GENERALIZED CALOGERO MODELS AND THEIR COLLECTIVE FIELD FORMULATION*

Joshua Feinberg

Physics Department, University of Haifa at Oranim, Tivon 36006, Israel
and
Physics Department, Technion, Haifa 32000, Israel

(Received October 2, 2007)

The relation between the Calogero model, namely, a system of N identical particles in one dimension with inverse-square interactions, and the three classical types of quantum-mechanical matrix models is well-known. In this talk I explore various generalized Calogero models and identify the quantum mechanical matrix model they correspond to at special values of their couplings. I also present and briefly discuss the collective field formulation of these generalized Calogero models.

PACS numbers: 03.65.Sq, 05.45.Yv, 11.10.Kk, 11.15.Pg

1. Introduction

Consider the $U(N)$-invariant Hamiltonian

$$H_{\rm MM} = -{\rm tr}\,\frac{\partial^2}{\partial\phi^2} + {\rm tr}\,V(\phi) \tag{1.1}$$

governing the dynamics of an $N\times N$ complex hermitian matrix ϕ. Let us diagonalize ϕ by the unitary matrix U as $\phi = U^\dagger X U$, with $X = {\rm diag}(x_1,\ldots,x_N)$, and split (1.1) into its $U(N)$-singlet and $U(N)$-angular momentum pieces as

$$H_{\rm MM} = -H_{\rm singlet} + \text{angular momentum piece}\,. \tag{1.2}$$

Here

$$H_{\rm singlet} = -\sum_{i=1}^N \frac{1}{\Delta(x)^2}\frac{\partial}{\partial x_i}\Delta(x)^2\frac{\partial}{\partial x_i} + \sum_{i=1}^N V(x_i)\,, \tag{1.3}$$

* Presented at the Conference on Random Matrix Theory, “From Fundamental Physics to Applications”, Kraków, Poland, May 2–6, 2007.

where $\Delta(x) = \prod_{i>j}(x_i - x_j)$ is the Vandermonde determinant. The

$$\text{angular momentum piece} = -\sum_{i\neq j} \frac{\frac{\partial}{\partial R_{ij}}\frac{\partial}{\partial R^*_{ij}}}{(x_i - x_j)^2}\,, \tag{1.4}$$

where $dR = UdU^\dagger$ is the right-invariant form.

These formulas may be readily generalized to include all three classical types of matrix models, where, in particular,

$$H_{\text{singlet}} = -\sum_{i=1}^{N} \frac{1}{|\Delta(x)|^\beta}\frac{\partial}{\partial x_i}|\Delta(x)|^\beta\frac{\partial}{\partial x_i} + \sum_{i=1}^{N} V(x_i)\,, \tag{1.5}$$

with $\beta := 1, 2, 4$ corresponding to real-symmetric, complex hermitian, and quaternionic-self-dual matrices, respectively.

The relation between the matrix-model Hamiltonians (1.5) and the Calogero model (CM) [1, 2]

$$H_{\text{c}} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \lambda(\lambda - 1)\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}\,, \tag{1.6}$$

describing N identical particles in one dimensions, subjected to inverse square interactions with dimensionless coupling λ, is well-known. For $\lambda = \beta/2 := 1/2, 1, 2$, H_{c} and H_{singlet} are related by a similarity transformation

$$H_{\text{c}} = \Delta(x)^\lambda\, H_{\text{singlet}}\, \Delta(x)^{-\lambda}\,. \tag{1.7}$$

This similarity transformation can be established by using the identity

$$\begin{aligned}&\frac{1}{\Delta(x)^\lambda}\left(\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} - \lambda(\lambda-1)\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}\right)\Delta(x)^\lambda\\ &= \sum_{i=1}^{N}\frac{1}{\Delta(x)^{2\lambda}}\frac{\partial}{\partial x_i}\Delta(x)^{2\lambda}\frac{\partial}{\partial x_i}\,.\end{aligned} \tag{1.8}$$

For recent reviews on the Calogero and Calogero–Sutherland models see [3, 4]. It is appropriate to mention at this point the recent review on the collective-field and other continuum approaches to the spin-Calogero–Sutherland model [5].

