CURIOUS SPACETIME SINGULARITIES*

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This paper briefly discusses the nature of curvature singularities in asymptotically flat van Stockum spacetimes.

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1. Introduction

In this paper we address the issue of the nature of curvature singularities in asymptotically flat van Stockum spacetimes. The spacetimes were first considered by van Stockum [1]. These are stationary and cylindrically symmetric spacetimes of dust moving along orbits of the time translation Killing vector field. Such flow is rigid and its angular velocity of rotation with respect to distant stationary observers ("fixed stars") is identically zero as if elements of the fluid were "frozen into" background. Despite the fact, vorticity scalar of the flow which measures local rotation of the fluid is a non-vanishing function on spacetime. It is proportional to proper energy density of dust. One may view the motion of the fluid as resulting from differential dragging of locally non-rotating observers who provide local standards of rest. In other words, space warps about symmetry axis of the system due to presence of angular momentum, thus effective motion of the fluid relative to space on circular orbits is still possible. The only Newtonian limit of the flow is empty flat spacetime.

Without a careful analysis a class of asymptotically flat solutions of van Stockum flow may be wrongly interpreted as globally regular. Bonnor's solution found in [2] is an example of such a solution. Bonnor pointed out that the spacetimes should have zero total mass. It was shown later in [3] that all asymptotically flat van Stockum spacetimes contain singularities

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of curvature of which support has measure zero. Active masses of these singularities are negative and balance positive masses distributed smoothly in other regions so as total mass of these spacetimes is zero.

What is the nature of these singularities? A curvature singularity is usually associated with diverging of some invariants built from the components of the Riemann tensor. The Schwarzschild singularity has this property since $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}=48M^2r^{-6}\to\infty$ as $r\to 0$. But it is also possible that such limits are everywhere bounded and spacetime may be still singular. For illustration let us consider a cube all of whose side walls are flat. There exists a continuous deformation that transforms the cube onto the unit sphere thereby the two figures become the same object, at least from the point of view of topology. As so, they have the same Euler characteristic χ which cannot be changed by continuous deformations. The invariant is defined by $4\pi\chi=\int_S\mathcal{K}\,dS$, where $\mathcal{K}={R^{12}}_{12}$ is the Gauss curvature of the deformed closed surfaces S. In this way we come to the conclusion that $\mathcal{K}$, which is constant on the sphere, tends to a distribution (an element of a broader class of functions containing Dirac δ function) in the limit as the sphere becomes the cube, and that the cube is almost everywhere flat except for a measure zero set were "infinite" curvature is localized.

Singularities of this kind may be easily overlooked while solving Einstein's equations. However, some integral characteristics may help us to detect their presence. In the following section a pedagogical example of this will be presented.

2. More about singularities in van Stockum spacetimes

The most general line element of van Stockum spacetime in "cylindrical" coordinates (t,ρ,ϕ,z) reads

$$ds^2=-dt^2+2\,\Pi\,(\rho,z)dtd\phi+(\rho^2-\Pi^2(\rho,z))d\phi^2+e^{2\Psi(\rho,z)}\left(d\rho^2+dz^2\right)\,.$$

Structure function Π has the interpretation of specific angular momentum, and Ψ is related to the Gauss curvature $\mathcal{K}=-e^{-2\Psi}\left(\Psi_{,\rho\rho}+\Psi_{,zz}\right)$ of two dimensional sections of constant t and ϕ with the line element $e^{2\Psi}\left(d\rho^2+dz^2\right)$.

The interesting feature of asymptotically flat van Stockum spacetimes is that their total mass is zero, however, the total angular momentum may be still nonzero. This is seen directly from the form of the above line element as asymptotical flatness implies that $\Pi\sim 2Jr^{-1}\sin^2\theta$ and $\Psi\sim\mathcal{O}(r^{-4})$ where r and θ are ordinary spherical coordinates. Another way of finding total mass is to calculate Komar mass, which for the above line element reduces to the form

$$M=\lim_{r\to\infty}\frac{1}{8\pi}\int\int\left(\frac{\Pi\,\partial_r\Pi}{r^2\sin^2\theta}\right)r^2\sin\theta d\theta d\phi\,,$$

hence $M = 0$ for $\Pi \sim \mathcal{O}(r^{-1})$. On the other hand, Einstein's equations imply that, for sufficiently smooth solutions, energy density $\mathcal{D}$ is positive

$$\mathcal{D} = e^{-2\Psi} \frac{\Pi_{,\rho}^2 + \Pi_{,z}^2}{8\pi\rho^2} \geqslant 0\,,$$

and that

$$\Psi_{,\rho} = \frac{\Pi_{,z}^2 - \Pi_{,\rho}^2}{4\rho}\,, \qquad \Psi_{,z} = -\frac{\Pi_{,\rho}\Pi_{,z}}{2\rho}\,.$$

