ON STATISTICAL MECHANICS
OF A SINGLE PARTICLE IN HIGH-DIMENSIONAL
RANDOM LANDSCAPES*

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We discuss recent results of the replica approach to statistical mechanics of a single classical particle placed in a random $N(\gg 1)$-dimensional Gaussian landscape. The particular attention is paid to the case of landscapes with logarithmically growing correlations and to its recent generalizations. Those landscapes give rise to a rich multifractal spatial structure of the associated Boltzmann–Gibbs measure. We also briefly mention related results on counting stationary points of random Gaussian surfaces, as well as ongoing research on statistical mechanics in a random landscape constructed locally by adding many squared Gaussian-distributed terms.

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One of the simplest models with quenched disorder — a single classical particle subject to a superposition of random Gaussian potential $V(\boldsymbol{x})$ and a non-random confining potential $V_{\mathrm{con}}(\boldsymbol{x})$, with $\boldsymbol{x} \in \mathbb{R}^N$ — turns out to be a surprisingly rich system, characterized by a non-trivial dynamical behavior as well as interesting thermodynamics. Denoting the total potential energy as $\mathcal{H}(\boldsymbol{x}) = V_{\mathrm{con}}(\boldsymbol{x}) + V(\boldsymbol{x})$, the statistical mechanics of the model is controlled by the free energy:

$$F_N = -\beta^{-1} \langle \ln Z(\beta) \rangle_V \,, \qquad Z(\beta) = \int\limits_{|\boldsymbol{x}| \leq L} \exp -\beta \mathcal{H}(\boldsymbol{x}) \, d\boldsymbol{x} \,, \qquad (1)$$

as a function of the inverse temperature $\beta = 1/T$, and the sample size L, with brackets standing for the averaging over the Gaussian potential distribution.

The covariance function of the random part is usually chosen in the form ensuring stationarity and well-defined large-N limit:

$$\langle V\left(\boldsymbol{x}_1\right) V\left(\boldsymbol{x}_2\right)\rangle_V = N\, f_V\left(\frac{1}{2N}(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2\right) . \tag{2}$$

Important information about the structure of the Gibbs–Boltzmann equilibrium measure $p_\beta(\boldsymbol{x}) = \frac{1}{Z(\beta)} \exp -\beta\mathcal{H}(\boldsymbol{x})$ can be extracted from the knowledge of moments

$$m_q = \int\limits_{|\boldsymbol{x}|\leq L} p_\beta^q(\boldsymbol{x})\, d\boldsymbol{x} = \frac{Z(\beta q)}{[Z(\beta)]^q} . \tag{3}$$

In the thermodynamic limit of the sample volume $V_L \to \infty$ one expects typically

$$m_q \sim V_L^{-\tau_q} , \tag{4}$$

where the set of exponents τ_q reflects the spatial organization of the Gibbs–Boltzmann weights. For example, if the weights are of the same order of magnitude across the sample volume, the normalization condition implies locally $p_\beta(\boldsymbol{x}) \sim V_L^{-1}$ and a simple power counting predicts the exponents $\tau_q = q - 1$. In such a situation it is conventional to speak about a *delocalised* measure. The opposite case of a fully *localized* measure describes the situation when essential Gibbs–Boltzmann weights concentrate in the thermodynamic limit in a domain with the finite total volume $V_\xi \ll V_L \to \infty$, and are vanishingly small outside that domain. This situation is obviously characterized by trivial exponents $\tau_{q>0} = 0$ and $\tau_{q<0} = \infty$. Finally, in many interesting situations the exponents τ_q may depend on q nonlinearly, and in this case one commonly refers to the *multifractality* of the measure. Equations (3) and (4) imply the following expression for the characteristic exponents τ_q in the general case

$$\tau_q = |q|\beta\mathcal{F}(|q|\beta) - q\beta\mathcal{F}(\beta) \tag{5}$$

relating them to the appropriately normalized free energy of the system:

$$\mathcal{F}(\beta) = -\lim_{V_L\to\infty} \frac{\ln Z(\beta)}{\beta \ln V_L} . \tag{6}$$

