TRANSITION OF AN EXTENDED OBJECT ACROSS THE COSMOLOGICAL SINGULARITY*

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We summarize our results concerning p-brane propagation through the singularity of the compactified Milne (CM) space. In particular, we present classical and quantum dynamics of a string. We also present our preliminary results on the propagation of a membrane. The CM space seems to be a promising model of the neighborhood of the cosmological singularity deserving further examinations.

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1. Introduction

In the paper we address, to some extent, the question of mathematical consistency of a cyclic universe scenario. Such scenario enables to construct a promising framework that can be used to describe possibly all available cosmological data. The cyclic universe scenario postulates that:

- evolution of the universe is a sequence of quantum and classical phases,
- each quantum phase can be described in terms of quantum p-branes propagating in higher dimensional ($d > 4$) spacetime with the cosmic singularity,
- each classical phase can be obtained from quantum phase by changing topology of its spacetime, and *vice versa*.

We restrict our considerations to the neighborhood of the cosmological singularity. The basic criterion for the choice of the model of universe in the quantum phase is that a reasonable model should allow for propagation of a quantum p-brane (*i.e.*, particle, string, membrane, ...) from

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the pre-singularity to post-singularity epoch. If a quantum p-brane cannot go through the cosmic singularity, the evolution cannot be realized. As a model of the universe in the quantum phase we choose the compactified Milne (CM) space. It is the simplest model of the universe with the cosmic singularity that is implied by string/M theory (and the simplest example of a time dependent singular orbifold).

2. Model of the universe with a cosmic singularity

We begin with the visualization of $2d$ compactified Milne space. It is given by the isometric embedding of $2d$ CM space into $3d$ Minkowski space

$$y^0(t,\theta) = t\sqrt{1+r^2}\,, \qquad r \in R^1\,, \tag{1a}$$
$$y^1(t,\theta) = rt\sin(\theta/r)\,, \tag{1b}$$
$$y^2(t,\theta) = rt\cos(\theta/r)\,. \tag{1c}$$

The surface defined by the equation $[r^2/(1+r^2)](y^0)^2 - (y^1)^2 - (y^2)^2 = 0$ is presented in Fig. 1, and the induced metric on it reads

$$ds^2 = -dt^2 + t^2 d\theta^2\,, \qquad (t,\theta) \in R^1 \times S^1\,. \tag{2}$$

It is locally isometric with $2d$ Minkowski space (except for $t = 0$)

$$ds^2 = -(dx^0)^2 + (dx^1)^2\,, \qquad x^0 := t\cosh\theta\,, \quad x^1 := t\sinh\theta\,. \tag{3}$$

A line element of the $d+1$ dimensional compactified Milne space reads

$$ds^2 = -dt^2 + dx^k dx_k + t^2 d\theta^2\,, \qquad \left(t, x^k\right) \in R^1 \times R^{d-1}\,, \quad \theta \in S^1\,. \tag{4}$$

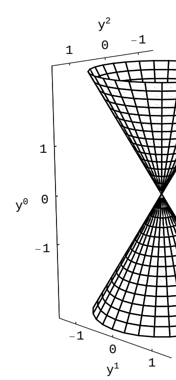

Fig. 1. Compactified $2d$ Milne space in $3d$ Minkowski space ($r = 0.7$).

For $t = 0$ one term in the metric (4) disappears so the CM space may be used to model a big-crunch/big-bang type of the cosmological singularity.

The compactified Milne space is not a manifold, but an orbifold due to the vertex at $t = 0$. All Riemann's tensor components vanish for $t \neq 0$. The singularity at $t = 0$ is of removable type: any time-like geodesic with $t < 0$ can be extended to some time-like geodesic with $t > 0$. The extension cannot be unique due to the Cauchy problem at $t = 0$ for the geodesic equation (compact dimension shrinks away and reappears at $t = 0$).

Orbifolding S^1 to the segment S^1/Z_2 gives the model of the two flat parallel "end of the world" branes [1] which collide and re-emerge at $t = 0$.