In the next section I shall describe two possible generalizations of the relation (1.7). The first one involves a two-dimensional version of the Calogero model and its relation to normal matrices (or more generally, a d-dimensional version of the Calogero model and its relation to d-commuting hermitian matrices). The second generalization extends (1.6) into two-species of identical particles, and it is related to dynamics over a certain symmetric super-space.

2. Generalized Calogero models and their relation to Matrix Models

2.1. Generalization into d-dimensions

The Calogero–Marchioro model [6] is a d-dimensional generalization of the Calogero model (1.6). Consider N identical particles in d-dimensions with coordinates $\boldsymbol{r}_i \in I\!R^d$ $i = 1, \ldots, N$. The Calogero–Marchioro Hamiltonian is given by

$$-\frac{1}{2}\sum_{i=1}^{N} \nabla_i^2 + g\sum_{i<j} \frac{1}{\boldsymbol{r}_{ij}^2} + G\sum_i \sum_{j<k}{}' \frac{\boldsymbol{r}_{ij}\cdot\boldsymbol{r}_{ik}}{\boldsymbol{r}_{ij}^2\boldsymbol{r}_{ik}^2}\,, \tag{2.1}$$

to which we may add a confining term $\frac{m^2}{2}\sum_i \boldsymbol{r}_i^2$. Here $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$. An important feature of (2.1) is the three-body interaction term. (This term vanishes identically at $d = 1$.) This model was discussed during the past few years in [7, 8], and more recently in [9, 10].

At $g = G = 1$, (2.1) is related to the singlet sector of a quantum-mechanical Hamiltonian describing d-commuting $N \times N$ hermitian matrices $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_d)$, $[\phi_a\,, \phi_b] = 0$, $\phi_a^\dagger = \phi_a$ in a way similar to (1.7).

Let us collect these d commuting matrices into a d-dimensional vector of matrices $\boldsymbol{\phi}$. These commuting matrices are diagonalizable by a common unitary matrix $U \in U(N)$ as $\boldsymbol{\phi} = U^\dagger\, \mathrm{diag}\,(\boldsymbol{r}_1\,, \ldots, \boldsymbol{r}_N)\, U$, $\boldsymbol{r}_i \in I\!R^d$ $i = 1, \ldots, N$. The corresponding Hamiltonian of this matrix model is [10][1]

$$H_{\mathrm{MM}} = -\frac{1}{2}\sum_{i=1}^{N} \frac{1}{\Delta(\boldsymbol{r})^2}\nabla_i\cdot\Delta(\boldsymbol{r})^2\,\nabla_i + \sum_i V(\boldsymbol{r}_i^2) - \sum_{i\neq j}\frac{\frac{\partial}{\partial R_{ij}}\frac{\partial}{\partial R_{ij}^*}}{\boldsymbol{r}_{ij}^2}\,, \tag{2.2}$$

where $\Delta(\boldsymbol{r})^2 = \prod\limits_{i>j}\boldsymbol{r}_{ij}^2$. The case $d = 2$ corresponds to normal matrices, namely, complex matrices ϕ such that $[\phi, \phi^\dagger] = 0$, since a normal matrix can always be written as $\phi = \phi_1 + i\phi_2$, with $\phi_1\,, \phi_2$ being commuting hermitian matrices. The quantum mechanical model of normal matrices was recently analyzed in detail in [10].

As was mentioned above, The Calogero–Marchioro Hamiltonian (2.1), at $g = G = 1$, can be mapped, by a similarity transformation, into the singlet sector of (2.2) (with $V(\boldsymbol{r}^2) \propto \boldsymbol{r}^2$).

[1] The discussion in [10] is for $d = 2$. The generalization to arbitrary d is immediate.

2.2. The multi-species Calogero model in one-dimension

Calogero's original model describes N indistinguishable particles on the line which interact through an inverse-square two-body interaction. It is well-known, however, that the CM may alternatively be interpreted in terms of N free particles obeying generalized exclusion statistics [3, 4, 12–15].

Haldane's formulation of statistics [12] may be extended to systems made of different species of particles, in which the interspecies statistical coupling depends on the species being coupled. This may be implemented in a multi-species generalization of the CM in which particles have different masses and different couplings to each other [16, 18–20].