For the equations to hold one needs also that the integrability condition

$$\Pi_{,\rho\rho} - \frac{\Pi_{,\rho}}{\rho} + \Pi_{,zz} = 0 \tag{2.1}$$

is satisfied, since second partial derivatives of Ψ should commute. Then $\mathcal{D}$ can be expressed also as

$$\mathcal{D} = -\frac{1}{2\pi} e^{-2\Psi} \left(\Psi_{,\rho\rho} + \Psi_{,zz}\right) = \frac{\mathcal{K}}{2\pi}\,. \tag{2.2}$$

This equation in conjunction with the previous expression for $\mathcal{D}$ says that Gauss curvature of surface (ρ, z) should be nonnegative.

The integrability condition is quite simple to solve exactly, and the corresponding energy density can be easily found. Among other solutions, a vast of asymptotically flat spacetimes can be constructed that may contain also closed time-like curves (coordinate ϕ is timelike if $|\Pi| > \rho$).

But how can it be possible to have simultaneously zero total mass and everywhere nonnegative proper energy density? To answer this question it should be remarked that the above reduced set of Einstein's equations was derived with the assumption that second partial derivatives of metric functions Π and Ψ were continuous. Apparently, global solutions of Einstein's equations of asymptotically flat van Stockum flow with the assumed differentiability class do not exist at all. Thus asymptotically flat van Stockum spacetimes provide a concrete example showing that sometimes formal calculations in physics may concern mathematically nonexisting objects and that sometimes mathematical pedantry in physics is indispensable.

Bellow we show that asymptotical flatness excludes the existence of star-like van Stockum spacetimes, that is, with asymptotically flat and simply connected spaces and with everywhere positive and integrable proper energy density. By integrating both sides of equation (2.2) over a hypersurface H of constant time t we get

$$0 < \int\limits_H \sqrt{-g}\mathcal{D} d\rho\, d\phi\, dz = \int\limits_{\mathbb{R}^2} d\left(\rho^2 \left[\partial_z\left(\frac{\Psi}{\rho}\right) d\rho - \partial_\rho\left(\frac{\Psi}{\rho}\right) dz\right]\right)\,,$$

where integration over azimuthal angle was carried out. By assumption Ψ has continuous second derivatives on $\mathbb{R}^2$. Then, by virtue of Stokes theorem, the surface integral on the right hand side can be transformed to a curvilinear integral over a boundary of $\mathbb{R}^2$. By asymptotic flatness $\Pi \sim \mathcal{O}(r^{-1}) \Rightarrow \Psi \sim \mathcal{O}(r^{-4})$, therefore the curvilinear integral vanishes. This in turn contradicts positivity of total mass defined by the volume integral on the left side. In this way we arrive to the conclusion that the region where the reduced set of equations is equivalent to Einstein's equations is not simply connected. Moreover, from the theory of elliptic equations on $\mathbb{R}^2$ it follows that solutions of (2.1) are twice differentiable everywhere except for a measure zero subset of $\mathbb{R}^2$ where solutions exist in generalized sense. Thus the reduced set of equations is equivalent to Einstein's equations not everywhere but almost everywhere, that is, except for a measure zero set of $\mathbb{R}^2$. Since total mass of an asymptotically flat van Stockum spacetime is zero and active mass M of its regularity regions is positive, active mass of the measure zero set (being also the set of curvature singularity) is equal to minus M. For more information concerning this issue, see [3].

There exists also another way of seeing that the negative mass singularities are present by recalling the positive mass theorem. It implies that, provided the dominant energy condition is satisfied, the only globally regular and asymptotically flat spacetime with vanishing total mass is the Minkowski spacetime. The spacetimes of concern here are not Minkowskian, therefore they must contain regions where the assumptions of the theorem are not satisfied. These are the singularities with measure zero support in asymptotically flat van Stockum spacetimes that violate the dominant energy condition.

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