The investigation of multifractal measures of diverse origin is a very active field of research in various branches of physics for more than two decades. In recent years important insights were obtained for disorder-generated multifractality, see [1] for a comprehensive discussion in the context of the Anderson localization transition, and also [2] for an example related to statistical

mechanics with disorder. The multifractality of random Gibbs–Boltzmann measures in a context related to ours appeared in the insightful paper [3]. A popular way of characterizing multifractality invokes the so-called singularity spectrum function $f(\alpha)$. The latter function is used to characterize the number $dN(\alpha) = V_L^{f(\alpha)} d\alpha$ of sites in the sample where the local Gibbs–Boltzmann measure scales as $p_\beta(\boldsymbol{r}) \sim V_L^{-\alpha}$ in the thermodynamic limit. The definition allows to extract the typical characteristic exponents τ_q as, see *e.g.* [1]

$$\tau_q = -\lim_{V_L \to \infty} \frac{\ln \int\limits_{f(\alpha) \geq 0} e^{-\ln V_L [\alpha q - f(\alpha)]} \, d\alpha}{\ln V_L} \, . \tag{7}$$

Performing the α-integration by the Laplace method we obtain that the positive values of the multifractality spectrum $f(\alpha)$ are related by the Legendre transform to the set of exponents τ_q:

$$\tau_q = \alpha_* q - f(\alpha_*), \qquad q = f'(\alpha_*). \tag{8}$$

Thus, the knowledge of the free energy $\mathcal{F}(\beta)$ in (6) allows one to characterize the positive part of the multifractality spectrum of the Boltzmann–Gibbs measure.

Early works by Mezard and Parisi [4], and Engel [5] used the replica trick to calculate the free energy Eq. (1) of an infinite system, $L = \infty$, confined by the simplest parabolic potential $V_{\rm con}(\boldsymbol{x}) = \frac{1}{2}\mu \boldsymbol{x}^2, \mu > 0$. Employing the so-called Gaussian Variational Ansatz (GVA) the authors revealed the existence of a low-temperature phase with broken replica symmetry, hence broken ergodicity. They were followed by Franz and Mezard [6] and Cugliandolo and Le Doussal [7] papers on the corresponding dynamics revealing long-time relaxation, aging, and other effects typical for glassy type of behaviour at low enough temperatures. The nature of the low-temperature phase was found to be very essentially dependent on the type of correlations in the random potential, specified via the covariance function described in Eq. (2). Namely, if the covariance $f_V(u)$ decayed to zero at large arguments u, the description of the low temperature phase was found to require only the so-called one-step replica symmetry breaking (1RSB) Parisi pattern. This effect correctly captures the statistics of the low-lying minima of associated Gaussian energy landscapes [8].

In contrast, for the case of long-ranged correlated potentials with $f_V(u)$ growing with x as a power-law[1] the full infinite-hierarchy Parisi scheme of replica symmetry breaking (FRSB) had to be used instead.

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[1] To be more precise, at large separations we require the structure function $\left\langle [V(\boldsymbol{x}_1) - V(\boldsymbol{x}_2)]^2 \right\rangle_V \propto (\boldsymbol{x}_1 - \boldsymbol{x}_2)^{2\gamma}, \, 0 < \gamma < 1$. However, one can easily satisfy oneself that in the present model under consideration the difference between the structure function and the covariance is immaterial for the free-energy calculations. This will be no longer the case for the model discussed in the end of this article.

Based on formal analogies with the Hartree–Fock method Mezard and Parisi [4] argued that GVA-based calculations should become exact in the limit of infinite spatial dimension N. In a recent paper [9] the replicated problem was reconsidered in much detail by an alternative method which directly exposed the degrees of freedom relevant in the limit $N \to \infty$, and in this way allowed to employ the Laplace (a.k.a. saddle-point) evaluation of the integrals. The results obtained in [9] by this method for the parabolic confinement case fully reproduced those obtained by GVA in [4, 5].