Quite a few such generalized multi-sepcies Calogero models exist, but contrary to the original CM, knowledge about their exact solvability was rather tenuous. The recent breakthrough in this front derives from the papers [21–24]. The authors of [21] introduced deformed Calogero models, related to root systems of super-algebras, and gave effectively a proof of their integrability. In [22] they presented a more conceptual proof by using shifted super-Jack polynomials. In related developments, the authors of [23, 24] introduced a supersymmetric generalizations of the CM which was based on Jacobians for the radial coordinates on certain super-spaces. Both aforementioned models are closely related to the multi-family generalization of the CM introduced in [25, 26].

Motivated by these developments, the latter model was investigated in [11] in the limit in which each family contains a large number of particles. In this limit, the high-density limit, the system is amenable to large-N collective-field formulation which will be discussed in Sections 3 and 4.

The two-species Calogero model is defined by the Hamiltonian [26]

$$\begin{aligned} H = &- \frac{1}{2m_1}\sum_{i=1}^{N_1}\frac{\partial^2}{\partial x_i^2} + \frac{\lambda_1(\lambda_1-1)}{2m_1}\sum_{i\neq j}^{N_1}\frac{1}{(x_i-x_j)^2} \\ &- \frac{1}{2m_2}\sum_{i=1}^{N_2}\frac{\partial^2}{\partial x_\alpha^2} + \frac{\lambda_2(\lambda_2-1)}{2m_2}\sum_{\alpha\neq\beta}^{N_2}\frac{1}{(x_\alpha-x_\beta)^2} \\ &+ \frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\lambda_{12}(\lambda_{12}-1)\sum_{i=1}^{N_1}\sum_{\alpha=1}^{N_2}\frac{1}{(x_i-x_\beta)^2}\,. \end{aligned} \tag{2.3}$$

Here, the first family contains N_1 particles of mass m_1 at positions $x_i, i = 1, 2, \ldots, N_1$, and the second one contains N_2 particles of mass m_2 at positions x_α, $\alpha = 1, 2, \ldots, N_2$. All particles interact via two-body inverse-square potentials. The interaction strengths within each family are parametrized by the coupling constants λ_1 and λ_2, respectively. The interaction strength between particles of the first and the second family is parametrized by λ_{12}.

In (2.3) we imposed the restriction that there be no three-body interactions, which requires [11, 25–29]

$$\frac{\lambda_1}{{m_1}^2} = \frac{\lambda_2}{{m_2}^2} = \frac{\lambda_{12}}{m_1 m_2}\,. \tag{2.4}$$

It follows from (2.4) that

$$\lambda_{12}^2 = \lambda_1 \lambda_2\,. \tag{2.5}$$

Finally, let me briefly mention that the F-species generalization of (2.3) was also discussed in [11]. In this generic model, the a-th family consists of N_a particles of mass m_a, which interact among themselves with couplings parametrized by λ_{aa} , and with the particles of the b-th family with couplings parametrized by λ_{ab} . The absence of 3-body interactions is guaranteed by $\lambda_{ab}/(m_a m_b) =$ const. for all a, b. Thus $\lambda_{aa}\lambda_{bb} = \lambda_{ab}^2$.

3. Collective field formulation

In this section I shall present a brief and non-technical summary of the collective field formalism [30, 31] in the context of a concrete example — the d-matrix model (2.2). The case $d = 2$ (*i.e.*, the normal matrix model) was discussed in great detail in [10], which contains a pedagogical review of the collective field formalism, in a way which is somewhat complimentary to the presentation in the textbook [30].

Thus, consider the d-dimensional N-body system of (2.2), with particles' coordinates $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ and their conjugate momenta. We shall further assume that N is large, such that the large-density limit may be assumed. Our objective then is to transform these canonical variables into (an overcomplete) continuous set of commuting density operators

$$\rho(\boldsymbol{r}) = \sum_{i=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_i) \tag{3.1}$$

and their conjugate momenta operators $\Pi(\boldsymbol{r})$. The ensuing formalism is a certain avatar of the old Bohm–Pines theory of plasma oscillations [32] .