3. Classical dynamics of a p-brane

The Polyakov action integral for a test p-brane embedded in a fixed background spacetime with metric $g_{\mu\nu}$ reads

$$S_p = -\frac{1}{2}\mu_p \int d^{p+1}\sigma\sqrt{-\gamma}\;\left[\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - p + 1\right] , \tag{5}$$

where μ_p is mass per unit p-volume, $(\sigma^a) \equiv (\sigma^0, \sigma^1, \ldots, \sigma^p)$ are p-brane world-volume coordinates, γ_{ab} is p-brane world-volume metric, $\gamma := det[\gamma_{ab}]$, $(X^\mu) \equiv (T, X^k, \Theta) \equiv (T, X^1, \ldots, X^{d-1}, \Theta)$ are embedding functions of p-brane, *i.e.* $X^\mu = X^\mu(\sigma^0, \ldots, \sigma^p)$, corresponding to $(t, x^1, \ldots, x^{d-1}, \theta)$ directions of $d+1$ dimensional background spacetime.

The total Hamiltonian, H_{T}, corresponding to the Polyakov action has the form [2]:

$$H_{\mathrm{T}} \;=\; \int d^p\sigma \mathcal{H}_{\mathrm{T}} \,, \tag{6a}$$

$$\mathcal{H}_{\mathrm{T}} \;:=\; AC + A^i C_i \,, \qquad i = 1, \ldots, p \,, \tag{6b}$$

where $A = A(\sigma^a)$ and $A^i = A^i(\sigma^a)$ are any "regular" functions. C and C_i are the first-class constraints

$$C \;:=\; \Pi_\mu \Pi_\nu g^{\mu\nu} + \mu_p^2 \det\left[\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}\right] \approx 0 \,, \tag{7a}$$

$$C_i \;:=\; \partial_i X^\mu \Pi_\mu \approx 0 \,. \tag{7b}$$

Thus, H_{T} does not generate time translations, but only gauge transformations.

Hamilton's equations are

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} = \{X^\mu, H_{\mathrm{T}}\} \,, \qquad \dot{\Pi}_\mu \equiv \frac{\partial \Pi_\mu}{\partial \tau} = \{\Pi_\mu, H_{\mathrm{T}}\} \,, \qquad \tau \equiv \sigma^0 \,, \tag{8}$$

where

$$\{\cdot,\cdot\} := \int d^p\sigma \Big(\frac{\partial \cdot}{\partial X^\mu} \frac{\partial \cdot}{\partial \Pi_\mu} - \frac{\partial \cdot}{\partial \Pi_\mu} \frac{\partial \cdot}{\partial X^\mu} \Big) \,. \tag{9}$$

The degrees of freedom read

$$n_\mathrm{c} =: 2n_\mathrm{p} = 2(d-p)\,,$$

where n_c = number of independent canonical variables, n_p = number of physical degrees of freedom, $d+1$ = dimension of spacetime, $p+1$ = number of constraints, p = dimension of p-brane.

4. Dynamics of a particle (0-brane)

Classical dynamics of a test particle in the CM space is unstable, however it can be quantized, *i.e.* there exists mathematically well defined quantum dynamics of a particle. For more details see [3, 4].

5. Dynamics of a string (1-brane)

5.1. Classical dynamics of a string

We consider dynamics of a string winding around the θ-dimension in its lowest energy mode. The string in such a state is defined by the conditions

$$\sigma^p := \theta \equiv \Theta \qquad \text{and} \qquad \partial_\theta X^\mu = 0 = \partial_\theta \Pi_\mu \,. \tag{10}$$

In the mode (10) the constraints (7) read

$$C = \Pi_\mu(\tau)\, \Pi_\nu(\tau)\, \eta^{\mu\nu} + \breve{\mu}_1^2\, t^2(\tau) \approx 0\,, \qquad C_1 = 0\,, \tag{11}$$

where $\breve{\mu}_1 \equiv \theta_0 \mu_1$, and where $\theta_0 = 2\pi$ for S^1 and $\theta_0 = \pi$ for S^1/Z_2 compactifications, respectively. The equations of motion are

$$\dot{\Pi}_t(\tau) \;=\; -2A(\tau)\, \breve{\mu}_1^2\, T(\tau)\,, \qquad \dot{\Pi}_k(\tau) = 0\,, \tag{12}$$

$$\dot{T}(\tau) \;=\; -2A(\tau)\, \Pi_t(\tau)\,, \qquad \dot{X}^k(\tau) = 2A(\tau)\, \Pi_k(\tau)\,, \tag{13}$$

where $A = A(\tau)$ is any function.

