ON LIE SYMMETRIES OF CERTAIN SPHERICALLY SYMMETRIC SYSTEMS IN GENERAL RELATIVITY*

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We discuss certain aspects of Lie-point symmetries in spherically symmetric systems of gravitational physics. Lie symmetries are helpful in solving differential equations. General concepts and a few examples are given: perfect fluid in shearfree motion, the conformal Weyl theory and a higher derivative gravity which is equivalent to General Relativity coupled to certain nonlinear spin-2 field theory.

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1. Introduction

Lie groups play an outstanding role in modern mathematical physics. Origins of the theory comes from developing methods for solving nonlinear differential equations by great mathematicians such as Gustav Jacobi and Sophus Lie. Nowdays the Lie symmetries method is a powerful technique for solving nonlinear ordinary (ODEs) as well as partial differential equations (PDEs), or at least for detecting integrable regimes of systems of differential equations (DEs). The method is based on a geometrical approach to a system of N-th order DEs, which defines a solution submanifold in a N-jet space representing all derivatives of the dependent variables up to N-th order.

For brevity, we do not discuss in details various methods of solving DEs by the symmetry approach, sometimes it is helpful enough to know only an analitic expression of a symmetry in order to discover an integration strategy. Our aim is to present shortly a general scheme of the Lie method

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restricted to a recipe for computation of point symmetries of ordinary differential equations. The main algorithm is illustrated with interesting examples of spherically symmetric systems.

1.1. Basic concepts

Let v be a vector field in a space of dependent and independent variables

$$v \equiv \xi(x,u)\partial_x + \sum_{j=1}^{m} \phi_j(x,u)\partial_{u_j} \,.$$

Let Σ be a system of ODEs

$$\Sigma = \{\psi_1 = 0, \psi_2 = 0 \ldots, \psi_r = 0\} \ ,$$

with one independent variable x and m dependent variables

$$u = (u_1, u_2, \ldots u_m)$$

and with derivatives up to the N-th order $\left\{u_k^{(n)}, n \le N\right\}$. Let us consider one-parameter Lie group of transformations

$$\begin{aligned} x^\star &= \Psi\left(x,u,\varepsilon\right), \\ u^\star &= \Phi\left(x,u,\varepsilon\right) \ , \end{aligned}$$

under which Σ must be invariant. The group action is infinitesimally given by

$$\begin{aligned} x^\star &= x + \varepsilon\xi(x,u) + O(\varepsilon^2)\,, \\ u_j^\star &= u_j + \varepsilon\phi_j(x,u) + O(\varepsilon^2)\,, \qquad j = 1, \ldots, m\,. \end{aligned}$$

In order to determine the functions ξ, ϕ_j we require that the previous transformation leaves invariant the space of solutions of Σ:

$$S_\Sigma = \{u : \psi_1 = 0, \psi_2 = 0 \ldots, \psi_r = 0\} \ .$$

This is equivalent to the following condition:

$$\mathrm{pr}^{(N)} v(\psi_i)|_\Sigma \ = \ 0 \qquad \text{for all} \qquad i = 1, \ldots, r\,, \tag{1}$$

$$\text{where} \quad \mathrm{pr}^{(N)} v \ = \ \xi\partial_x + \sum_{k=1}^{m} \phi_k \partial_{u_k} + \sum_{k=1}^{m}\sum_{n=1}^{N} \phi_k^{[n]} \partial_{u_k^{(n)}} \tag{2}$$

is a N-th prolongation of the vector field v (an extension of v to the N-jet space) [1,2] and $\phi_k^{[n]}$ is defined recursively by

$$\phi_k^{[n+1]} = D_x \phi_k^{[n]} - u_k^{(n+1)} D_x \xi \,, \tag{3}$$

where D_x is the total derivative with respect to x. The condition (1) yields a system of linear partial differential equations for ξ, ϕ_j, called a determining equations system, which has to be satisfied for all solutions of Σ. Sometimes such a system appears to be easier to solve then the original one. The above steps form an algorithmic procedure which may be successfully implemented in a computer algebra system.

The main application of Lie-point symmetries is searching for exact solutions or testing integrability of differential equations in general. A N-th order ODE which has a symmetry can be reduced to a $N-1$-th order ODE. A classical Lie's theorem says that if a N-th order ODE admits a N-dimensional solvable Lie group (of point-symmetries) that acts transitively in the space of first integrals then the solutions can be given in terms of N line integrals [1]. It should be noticed that differential equations may have other types of symmetries:

- generalized (or Lie–Backlund), ξ, ϕ_j depend on derivatives of dependent variables.
- nonlocal symmetries, ξ, ϕ_j depend also on integrals of variables.