Evidently, the transformation of $d \cdot N$ d.o.f. to a continuum is ill-defined and needs to be regulated. One possible regularization is to work in Fourier space, where the system is restricted to its lowest dN modes. In the regulated theory, the transformation to density variables is canonical. Collective field theory is a high density, continuum formalism. It probes the many-body system at length scales much larger than the mean interparticle spacing $1/(\bar{\rho})^{1/d}$ (where $\bar{\rho}$ is the mean bulk density). In this way we obtain a continuous hydrodynamic description of the many-body system.

The collective Hamiltonian is obtained as a functional of $\rho(\boldsymbol{r})$ and its conjugate momentum $\Pi(\boldsymbol{r})$. It acts on wave functionals of $\rho(\boldsymbol{r})$ which can be traced back only to purely symmetric factors of the original many-body wave functions $\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$. These symmetric factors may be obtained by stripping off of Ψ any permutation non-invariant factors (*e.g.* — Jastrow factors) by means of similarity transformations.

The collective Hamiltonian governing the dynamics of $\rho(\boldsymbol{r})$ is

$$\begin{aligned} H_{\text{coll}} &= \tfrac{1}{2}\int d\boldsymbol{r}\, \nabla \Pi(\boldsymbol{r}) \cdot \rho(\boldsymbol{r})\, \nabla \Pi(\boldsymbol{r}) + \frac{1}{2}\int d\boldsymbol{r}\, \rho(\boldsymbol{r})\, \boldsymbol{E}^2(\boldsymbol{r}) \\ &\quad + \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) V(\boldsymbol{r}) - \mu \left(\int d\boldsymbol{r}\, \rho(\boldsymbol{r}) - N \right) + \text{a singular term}\,. \end{aligned} \tag{3.2}$$

The singular piece in (3.2) need not concern us here. A particularly important piece of (3.2) is the so-called *collective potential*

$$V_{\text{coll}} = \tfrac{1}{2}\int d\boldsymbol{r}\, \rho(\boldsymbol{r})\, \boldsymbol{E}^2(\boldsymbol{r})\,, \tag{3.3}$$

where

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_i \frac{\boldsymbol{r} - \boldsymbol{r}_i}{(\boldsymbol{r} - \boldsymbol{r}_i)^2 + s^2}\,, \qquad s \to 0\,, \tag{3.4}$$

(note that it is an "electric field" in $d = 2$). The quantity

$$V_{\text{ext}}(\boldsymbol{r}) = \int d\boldsymbol{r}\, \rho(\boldsymbol{r})\, V(\boldsymbol{r}) \tag{3.5}$$

is the *external potential*, originated in the interaction of the particles with the external field $V(\boldsymbol{r})$. Finally, μ is a Lagrange multiplier (the chemical potential) enforcing the constraint $\int d\boldsymbol{r}\, \rho(\boldsymbol{r}) = N$.

By construction H_{coll} is symmetric with respect to the flat integration measure $d\mu\left(\{\rho(\boldsymbol{r})\}\right) = \prod_{\boldsymbol{r}} d\rho(\boldsymbol{r})$ (restricet to $(\rho(\boldsymbol{r}) \geq 0)$).

The density field $\rho(\boldsymbol{r}, t)$ and its momentum $\Pi(\boldsymbol{r}, t)$ obey the canonical commutation relations

$$\left[\rho(\boldsymbol{r}, t)\,, \Pi(\boldsymbol{r}', t)\right] = i\delta(\boldsymbol{r} - \boldsymbol{r}')\,. \tag{3.6}$$

(Here we ignored the fact that the Fourier zero-mode of ρ is not dynamical. This distinction is irrelevant in the large volume limit.) Therefore, the Heisenberg equations of motion of these fields are

$$\begin{aligned} \partial_t \rho(\boldsymbol{r},t) &= -\nabla\cdot\left(\rho(\boldsymbol{r},t)\nabla\Pi(\boldsymbol{r},t)\right) \\ \partial_t \Pi(\boldsymbol{r},t) &= -\tfrac{1}{2}\left(\nabla\Pi(\boldsymbol{r},t)\right)^2 - \frac{\delta}{\delta\rho(\boldsymbol{r},t)} \\ &\quad \times \int d\boldsymbol{r}'\left[\tfrac{1}{2}\,\rho(\boldsymbol{r}',t)\boldsymbol{E}^2(\boldsymbol{r}',t) + \rho(\boldsymbol{r}',t)\,\left(V(\boldsymbol{r}')-\mu\right)\right]. \end{aligned} \tag{3.7}$$