The method of [9] also works for statistical mechanics of a single particle inside any spherical sample $|\boldsymbol{x}| < L$ of a given radius L which makes it particularly suitable for studying, *e.g.*, multifractality of the associated Gibbs–Boltzmann measure. To this end it is easy to understand that the radius L must be scaled with the dimension as $L = R\sqrt{N}$ to ensure nontrivial results when $N \to \infty$. The effective size $R < \infty$ (which is actually half of the length of an edge of the cube inscribed in this sphere) can be used as an additional control parameter of the model. In particular, the chosen scaling $L = R\sqrt{N}$ ensures that the sample volume $V_L = \pi^{N/2} \frac{L^N}{\Gamma(N/2+1)}$ retains in the limit $N \to \infty$ the natural scaling with size R and dimension N. Namely, for $R \gg 1$ we have $\ln V_L = N \ln R+$ smaller terms, which is very essential for the analysis of multifractality.

In what follows we thus concentrate on the case of no confinement potential $V_{\mathrm{con}}(\boldsymbol{x}) = 0$, and choose R to have any fixed value (eventually we will be interested in a kind of thermodynamic limit $R \to \infty$). One of the observations made in [9] is the existence of a simple mathematical criterion which formally differentiates between the short-range correlated potentials and their long-ranged counterparts. Namely, assume the covariance function $f_V(u)$ in (2) to satisfy technical conditions $f'_V(u) < 0$, $f''_V(u) > 0$ and $f'''_V(u) < 0$ for all $u \geq 0$, and also $f'_V(u) \to 0$ when $u \to \infty$. The criterion is based on considering a combination $A(u)$ expressed in terms of $f(u)$ as[2]

$$A(u) = \frac{\frac{3}{2}\left[f'''_V(u)\right]^2 - f''_V(u)f''''_V(u)}{[f''_V(u)]^2} \,, \tag{9}$$

where dashes indicate the order of derivatives taken. Then any potential satisfying $A(u) > 0$, $\forall u \geq 0$ (this family includes, *e.g.*, the potentials with $f_V(u) = \exp\left[-(a + bu)^\alpha\right]$, such that $a > 0, b > 0$ and $0 < \alpha \leq 1$) turned out to have the low-temperature phase which is necessarily of 1RSB type. The standard replica stability analysis of this 1RSB low-temperature phase revealed that the stability is controlled by two eigenmodes, denoted in [9] as Λ_0^* and Λ_K^* (see equations (B.29) and (B.30) of the Appendix B of that

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[2] This eventually coincides, up to the overall sign, with the standard definition of the so-called Schwarzian derivative $\{f'(u), u\}$.

paper). If both are positive, all other eigenvalues of the stability matrix are positive and the 1RSB solution corresponds to an extremum of the free energy functional stable with respect to small variations. And those two eigenvalues were indeed found to be strictly positive as long as $A(u) > 0$.

The situation was found to be very different for the potentials with $A(u) < 0, \forall u \geq 0$ (this family includes, most notably, the power-law-correlated potentials with the covariance of the form $f_V(u) = f(0) - g^2 \times (u+a)^\gamma, f(0) > g^2 a^\gamma, 0 < \gamma < 1$). The low temperature phase is now of FRSB type, and it is only "marginally stable". Indeed, the stability matrix for this type of the replica symmetry breaking can be shown to contain always a family of zero "replicon" modes, see *e.g.* [10] for a calculation in the framework of GVA.