1.2. Examples

The simplest second order ODE $u'' = 0$ has an eight dimensional group of Lie-point symmetries, using (1) and (2) for $\Sigma = \{u'' = 0\}$ one obtains a system of determining equations, its solutions represent eight independent generators of symmetries, the most general one is:

$$\begin{aligned} v \;=\; & \left(a_1 + a_2 x + a_3 u + a_4 xu + a_5 x^2\right) \frac{\partial}{\partial x} \\ & + \left(a_6 + a_7 x + a_8 u + a_5 xu + a_4 u^2\right) \frac{\partial}{\partial u} \,, \end{aligned}$$

where a_i are arbitrary constants. One can show that every second order ODE admits at the most an eight-dimensional Lie group of point symmetries while a linear n-th order equation admits at least n-dimensional one. Next we give an example of a second order ODE which has no Lie-point symmetries:

$$u'' = xu + e^{u'} + e^{-u'} \,.$$

2. Perfect fluid in shearfree motion

Assume the line element

$$g = -e^{2\nu(r,t)}dt^2 + e^{2\lambda(r,t)}\left[dr^2 + r^2 d\Omega^2\right] . \quad (4)$$

Energy momentum tensor is given by:

$$T^{\alpha}{}_{\beta} = (\mu + p)u^{\alpha}u_{\beta} + p\delta^{\alpha}{}_{\beta} ,$$

where u^{α} is a 4-velocity of the fluid, μ is mass density and p is pressure. Kustaanheimo and Qvist [3] showed that for the metric function

$$u(x,t) = e^{-\lambda(r,t)}, \qquad x \equiv r^2 ,$$

the field equations reduce to one ordinary differential equation

$$u'' = F(x)u^2 . \quad (5)$$

The second metric function is then

$$e^{\nu(r,t)} = \lambda_{,t}e^{-f(t)} ,$$

where $f(t)$ is an arbitrary function connected with the freedom of scaling t. The function $F(x)$ depends on the equation of state of the fluid, and mass density μ as well as pressure p can be computed from λ and f.

2.1. The symmetry approach

Using the procedure (1), (2) one finds that for given $F(x)$, the equation admits a symmetry if there exists $B(x)$ which satisfies

$$F\left(\frac{5}{2}B' + c\right) + BF' = 0 , \qquad B''' = 4(dx + e)F ,$$

where c, d and e are arbitrary constants [4].

For the symmetry approach to the differential equation $u'' = F\,(x)\,u^2$ we encounter two different tasks. The first to determine all functions $F\,(x)$ for which $u'' = F(x)u^2$ admits a symmetry, the second: for given $F(x)$, solve the differential equation.

The first task: two cases have to be considered:

1. $u'' = F\,(x)\,u^2$ admits one symmetry. All functions F belonging to symmetries with either $c = 0$ or $dx + e = 0$ are known and given by

$$B'' = B^{-5/2}$$

for $c = 0$, and by

$$B = \alpha x^2 + 2\beta x + \gamma\,,$$
$$F = B^{-5/2}\exp\left(-c\int\frac{dx}{B}\right)$$

for $dx + e = 0$; for the general case $c\,(dx+e) \neq 0$ no solution is known which admits only one symmetry.

2. $u'' = F\,(x)\,u^2$ admits two symmetries. All these functions F are known; they satisfy

$$2\left[F^{-4/5}\left(F^{-1/5}\right)''\right]'' = F^{-3/5}\left[\left(F^{-1/5}\right)''\right]^2 .$$

The second task: $F\,(x)$ is given and one solves the differential equation. Then three main cases can occur:

1. $u'' = F\,(x)\,u^2$ does not admit a symmetry. Then the symmetry approach does not help.

2. $u'' = F\,(x)\,u^2$ admits one symmetry. Then $u'' = F\,(x)\,u^2$ can be transformed into $\ddot{Y} = -2c\dot{Y} + Y^2 + b_1 Y + b_2$ by means of the transformation

$$Y = \mathrm{const}\,u\,(x) + v\,(x)\ , \quad XY = 0\,, \quad u\,(x) = \frac{1}{\sqrt{B}}\exp\left(-\int\frac{c}{B}dx\right)\,,$$

$$v\,(x) = -\frac{1}{4}BB'' + \frac{1}{8}\left(B'\right)^2 + v_0\,,$$

and

$$s = \int\frac{dx}{B}\,, \qquad Xs = 1\,.$$

If the symmetry has $c = 0$, then this equation can be solved by quadratures. For $c \neq 0$, no solution of $u'' = F\,(x)\,u^2$ is known.