By construction of the collective field formalism, the Heisenberg equations (3.7) lend themselves to hydrodynamical interpretation. To make this explicit we define the velocity field

$$\boldsymbol{v}(\boldsymbol{r},t) = \nabla\Pi(\boldsymbol{r},t) \tag{3.8}$$

in terms of which (3.7) and may be written (after taking the gradient of the second equation in (3.7)) as

$$\begin{aligned} \partial\rho \;&+\; \nabla\cdot(\rho\boldsymbol{v}) = 0\,, \\ \partial_t\boldsymbol{v} \;&+\; (v\cdot\nabla)\boldsymbol{v} = -\nabla W\,. \end{aligned} \tag{3.9}$$

Here

$$W(\boldsymbol{r}) = \frac{\delta}{\delta\rho(\boldsymbol{r},t)}\int d\boldsymbol{r}'\left[\tfrac{1}{2}\,\rho(\boldsymbol{r}',t)\,\boldsymbol{E}^2(\boldsymbol{r}',t) + \rho(\boldsymbol{r}',t)\,\left(V(\boldsymbol{r}')-\mu\right)\right] \tag{3.10}$$

is the enthalpy density of the fluid, and the equations (3.9) really describe the isentropic flow of an Eulerian fluid.

As was mentioned above, the case $d = 2$, corresponding to the large-N limit of quantized normal matrices, was studied in detail in [10] for an *arbitrary* confining potential $V(\boldsymbol{r})$. There, the problem was reduced to 2d nonlinear electrostatics, and the ground state energy and eigenvalue density were obtained explicitly in terms of the couplings in the confining potential. Furthermore, certain quantum phase transitions (disk-annulus transitions in the shape of the ground state's $\rho(\boldsymbol{r})$) were studied in [10] explicitly.

Collective field theory of the 2d Calogero–Marchioro model (2.1) (with emphasis on the special point $g = G = 1$) was studied recently in [33], including analysis of quadratic fluctuations (*i.e.*, leading $1/N$ corrections) around the uniform density ground state.

4. Collective field formulation of the two-family Calogero model

Collective field formulation of the conventional one-family Calogero model appeared originally in [34], as a generalization of the collective field formulation of quantized quaternionic self-dual matrices. Fluctuations (leading $1/N$ corrections) around the infinite-N ground state were later studied in [35]. For a recent review, see [36].

In what follows I shall concentrate on the collective field formulation of the two-family model, following [11]. Recall the Hamiltonian (2.3) (subjected to the constraints (2.4) and (2.5)). Due to the singular interactions at coincidence points, we expect wave functions to vanish at these points. Thus, these many-body wave functions should contain Jastrow factors

$$\Pi_1=\prod_{i>j}^{N_1}(x_i-x_j)^{\lambda_1}\,,\quad \Pi_2=\prod_{\alpha>\beta}^{N_2}(x_\alpha-x_\beta)^{\lambda_2}\,,\quad \Pi_{12}=\prod_{i,\alpha}^{N_1,N_2}(x_i-x_\alpha)^{\lambda_{12}}\,, \tag{4.1}$$

and, therefore, have the general form

$$\Psi(\{x_i\},\{x_\alpha\})=\Pi_1\Pi_2\Pi_{12}\chi_s(\{x_i\},\{x_\alpha\})\,, \tag{4.2}$$

where $\chi_s(\{x_i\},\{x_\alpha\})$ is completely symmetric. As was explained in the previous section, the collective field Hamiltonian, being a functional of the totally symmetric density fields, acts only on the continuum limit of the symmetric factor $\chi_s(\{x_i\},\{x_\alpha\})$. Thus, we have to strip away the Jastrow factors in (4.2), which induces the similarity transformation [37]

$$\tilde{H}=\Pi_{12}^{-1}\Pi_2^{-1}\Pi_1^{-1}\,H\,\Pi_1\Pi_2\Pi_{12} \tag{4.3}$$

on the Hamiltonian. $\tilde{H}$ acts on totally symmetric functions χ_s and thus lends itself to collective field formulation. The relevant collective fields are