In the gauge $A(\tau) = 1$, the solutions are

$$\Pi_t(\tau) \;=\; b_1 \exp(2\breve{\mu}_1\tau) + b_2 \exp(-2\breve{\mu}_1\tau)\,, \quad \Pi_k(\tau) = \Pi_{0k}\,, \tag{14}$$

$$T(\tau) \;=\; a_1 \exp(2\breve{\mu}_1\tau) + a_2 \exp(-2\breve{\mu}_1\tau)\,, \quad X^k(\tau) = X_0^k + 2\Pi_{0k}\tau\,, \tag{15}$$

where $b_1, b_2, \Pi_{0k}, a_1, a_2, X_0^k \in R$. Elimination of τ leads finally to

$$X^k(t) = X_0^k + \frac{\Pi_0^k}{\breve{\mu}_1} \sinh^{-1} \left(\frac{\breve{\mu}_1}{\sqrt{\Pi_0^k \Pi_{0k}}}\, t \right) , \tag{16}$$

where $t(\tau) \equiv T(\tau)$ plays the role of an evolution parameter. The solution (16) is smooth at $t = 0$, and describes stable propagation of a string across the cosmic singularity.

5.2. Quantum dynamics of a string

In the gauge $A = 1$, the Hamiltonian of a string is

$$H_{\mathrm{T}} = C = \Pi_\mu(\tau)\, \Pi_\nu(\tau)\, \eta^{\mu\nu} + \breve{\mu}_1^2\, t^2 . \tag{17}$$

The quantum Hamiltonian corresponding to (17) has the form

$$\hat{H}_{\mathrm{T}} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^k \partial X_k} + \breve{\mu}_1^2 t^2 , \qquad t \equiv T \tag{18}$$

(we use the Laplace–Beltrami mapping). According to Dirac's quantization method physical states ψ should satisfy the equation

$$\hat{H}_{\mathrm{T}}\, \psi(t, X^k) = 0 . \tag{19}$$

Eq. (19) has the form of the Klein–Gordon equation. Due to this analogy we interpret t as an evolution parameter in our quantum description. To solve (19) we make the substitution

$$\psi(t, X^1, \ldots, X^{d-1}) = F(t)\, G_1(X^1)\, G_2(X^2) \ldots G_{d-1}(X^{d-1}) , \tag{20}$$

which turns (19) into the following set of equations

$$\frac{d^2 G_k(q_k, X_k)}{d X_k^2} + q_k^2\, G_k(q_k, X_k) \;=\; 0 , \qquad k = 1, \ldots, d-1 , \tag{21}$$

$$\frac{d^2 F(q,t)}{dt^2} + (\breve{\mu}_1^2 t^2 + q^2)\, F(q,t) \;=\; 0 , \qquad q^2 := q_1^2 + \ldots + q_{d-1}^2 , \tag{22}$$

where $q_k^2, q^2 \in R$ are the separation constants.

Two independent solutions to (21) have the form

$$G_{1k}(q_k, X_k) = \cos(q_k X^k) , \qquad G_{2k}(q_k, X_k) = \sin(q_k X^k) , \tag{23}$$

(no summation in $q_k X^k$ with respect to k).

