r = \text{const.})) = 0$).

The equation for ζ can be derived from equation (1)

$$\left(\partial_t^L\right)^2\zeta = \frac{2m(R)\zeta}{R^3} + \frac{1}{\varrho}\partial_R\left(a^2\varrho\left(\partial_R\zeta + \frac{2\zeta}{R}\right)\right) - \frac{2\zeta}{\varrho}\partial_R p\,. \tag{3}$$

Balazs tried to find solutions of (3) in the form $\zeta(R(r), t) = \exp(i\omega t)\tilde{\zeta}(R(r))$, where ω^2 is positive and modulus $\tilde{\zeta}(R(r))$ is time independent. Unfortunately this strategy fails. Instead, one can define an energy

$$E = \int\limits_V dV\varrho\left(\frac{1}{2}(\partial_t\zeta)^2 + \frac{1}{2}(\partial_R\zeta)^2\left(a^2 - U^2\right) + \frac{\zeta^2}{R^2}\left(a^2 - \frac{m}{R} - R\partial_R a^2\right)\right),$$

where V is an annulus between R_* and R_∞. For small perturbations (we are dealing with linearised theory) it suffices to consider the region outside sonic radius as all perturbations in the supersonic zone should be eventually washed out by the central black hole.

In order to show stability we will compute the time derivative of E hoping for $\partial_t \leq 0$. If additionally E were proven to be positive definite, we could exclude long-term exponential growth of the linear modes ζ.

The first task can be accomplished easily. Time derivative of the energy E can be computed to yield

$$\begin{aligned}\partial_t E = &- \int\limits_V dV\zeta^2\frac{\partial_t m(R)}{R^3}\\ &+ 4\pi\left[R^2\varrho\left(\partial_t\zeta\partial_R\zeta\left(a^2 - U^2\right) - U\left(\partial_t\zeta\right)^2\right)\right]_{R_*}^{R_\infty}\,.\end{aligned} \tag{4}$$

A careful inspection of the boundary terms appearing on the right-hand side of (4) leads to the conclusion that indeed $\partial_t E \leq 0$.

The question of the positivity of the energy E is slightly more subtle. The identity

$$\begin{aligned}\varrho\left(a^2 - \frac{m}{R} - R\partial_R a^2\right) = &- \partial_R\left[\varrho R\left(a^2 - \frac{m}{2R}\right)\right]\\ &+ \frac{2\varrho}{a^2 - U^2}\left(a^2 - \frac{m}{2R}\right)^2 - 2\pi R^2\varrho\end{aligned}$$

can be used to show that

$$E = \tilde{E} - \left[4\pi R\zeta^2 \varrho \left(a^2 - \frac{m}{2R}\right)\right]_{R_*}^{R_\infty} , \tag{5}$$

where we have introduced

$$\tilde{E} = \frac{1}{2}\int\limits_V dV \left(X^2 + Y^2\right) - 2\pi \int\limits_V dV \zeta^2 \varrho^2 . \tag{6}$$

Symbols X and Y are used here to abbreviate the following expressions

$$\begin{aligned} X &= \sqrt{\varrho}\partial_t \zeta , \\ Y &= \sqrt{\varrho}\left(\frac{2a^2R - m}{R^2\sqrt{a^2 - U^2}}\zeta + \sqrt{a^2 - U^2}\partial_R \zeta\right). \end{aligned}$$

Although the boundary terms in (5) are nonnegative, the last term in (6) is strictly negative. It appears solely due to the selfgravity of the fluid. Thus, as long as we are restricting ourselves to the test-fluid case, E becomes positive definite and, in accordance with Moncrief, we conclude that the flow is stable. The selfgravitating case, however, remains still open.

The reader interested in the details of the above analysis can consult [7].

6. Numerical analysis of stability

The problems signalled in the preceding section led us to numerical investigations of the stability of selfgravitating accretion. Preliminary results that we present here suggest stability also in the selfgravitating regime, despite difficulties encountered when trying to solve the problem by analytical means.

In order to investigate the stability of the selfgravitating accretion we have restricted ourselves to the Newtonian case and adapted the Prometheus code by B. Fryxell and E. Mueller to our purposes. Prometheus is a general hydrodynamical code based on the modern high-resolution, shock-capturing PPM scheme which has originated in the work of Collela and Woodward [3]. It has been extensively used for simulating such astrophysical phenomena as supernova explosions; it is capable of simulating selfgravitating flows in both spherically and axially symmetric cases.

As the initial data we have used solutions to the Newtonian equations for the steady flow that include selfgravitation of the accreting fluid. An equation of state was a polytrope $p = K\varrho^\Gamma$. Some point mass was assumed to exist at $R = 0$ to imitate the central black hole. At the inner boundary the outflow conditions were assumed, while the distant outer boundary was kept fixed using values obtained from the initial solution (inflow boundary). The latter turned out to be necessary — setting the numerical outflow conditions also at the outer boundary can lead to instabilities even in the test-fluid case.

Such solution can be evolved in a stable way leading to small inaccuracies at the level of the code precision. In order to investigate the stability of the flow, an initial perturbation was applied to the velocity field.

This can be done in the spherically symmetric case, axial symmetry or in full three spatial dimensions, depending on computational resources. Computations discussed here have been performed in spherical symmetry, however preliminary results obtained for axially symmetric perturbations lead to the same conclusions [8].