$$\begin{aligned}\rho_1(x) &= \sum_{i=1}^{N_1}\delta(x-x_i)\,, \qquad \pi_1(x)=-i\frac{\delta}{\delta\rho_1(x)}\,,\\ \rho_2(x) &= \sum_{\alpha=1}^{N_2}\delta(x-x_\alpha)\,, \qquad \pi_2(x)=-i\frac{\delta}{\delta\rho_2(x)}\,,\end{aligned} \tag{4.4}$$

and they are subjected to the normalization conditions

$$\int dx\,\rho_1(x)=N_1\,,\quad \int dx\,\rho_2(x)=N_2\,. \tag{4.5}$$

The collective Hamiltonian thus obtained from $\tilde{H}$ is

$$\begin{aligned} H_{\text{coll}} &= \frac{1}{2m_1}\int dx\, \Pi_1'(x)\rho_1(x)\Pi_1'(x) + \frac{1}{2m_1}\int dx\,\rho_1(x) \\ &\times \left(\frac{\lambda_1 - 1}{2}\frac{\partial_x\rho_1}{\rho_1} + \lambda_1 ⨍ \frac{dy\rho_1(y)}{x-y} + \lambda_{12} ⨍ \frac{dy\rho_2(y)}{x-y}\right)^2 \\ &+ \frac{1}{2m_2}\int dx\, \Pi_2'(x)\rho_2(x)\Pi_2'(x) + \frac{1}{2m_2}\int dx\,\rho_2(x) \\ &\times \left(\frac{\lambda_2 - 1}{2}\frac{\partial_x\rho_2}{\rho_2} + \lambda_2 ⨍ \frac{dy\rho_2(y)}{x-y} + \lambda_{12} ⨍ \frac{dy\rho_1(y)}{x-y}\right)^2 \\ &+ \mu_1\left(\int dx\,\rho_1(x) - N_1\right) + \mu_2\left(\int dx\,\rho_2(x) - N_2\right) \\ &+ \text{a singular term}\,. \end{aligned} \tag{4.6}$$

4.1. Exact duality symmetries of H_{coll} in (4.6)

The collective-field theory provides a natural framework for analyzing symmetries of the system which cannot be seen directly in the original (finite) N-particle quantum system. An important example in this respect is the strong-weak coupling duality symmetry of the one-family Calogero model discussed in [38]. In [11] this approach was generalized to the multi-family Calogero model. As we shall see momentarily, the collective-field Hamiltonian (4.6) is invariant under certain duality transformations, which interchange, among other things, particles and antiparticles, and thus generalize the duality symmetry [38] of the ordinary Calogero model. These dualities, which form an Abelian group, were all identified and studied in [11]. In particular, the results of [11] enable us to find the conditions under which collective quasi-particles describing density fluctuations in the F-family Calogero model can be identified with those of an effective one-family Calogero model. (In what follows I shall only discuss the two-family case explicitly.) As a by-product, this may help to better understand the exact solvability of some of the recently proposed two-family Calogero models [21–24]. It should be stressed that the duality relations derived and discussed in [11] are *exact* symmetries of the collective-field Hamiltonian, as opposed to the approximate duality symmetries discussed in [39, 40].

Let us now introduce these duality transformations and the corresponding symmetries of the collective-field Hamiltonian (4.6). It is straightforward to check that (4.6) is invariant under the following set of transformations[2,3] of the parameters:

$$\tilde{\lambda}_1 = \frac{1}{\lambda_1}\,, \quad \tilde{\lambda}_2 = \frac{1}{\lambda_2}\,, \quad \tilde{m}_1 = -\frac{m_1}{\lambda_1}\,, \quad \tilde{m}_2 = -\frac{m_2}{\lambda_2}\,, \quad \tilde{\lambda}_{12} = \frac{1}{\lambda_{12}}\,, \tag{4.7}$$

and of the operators:

$$\tilde{\rho}_1 = -\lambda_1\rho_1\,, \quad \tilde{\rho}_2 = -\lambda_2\rho_2\,, \quad \tilde{\pi}_1 = -\frac{\pi_1}{\lambda_1}\,, \quad \tilde{\pi}_2 = -\frac{\pi_2}{\lambda_2}\,. \tag{4.8}$$