Two independent solutions of (22) read

$$\tilde{F}_1(q,t) \;=\; \exp\left(\frac{-i\breve{\mu}_1 t^2}{2}\right) H\Big(-\frac{\breve{\mu}_1+iq^2}{2\breve{\mu}_1}, (-1)^{1/4}\,\sqrt{\breve{\mu}_1}\,t\Big)\,, \tag{24}$$

$$F_2(q,t) \;=\; \exp\left(\frac{-i\breve{\mu}_1 t^2}{2}\right) {}_1F_1\Big(\frac{\breve{\mu}_1+iq^2}{4\breve{\mu}_1}, \frac{1}{2}, i\breve{\mu}_1 t^2\Big)\,, \tag{25}$$

where $H(a,t)$ is the Hermite function and ${}_1F_1(a,b,t)$ denotes the Kummer function.

Now we construct the Hilbert space, $\mathcal{H}$, based on the solutions (23)–(25). We demand that the solutions are bounded functions on $R \times [-t_0, t_0]$. The function $F_2(q,t)$ is bounded, whereas $\tilde{F}_1(q,t)$ blows up as $|q| \rightarrow \infty$. Therefore, we use the replacement

$$F_1(q,t) := \sqrt{q}\,\exp\left(-\frac{\pi}{8\breve{\mu}_1}q^2\right)\tilde{F}_1(q,t)\,. \tag{26}$$

Fig. 2 illustrates the solutions F_1 and F_2.

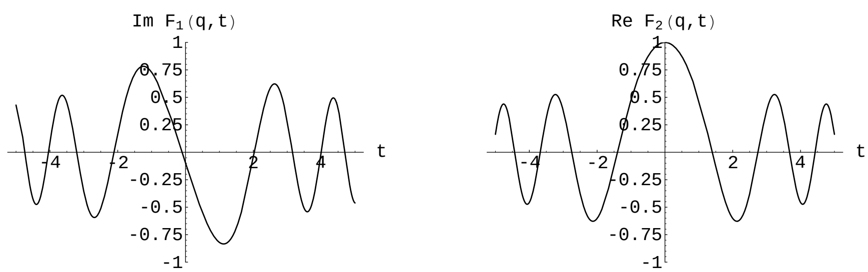

Fig. 2. Example of two independent bounded solutions to Eq. (22), for $q=1$, on $[-t_0, t_0]$.

We introduce generalized solutions by

$$h_s(t,\vec{X}) := \int\limits_{R^{d-1}} f(q_1,\ldots,q_{d-1}) F_s(q,t) \prod_k \exp(-iq_k X^k) dq_1 \ldots dq_{d-1}\,, \tag{27}$$

where $f \in L^2(R^{d-1}), s=1,2$ and where $q^2 = q_1^2 + \ldots q_{d-1}^2, (\vec{X}) \equiv (X^1, \ldots, X^{d-1})$. Eq. (27) includes (23) due to the term $\exp(-iq_k X^k)$, with $q_k \in R$. One can verify that $\hat{H}_{\rm T} h_s = 0$.

Eq. (27) defines a modified Fourier transform of the product fF_s. Thus, due to the Fourier transform theory it defines the mapping

$$L^2\left(R^{d-1}\right) \ni f \longrightarrow h_s \in L^2\left([-t_0, t_0] \times R^{d-1}\right)\,. \tag{28}$$

Replacing f by consecutive elements of a basis in $L^2(R^{d-1})$ creates, roughly speaking, a basis in the Hilbert space $\mathcal{H} \subset L^2([-t_0, t_0] \times R^{d-1})$.

Example: $L^2(R^{d-1}) := \bigotimes_{k=1}^{d-1} L_k^2(R)$, where $L_k^2(R) \equiv L^2(R)$ with the basis $f_n \in L^2(R)$ defined as

$$f_n(q) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp\left(\frac{-q^2}{2}\right) H_n(q)\,, \qquad n = 0, 1, 2, \dots\,, \tag{29}$$

where $H_n(q)$ is the Hermite polynomial. The orthonormal basis (29) can be used to define a sequence of vectors $\bigotimes_{k=1}^{d-1} f_{n_k}(q^k) \in L^2(R^{d-1})$, and further used to create a sequence of vectors in $\mathcal{H} \subset L^2([-t_0, t_0] \times R^{d-1})$, owing to (28). Obtained set of vectors can be used to build another set of independent vectors by a standard method, and turned into an orthonormal basis by making use of the Gram–Schmidt procedure. Completion of the span of such an orthonormal basis defines the Hilbert space $\mathcal{H} \subset L^2([-t_0, t_0] \times R^{d-1})$. For more details see [5].