Finally, the above criterion naturally singles out the random potentials satisfying $A(u) = 0, \forall u \geq 0$ as a boundary case between the two regimes. Denoting $\tilde{f}(x) = f_V''(u)$ and noticing that $A(u) = 0$ implies $\tilde{f}'/\tilde{f}^{3/2} = \mathrm{const}$, we find the function $f_V(u)$ to be equal to $f_V(u) = C_0^{-2} \ln (C_0 u + C_1) + C_2 u + C_3$, where C_i are arbitrary constants. The condition $f_V'(u) \to 0$ when $u \to \infty$ then selects the case of logarithmic correlations as the only possible, which we write as

$$f_V(u) = f_0 - g^2 \ln (u + a^2). \tag{10}$$

This latter choice turns out to be in many respects the most interesting situation. Indeed, in [9] it was shown that it leads to a phase diagram which combines features typical for the short-ranged behavior with others characteristic of the long-ranged disorder. As a particular interesting feature we would like to mention that although the low-temperature phase can be thought of as described by a special case of 1RSB breaking scheme, the relevant eigenvalues Λ_0^* and Λ_K^* of the stability matrix identically vanish everywhere in the low-temperature phase[3], rendering 1RSB phase in this special case marginally stable.

The qualitative difference between the three cases — short-ranged, long-ranged, and logarithmic, is most clearly seen in the thermodynamic limit of large sample size $R \to \infty$. One finds that for a typical short-ranged potential the domain of existence of 1RSB phase vanishes as long as $R \to \infty$. For example, the transition (de-Almeida–Thouless [11], AT) temperature signaling of instability of the replica-symmetric solution typically behaves as $T_{\mathrm{AT}}(R) \approx R^2 \sqrt{f_V''(R^2)}$ and rapidly tends to zero for decaying correlations. For any fixed temperature $T > 0$ in the limit $R \to \infty$ the system is effectively in the high-temperature replica symmetric phase, and the free

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[3] This fact, though not explicitly mentioned in [9], immediately follows from definitions (B.29) and (B.30) after substituting for $q_1 - q_0 = Q$ and $q_d - q_1 = y$ the expressions (74) and (79) of that paper.

energy behaves asymptotically like $F(T) \approx -TN\ln R$. Actually this result can be seen as a purely entropic contribution, and in particularly implies via Eqs. (5), (6) the trivial scaling of the exponents $\tau_q = q - 1$, corresponding to the totally de-localized Boltzmann–Gibbs measure.

In contrast, for a power-law growth of correlations one finds that the low-temperature glassy phase occupies bigger and bigger portion of the phase diagram with growing radius R. Indeed, the transition temperature can be shown to grow with R as $T_{\mathrm{AT}}(R) \sim R^\gamma$, and increasing the system size R at any fixed temperature $T > 0$ results in the free energy given asymptotically by the temperature-independent value $F(T)|_{R\to\infty} \sim -NR^\gamma$. This expression actually coincides with the typical minimum of the energy function for our system. The corresponding exponents $\tau_{q>0} = 0$. In a sense the system of this type is always "frozen" in the thermodynamic limit, and indeed the Boltzmann–Gibbs measure is localized on a few deep minima.

Only for the logarithmic case Eq. (9) the transition temperature tends in the thermodynamic limit to a finite value $T_{\mathrm{AT}}(R \to \infty) = g$, and the free energy asymptotics depends non-trivially on the temperature:

$$F(T)|_{R\to\infty} \approx -N\,\ln R \left\{ \begin{array}{ll} T(1+g^2/T^2), & T > g\,, \\ 2g, & T < g\,. \end{array} \right. \tag{11}$$

This is natural to interpret as a freezing transition, precisely of the same sort as appeared in the celebrated Random Energy Model (REM) by Derrida [14]. The same expression for the free energy appeared actually in studies of a zero-energy wavefunction for Dirac particles in dimension $N = 2$ and random magnetic field [3], after a mapping to a problem of statistical mechanics. The Boltzmann–Gibbs measure in this particular case is characterized via a set of non-trivial multifractality exponents τ_q, see [3] and also [12]. The ensuing multifractality spectrum $f(\alpha)$ is simple parabolic for all temperatures, and shows interesting "freezing" behavior for $\alpha \to 0$ at $T = g$, *i.e.* at the point of ergodicity breaking.