Let us denote the set of transformations (4.7) and (4.8) by T_{12}. These transformations are canonical, as they preserve the commutation relations (3.6). For obvious reasons, we refer to the transformations T_{12} as the strong-weak coupling duality transformation. Thus, we see that our Hamiltonian, expressed in terms of the new tilded parameters and operators, is identical in form to the original one, but with λ_1 and λ_2 and the inter-family coupling λ_{12} turned into their reciprocal values; with N_1 and N_2 turned, respectively, into $\tilde{N}_1 = -\lambda_1 N_1$ and $\tilde{N}_2 = -\lambda_2 N_2$ and, finally, with masses m_1 and m_2 turned into $-m_1/\lambda_1$ and $-m_2/\lambda_2$. The minus signs which occur in these identifications are all important: By drawing analogy to a similar situation in the one-family case [38,41], we interpret all negative values of the parameters and densities as those pertaining to holes, or anti-particles. Now, strictly speaking, since N_i and $\tilde{N}_i$ are integers, this interpretation is consistent only for rational couplings, as was discussed in [38, 41].

We further note that (4.6) is invariant also under two more sets of canonical duality transformations. The first one, which we denote by T_1, is comprised of the set of transformations of parameters

$$\tilde{\lambda}_1 = \lambda_1\,, \quad \tilde{\lambda}_2 = \frac{1}{\lambda_2}\,, \quad \tilde{m}_1 = m_1\,, \quad \tilde{m}_2 = -\frac{m_2}{\lambda_2}\,, \quad \tilde{\lambda}_{12} = -\frac{\lambda_{12}}{\lambda_2}\,, \tag{4.9}$$

and of the operators

$$\tilde{\rho}_1 = \rho_1\,, \quad \tilde{\rho}_2 = -\lambda_2\rho_2\,, \quad \tilde{\pi}_1 = \pi_1\,, \quad \tilde{\pi}_2 = -\frac{\pi_2}{\lambda_2}\,. \tag{4.10}$$

Negative values of masses, densities and momenta, as in the previous case, refer to holes. These transformations map the two-family Calogero model of

[2] Note that (4.7) and (4.8) do not constitute a symmetry of the original Hamiltonian (2.3).

[3] It should be mentioned that the $(\lambda_i - 1)\partial_x\rho_i/\rho_i$ terms in (4.6) are crucial in obtaining these transformations uniquely.

particles (positive m_1, m_2, ρ_1 and ρ_2) with inter-family interaction strength λ_{12} into the dual two-family Calogero model of particles (m_1, ρ_1) and holes $(\tilde{m}_2, \tilde{\rho}_2)$ with the inter-family interaction strength $-\lambda_{12}/\lambda_2$. The second (and last) set of duality symmetries of (4.6), which we denote by T_2, is obtained from (4.9) and (4.10) simply by permuting the family indices $1 \leftrightarrow 2$.

It is easy to check that the duality transformations T_1, T_2, T_{12}, together with the identity transformation I, form an Abelian group under composition, in which each element squares to I, and where $T_1T_2 = T_{12}$, $T_1T_{12} = T_2$ and $T_2T_{12} = T_1$. This is readily identified as Klein's four-group. The latter is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, where the two $\mathbb{Z}_2$ factors are $\{I, T_1\}$ and $\{I, T_2\}$.

4.1.1. Resemblance of special two-family Calogero models to single-family Calogero models

As an interesting application of the duality symmetry group, consider the special case of the two-family Calogero model (2.3) in which[4] $\lambda_2 = 1/\lambda_1$ and $m_2 = -m_1/\lambda_1$. From (2.4) we then find that $\lambda_{12} = (m_2/m_1)\lambda_1 = -1$. Note that the two particle families in this system are generically manifestly distinct. Nevertheless, this distinction is, in some sense, an illusion. To see this, note that by the duality transformations (4.9) and (4.10), namely, the element T_1 of the duality group, this system is equivalent to a two-family system with parameters $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_{12} = \lambda_1$, $\tilde{m}_1 = \tilde{m}_2 = m_1$ and densities $\tilde{\rho}_1 = \rho_1, \tilde{\rho}_2 = -\rho_2/\lambda_1$. In the latter dual system, the two families are identical! For this reason, we may refer to the two dimensional locus