3. $u'' = F(x)u^2$ admits two symmetries. Then its general solution can be given in terms of quadratures by standard group theoretical methods, or by transforming $u'' = F(x)u^2$ using the symmetry with $c = 0$ and then performing some quadratures. In the two-symmetry case F satisfies

$$2F^{-2/5}\left[F^{-2/5}\left(F^{-3/5}\left(F^{-1/5}\right)''\right)'\right]' + F^{-6/5}\left(\left(F^{-1/5}\right)''\right)^2 = 0\,.$$

3. Weyl Conformal Gravity

The Lagrangian density for the Conformal Gravity [5] is given by

$$\mathcal{L} = -\frac{1}{2}C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} = \frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu} - \frac{1}{2}L_{\rm GB}\,, \tag{6}$$

where $C_{\alpha\beta\mu\nu}$ is the Weyl tensor and $L_{\rm GB} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu} + R^2$, the Gauss–Bonnet term, is a total divergence in four dimensional spaces, *i.e.* it does not contribute to the field equations. The Bach equations which follow for the Lagrangian (6) are satisfied by all Einstein vacuum metrics and any conformally related metrics, they read

$$2\nabla^\alpha\nabla^\beta C_{\alpha\ \ \nu\beta}^{\ \ \mu} + R^{\alpha\beta}C_{\alpha\ \ \nu\beta}^{\ \ \mu} = B^\mu{}_\nu = 0\,, \tag{7}$$

where $B^\mu{}_\nu$ is the Bach tensor.
For the sake of simplicity we use a kind of isotropic coordinates:

$$g = -e^{2\nu(x)}d\tau^2 + e^{2(\nu(x)-\alpha(x))}\left[dx^2 + x^2 d\Omega^2\right]\,. \tag{8}$$

The only nonvanishing components of the Bach tensor for the metric (8) read

$$\begin{aligned}
B^0 &= \frac{e^{4(\alpha-\nu)}}{3}\left\{2\alpha'''' + \left(4\alpha' + \frac{8}{x}\right)\alpha''' + 3{\alpha''}^2 - 2{\alpha'}^2\alpha'' - {\alpha'}^4\right.\\
&\quad \left. + \frac{2{\alpha'}^3 + 22\alpha'\alpha''}{x} + \frac{5{\alpha'}^2}{x^2}\right\},\\
B^1{}_1 &= \frac{e^{4(\alpha-\nu)}}{3}\left\{2\left(\alpha' - \frac{1}{x}\right)\alpha''' - {\alpha''}^2 + 2{\alpha'}^2\alpha'' - {\alpha'}^4 + \frac{6{\alpha'}^3}{x}\right.\\
&\quad \left. - \frac{4\alpha'' + 9{\alpha'}^2}{x^2} + \frac{4\alpha'}{x^3}\right\},\\
B^2{}_2 &= \frac{e^{4(\alpha-\nu)}}{3}\left\{-\alpha'''' - 3\left(\alpha' + \frac{1}{x}\right)\alpha''' - {\alpha''}^2 + {\alpha'}^4 - \frac{11\alpha'\alpha'' + 4{\alpha'}^3}{x}\right.\\
&\quad \left. + \frac{2\alpha'' + 2{\alpha'}^2}{x^2} - \frac{2\alpha'}{x^3}\right\} = B^3{}_3\,.
\end{aligned}$$

By direct inspection one finds that there is a simple Lie symmetry

$$v = d_1\frac{\partial}{\partial\alpha} + d_2(x)\frac{\partial}{\partial\nu}\,, \tag{9}$$

which appears to be the conformal symmetry of the field equations. The Bach tensor is trace free

$$B^{\mu}{}_{\mu} = B^0 + B^1{}_1 + B^2{}_2 + B^3{}_3 = 0\,, \qquad B^2{}_2 = B^3{}_3\,,$$

conformally covariant of weight -1 *i.e.* if Ω^2 is a conformal factor

$$g_{\mu\nu} \to \widehat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

then

$$B_{\mu\nu} \to \widehat{B}_{\mu\nu} = \Omega^{-2} B_{\mu\nu}\,.$$

With help of (9) one can find the general solution, which in a slightly different parametrization [6] reads

$$g = -a(x)b^2(x)d\tau^2 + \frac{1}{a(x)}dx^2 + x^2 d\Omega^2\,,$$

where

$$\begin{aligned} a(x) &= c_1 + \frac{c_2}{x} + c_3 x + c_4 x^2\,, \text{ with a constraint} \quad 1 - c_1^2 + 3c_2c_3 = 0\,, \\ b &= \text{const}\,. \end{aligned}$$

We did not detect other Lie symmetries of the above system.