We thus see that our results have counterparts in the finite-dimensional systems. Actually, understanding the generic statistical-mechanical behaviour of disordered systems for finite N remains very challenging problem. To this end, rather detailed attempt of investigating our model for finite dimensions $N < \infty$ in the thermodynamic limit $L \to \infty$ was undertaken in a very insightful paper by Carpentier and Le Doussal [12]. That paper also can be warmly recommended for describing the present model in a broad physical context and elucidating its relevance for quite a few other interesting and important physical systems, as *e.g.* directed polymers on trees [13]. The work was based on employing a kind of real-space renormalization group (RG) treatment augmented with numerical simulations. The authors concluded that for finite spatial dimensions neither models with short-ranged,

nor with long-ranged correlations can display a true phase transition at finite temperatures $T > 0$. And only if correlations grow *logarithmically* with the distance, for such marginal situation the true REM-like freezing transition indeed happens at some finite $T > 0$ at any dimension $N \geq 1$. Fortunately, the logarithmic growth is not at all an academic oddity, but rather emerges in quite a few systems of actual physical interest, see [12] for a detailed discussion and further references.

We thus see that the picture following from the results of [9] for the thermodynamic limit (understood as $R \to \infty$) of the model in infinite dimension is in overall qualitative agreement with the $N < \infty$ renormalization group studies of the same model in the limit $L \to \infty$. Another fact which is perhaps worth mentioning is that a recent work [15] claimed that 1RSB low-temperature phase fails to survive in finite spatial dimensions, the fact being related to absence of marginally stable modes in the fluctuation spectrum. If one assumes that the validity of that claim extends beyond the particular model considered in [15], then in our case 1RSB phase in finite dimensions has no chance of survival for any short-range potentials, but in the logarithmic case it could survive due to the mentioned marginal stability. This picture would be indeed in agreement with the above-discussed RG results of [12]. We consider further work in this direction highly desirable, although it is clear that performing any perturbative expansion around $N = \infty$ limit is expected to be a rather technically challenging task.

We end up our presentation by giving a brief overview of a few most recent advances in understanding the statistical mechanics of a single particle in random high-dimensional potentials.

1. Multiscale logarithmic potential

As revealed by Bouchaud and the present author in [16, 17], the picture of potentials with short-ranged, long-ranged, and logarithmic correlations presented above is still incomplete, and misses a rich class of possible behavior that survives in the thermodynamic limit $R \to \infty$. Namely, given any increasing positive function $\Phi(y)$ for $0 < y < 1$, one can consider potential correlation functions $f_V(u)$ in the right-hand side of Eq. (2) which take the following scaling form

$$f_V(u) = -2 \ln R \; \Phi \left(\frac{\ln (u + a^2)}{2 \ln R} \right), \qquad 0 \leq u < R^2. \qquad (12)$$

This type of potential can be constructed by a superposition of several logarithmically correlated potentials of the type (10) with different cutoff scales a_i, and allowing those cutoff scales to depend on the system size R in a power-law way: $a_i \sim R^{\nu_i}$, $0 < \nu_i < 1$ [16].

The thermodynamics of such system in the limit $R \to \infty$ turns out to be precisely equivalent [16] to that of the celebrated Derrida's Generalized Random Energy Model (GREM) [18, 19]. The REM-like case Eq. (10) turns out to be only a (rather marginal) representative of this class: $\Phi(y) = g^2 y$.

The leading term in the equilibrium free energy turns out to be of the form $F(T) = N \ln R \, \mathcal{F}(T)$, where for $0 \leq T \leq T_{\mathrm{AT}} = \sqrt{\Phi'(1)}$

$$-\mathcal{F}(T) = T\nu_*(T) + \frac{[\Phi(\nu_*) - \Phi(0)]}{T} + 2 \int\limits_{\nu_*}^{1} \sqrt{\Phi'(y)} \, dy \,, \qquad (13)$$

where the parameter ν_* is related to the temperature T via the equation

$$T^2 = \Phi'(\nu_*) \,. \qquad (14)$$

For $T > T_{\mathrm{AT}}$ the free energy is instead given by

$$-\mathcal{F}(T) = T + \frac{[\Phi(1) - \Phi(0)]}{T} \,. \qquad (15)$$

These expressions for the free energy can be given a clear interpretation as describing a continuous sequence of "freezing transitions" of REM type, with freezing happening on smaller and smaller spatial scales [16, 17].