$$\lambda_2 = \frac{1}{\lambda_1}, \qquad m_2 = -\frac{m_1}{\lambda_1}, \qquad \lambda_{12} = -1, \tag{4.11}$$

in parameter space as the "surface of hidden identity", or SOHI. Thus, the special two-family Calogero model we started with resembles the single-family Calogero model specified by

$$\lambda = \lambda_1, \qquad m = m_1, \qquad \rho = \tilde{\rho}_1 + \tilde{\rho}_2 = \rho_1 - \frac{1}{\lambda_1}\rho_2. \tag{4.12}$$

Similarly, by inverting the roles of family indices $1 \leftrightarrow 2$ in the previous discussion, which leaves us on the SOHI (4.11), and then applying the duality transformation T_2, we shall conclude that the special two-family Calogero model we started with resembles the single-family Calogero model specified by

$$\lambda = \lambda_2, \qquad m = m_2, \qquad \rho = \rho_2 - \frac{1}{\lambda_2}\rho_1. \tag{4.13}$$

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[4] If we ignore inter-family coupling, we can think of this system as made of two single-family models, related by the one-family version of the strong-weak coupling duality, save for the relation between densities.

In both cases, the effective single-species collective field ρ actually shares the statistics λ and the mass m with the first or the second family, respectively. Note that these two cases can be mapped one each other by the duality transformation T_{12}. Thus, the SOHI (4.11) is left invariant under T_{12}. However, the latter does not act on it freely, as $\lambda_1 = \lambda_2 = -\lambda_{12} = 1$ and $m_1 = -m_2$ is a fixed line. Models lying on this line are comprised of particles and their antiparticles, and only particles and antiparticles interact (repulsively).

Note that we described the relation between the original special two-family models and the corresponding single-family models merely as "resemblance". They are certainly *not* identical! The density operator ρ appearing on the LHS of (4.12) and (4.13), which corresponds to the single-family Calogero model, is defined in a Hilbert space made of many-body wave functions which are completely symmetric in the coordinates of all particles. ρ_1 and ρ_2, on the other hand, are symmetric only in the coordinates of particles of each family separately. The best one could do is perhaps to consider the two-family system with identical families (the one dual to the special two-family systems we started with) as a one-family system divided into two parts, differing by some internal quantum number, in which one symmetrizes in each sector separately. However, this means one should also contrapt an actual physical context to justify such separate symmetrization.

Sergeev and Veselov [21] constructed supersymmetric extensions of the Calogero–Sutherland model which actually correspond to the two-family Calogero model (2.3) with $\lambda_1\lambda_2 = 1, \lambda_{12} = -1$ and $m_1 m_2 < 0$! They gave solutions in terms of deformed Jack polynomials. In a recent paper, Kohler and Guhr [23] introduced a supersymmetric generalization of the Calogero–Sutherland model. Their construction is based on Jacobians for the radial coordinates on certain superspaces. This approach allowed them to explicitly construct the solutions in terms of recursion formulae for a non-trivial ($\lambda_1\lambda_2 = 1$) one-parameter subspace in the (λ_1, λ_2) plane. The underlying model involves two kinds of interacting particles, one with the positive and the other one with the negative mass. Needless to say, this again corresponds to our two-family Calogero model with $\lambda_{12} = -1$. It is interesting to observe that the authors of Refs. [21–24] were probably unaware of the constraints (2.4). Namely, in their approaches, these constraints remain hidden, but still present, as can be easily checked by direct substitutions. Consequently, the two types of models discussed in [21–24], share the very same parametric structure, which enables one to transform them to the one-family Calogero model [11]. (This transformation will be discussed in the last section.) This connection then guarantees their exact integrability. Although the collective-field approach of [11], is applicable only to the multi-species Calogero system with an infinitely large number of particles within each family, it is most likely that the findings of [11] shed some light on the problem of their exact integrability in general.

Sections 2.2 and 4 are based on [11]. I thank V. Bardek and S. Meljanac for introducing me to the multi-species Calogero models and for fruitful collaboration on the subject.

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