Fig. 2 shows snapshots of the temporal evolution of the density profile. The outer boundary of the cloud was assumed to be located at $R_\infty = 2\times 10^6$, the central mass was set to $m_{\rm BH} = 3\times 10^3$ (gravitational units are used here; in the Newtonian case this corresponds to setting $G = 1$). The ratio $m_{\rm BH}/m_{\rm tot} = 3\%$. The accretion mass rate is so small that the change of the central mass could be neglected during the entire evolution. Other parameters of the unperturbed flow was as follows: $\varrho_\infty = 3\times 10^{-15}$, $U_\infty = -4\times 10^{-5}$; the parameters of the polytropic equation of state are: $K = 5\times 10^4$, $\Gamma = 1.4$.

For the results illustrated in Fig. 2 the initial perturbation was chosen to be a bell-shape profile of a sine function restricted to one half of its period. Such perturbation, initially located outside the sonic radius, produces two signals: one travelling outwards and one towards the centre. If the initial amplitude of the perturbation is very small, the signal propagating inwards passes through the sonic point and eventually disappears in the inner boundary (it falls into the central body). For large initial perturbations the situation is different. A shocked, discontinuous solution can develop due to the perturbation and we can observe some reflection of the signal that was initially propagating inwards. In both cases the behaviour is stable. The amplitudes of the perturbations decrease and after sufficient time we are left with the initial solution. This can be demonstrated for both accretion regimes: solutions with a heavy point mass in the centre and relatively small amount of accreting fluid and solutions with relatively small central mass as compared to the mass of the fluid.

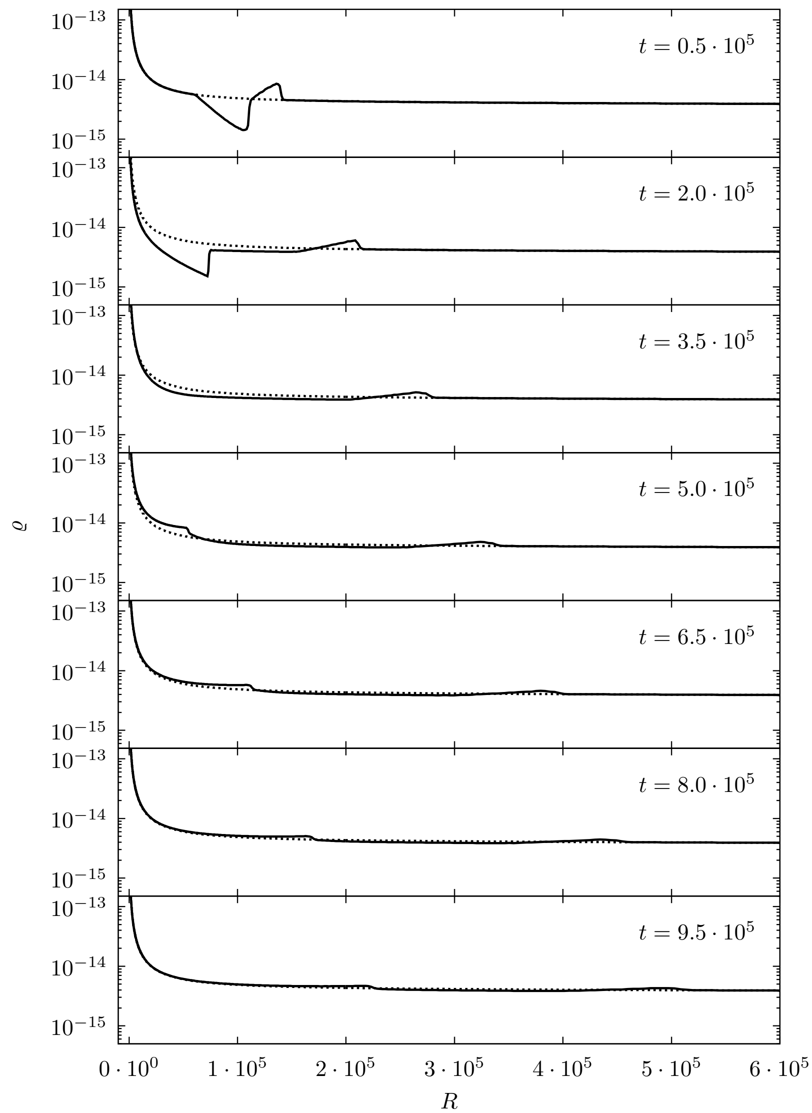

Fig. 2. Evolution of the perturbed density. The snapshots are placed in chronological order. The profile corresponding to the unperturbed flow is depicted with a dotted line.

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REFERENCES

[1] N.L. Balazs, *Mon. Not. R. Astron. Soc.* **160**, 79 (1972).

[2] H. Bondi, *Mon. Not. R. Astron. Soc.* **112**, 195 (1952).

[3] P. Colella, P.R. Woodward, *J. Comput. Phys.* **54**, 174 (1984).

[4] J. Karkowski, B. Kinasiewicz, P. Mach, E. Malec, Z. Świerczyński, *Phys. Rev.* **D73**, 021503(R) (2006).

[5] J. Karkowski, E. Malec, K. Roszkowski, to appear in *Astron. and Astrophys.* (2007).

[6] B. Kinasiewicz, P. Mach, *Acta Phys. Pol. B* **38**, 39 (2007).

[7] B. Kinasiewicz, P. Mach, E. Malec, *International Journal of Geometric Methods in Modern Physics* **4**, 197 (2007).

[8] P. Mach, E. Malec, in preparation.

[9] E. Malec, *Phys. Rev.* **D60**, 104043 (1999).

[10] F.C. Michel, *Astrophys. Space Sci.* **15**, 153 (1972).

[11] V. Moncrief, *Astrophys. J.* **235**, 1038 (1980).