6. Dynamics of a membrane (2-brane)

The physical phase space of a membrane (in the zero-mode, winding around the θ-dimension) is defined by the constraints

$$C \;=\; \mathit{\Pi}_\mu(\tau,\sigma)\, \mathit{\Pi}_\nu(\tau,\sigma)\, \eta^{\mu\nu} + \kappa^2\, t^2(\tau,\sigma) \acute{X}^\mu(\tau,\sigma) \acute{X}^\nu(\tau,\sigma)\, \eta_{\mu\nu} \approx 0\,, \tag{30}$$

$$C_1 \;=\; \acute{X}^\mu(\tau,\sigma)\, \mathit{\Pi}_\mu(\tau,\sigma) \approx 0\,, \qquad C_2 = 0\,, \tag{31}$$

where $\acute{X}^\mu := \partial X^\mu / \partial\sigma, \sigma \equiv \sigma^1$, and where $\kappa \equiv \pi\mu_2$. For some states of a membrane the expressions for C and C_1 are well defined [6]. To examine the algebra of constraints we "smear" the constraints as follows

$$\check{A} := \int_0^\pi d\sigma\; f(\sigma) A(\tau,\sigma)\,, \qquad f \in C_0^\infty[0,\pi]\,. \tag{32}$$

The Lie bracket is defined as

$$\{\check{A}, \check{B}\} := \int_0^\pi d\sigma \left(\frac{\partial \check{A}}{\partial X^\mu} \frac{\partial \check{B}}{\partial \mathit{\Pi}_\mu} - \frac{\partial \check{A}}{\partial \mathit{\Pi}_\mu} \frac{\partial \check{B}}{\partial X^\mu}\right). \tag{33}$$

Constraints in an integral form satisfy the algebra

$$\{\check{C}(f_1), \check{C}(f_2)\} = \int_0^\pi d\sigma\; (f_1 \acute{f}_2 - \acute{f}_1 f_2) 4\kappa^2 t^2(\tau,\sigma) C_1(\tau,\sigma)\,, \tag{34}$$

$$\{\check{C}_1(f_1), \check{C}_1(f_2)\} = \int\limits_0^\pi d\sigma\, (f_1 \acute{f}_2 - \acute{f}_1 f_2) C_1(\tau,\sigma)\,, \tag{35}$$

$$\{\check{C}(f_1), \check{C}_1(f_2)\} = \int\limits_0^\pi d\sigma\, (f_1 \acute{f}_2 - \acute{f}_1 f_2) C(\tau,\sigma)\,. \tag{36}$$

Quantization of the dynamics of a membrane means finding an essentially self-adjoint representation of this algebra on a dense subspace of a Hilbert space. However, the "structure constant", t^2, is not a constant, but a function on the phase space. Little is known about representations of such type of an algebra. Work is in progress.

7. Conclusions

The results we have obtained demonstrate that:

- classical dynamics of a particle can be quantized despite the fact that it is unstable,
- dynamics of a string in the zero-mode of winding string is well defined both at classical and quantum levels,
- quantizing dynamics of a membrane appears to be a challenge.

As our next steps we plan:

- quantization of dynamics of a membrane,
- obtaining classical phase from quantum phase, and *vice versa*,
- quantization of CM space (by making use of LQG methods) to see what type of singularity we are dealing with: Is it a big-crunch/big-bang (change of spacetime dimension) or big-bounce (no change of dimensionality of spacetime),
- making predictions for the CMB polarization spectra: tensor-to-scalar ratio and spectral index of the scalar perturbations, to compare with cosmological observations to be done by Planck, BPol, Spider and Polatron missions.

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