4. Higher-derivative gravity

Higher-derivative gravity theories given by Lagrangian $\mathcal{L} = R + aR^2 + bR_{\mu\nu}R^{\mu\nu}$ are dynamically equivalent to Einstein's General Relativity coupled to certain scalar and spin-2 field theories [7]. We restrict here only to the case when $a = 1/(3m^2)$ and $b = -1/(m^2)$, such a theory by means of a specific Legandre transformation may be recast to the Hamilton picture where it describes a spin-2 field interacting to the standard Einstein gravity.
Euler–Lagrange equations read

$$\begin{aligned} G^{\mu}{}_{\nu} + \frac{1}{m^2}\Big\{ &-\Box R^{\mu}{}_{\nu} + 2R_{\alpha}{}^{\mu}{}_{\nu}{}^{\beta}R^{\alpha}{}_{\beta} + \frac{1}{2}R^{\alpha}{}_{\beta}R^{\beta}{}_{\alpha}\delta^{\mu}{}_{\nu} \\ &+\frac{1}{6}(\Box R - R^2)\delta^{\mu}{}_{\nu} + \frac{1}{3}\nabla^{\mu}\nabla_{\nu}R + \frac{2}{3}RR^{\mu}{}_{\nu}\Big\} = 0 \end{aligned} \tag{10}$$

i.e. Einsten tensor $+ \; 1/n^2$ Bach tensor $= 0$.

The trace of this system implies vanishing of the Ricci scalar (since the Bach tensor is traceless):

$$R = 0\,,$$

therefore, the terms in the bracket may be easily reduced. For the line element

$$g = -e^{2\nu(x)}d\tau^2 + e^{2\lambda(x)}dx^2 + x^2 d\Omega^2\,, \tag{11}$$

where $x = m\,r$, $\tau = m\,t$ are dimensionless $G^1{}_1 + B^1{}_1 = 0$ with help of $R = 0$, $R^{'} = 0$, $R^{''} = 0$, reduces to

$$\left(\frac{\nu^{'}}{x} - \frac{1}{x^2}\right)\lambda^{''} + \frac{\nu^{'3} + \nu^{'2}\lambda^{'} - 2\nu^{'}\lambda^{'2}}{x} + \frac{-\frac{3}{2}\nu^{'2} + \nu^{'}\lambda^{'} + \frac{1}{2}\lambda^{'2}}{x^2}$$
$$+\left(1 - e^{2\lambda}\right)\left(\frac{\lambda^{'}}{x^3} - \frac{1}{x^4}\right) + e^{2\lambda}\left(\frac{\nu^{'}}{x} + \frac{1 - e^{2\lambda}}{2x^2}\right) = 0$$
$$-2e^{-2\lambda}\left\{\nu^{''} + \nu^{'2} - \nu^{'}\lambda^{'} + 2\frac{\nu^{'} - \lambda^{'}}{x} + \frac{1}{x^2}\right\} + \frac{2}{x^2} = 0\,.$$

Solving the system of partial differential equations from (1) one obtains that the only one symmetry vector field is

$$v = \text{const}\,\frac{\partial}{\partial \nu}\,,$$

which can be found immediately since there is no variable ν in the equations of the system but only $\nu^{'}$ and $\nu^{''}$. This does not help to solve our equations. A system with no Lie-point symmetries may have nonlocal symmetries or generalized Lie symmetries, which could provide the only way to find the general solution. The above system does not possess generalized symmetries depending on second order derivatives. We studied also third and fourth order subsidiary systems obtained from the original field equations and found no further symmetries.

5. Conclusions

When a direct approach to differential equations fails one can use the Lie symmetry method, which is one of the most powerful. In case of perfect fluid or conformal gravity existence of symmetries appears to be helpful for solving those difficult equations.

In the higher derivative theory of gravity the field equations have a trivial Lie-point symmetry, but one expects there exist non-local symmetries, however, they are more difficult to detect. We even could not decouple successfully the system of equations in contrary to the other two examples.

The main advantage of the computational recipe for Lie-point symmetries is that it can be used for any systems of differential equations, though sometimes its groups of Lie symmetries turn out to be trivial, nevertheless shedding some light on solvability of a given system.

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