The form Eq. (13,15) can give rise to a rather rich multifractal behavior of the Boltzmann–Gibbs measure [17]. The associated singularity spectrum $f(\alpha)$ calculated via Eq. (8) is positive in an interval $\alpha \in (\alpha_{\min}, \alpha_{\max})$, where the zeroes $\alpha_{\min}, \alpha_{\max}$ of the function $f(\alpha)$ are given by

$$\alpha_{\min} = -\beta \mathcal{F}(\beta) - 2\beta \int\limits_{0}^{1} \sqrt{\Phi'(y)} \, dy \,, \quad \alpha_{\max} = -\beta \mathcal{F}(\beta) + 2\beta \int\limits_{0}^{1} \sqrt{\Phi'(y)} \, dy \,.$$

The singularity spectrum is symmetric with respect to the midpoint of the interval of interest, $\alpha_m = (\alpha_{\min} + \alpha_{\max})/2 = -\beta \mathcal{F}(\beta) > 0$, where it has the maximum $f(\alpha_m) = 1$ as expected. Close to this maximum, namely, in the subinterval $\alpha \in (\alpha_-, \alpha_+)$ with $\alpha_\pm = \alpha_m \pm 2A(\beta)T/(T_{\mathrm{AT}})$, where $A(\beta) = \beta^2(\Phi(1) - \Phi(0))$ the singularity spectrum has a simple parabolic shape:

$$f(\alpha) = 1 - \frac{1}{4A(\beta)}(\alpha - \alpha_m)^2 \,, \qquad \alpha_- \leq \alpha \leq \alpha_+ \,. \qquad (16)$$

In particular, at the boundaries $f(\alpha_\pm) = 1 - \beta_{\mathrm{AT}}^2 (\Phi(1) - \Phi(0))$. Note that in the REM-like limit $\Phi(y) = g^2 y$ we have $\alpha_{\min/\max} \to \alpha_{-/+}$ and the parabolic behavior is the only surviving, in agreement with the results of [3, 12].

At the same time outside the interval of parabolicity the general GREM-like model shows a much richer multifractal structure manifesting itself via a quite unusual behavior of the singularity spectrum close to the zeros $\alpha_{\min}, \alpha_{\max}$. To illustrate this fact, we consider a broad class of functions $\Phi(y)$ behaving at small arguments $y \ll 1$ as $\Phi(y) \approx C^2 \, y^{2s+1}$ with $s \geq 0$ and the coefficient $0 < C < \infty$. In particular, in the limiting case $s \to 0$ we are back to the old REM-like model. Now we can extract the behavior of the $f(\alpha)$ when approaching the endpoints $\alpha_{\min}$ or $\alpha_{\max}$. It is given by

$$f(\alpha) \approx \frac{s+1}{s^{s/(s+1)}} \, \alpha_c^{s/(s+1)} |\alpha - \alpha_{\min/\max}|^{\frac{1}{s+1}} , \tag{17}$$

where

$$\alpha_c = \frac{2s^2}{(s+1)(2s+1)} (\beta C \sqrt{2s+1})^{-\frac{1}{s}} . \tag{18}$$

We see that for any $s > 0$ the derivative of the singularity spectrum diverges as $f'(\alpha) \sim |\alpha - \alpha_{\min/\max}|^{-\frac{s}{s+1}} \to \infty$. This is very different from the standard behavior observed in other disordered systems [1, 2]: $f'(\alpha) < \infty$ at zeros of $f(\alpha)$. At the level of multifractal exponents τ_q this feature is translated to a rather unusual behavior for large enough $|q|$, namely: $\tau_q - q\alpha_{\min} = -\alpha_c q^{-\frac{1}{s}}$ for $q > T/T_{\mathrm{AT}}$, and a similar formula for $q < -T/T_{\mathrm{AT}}$. Note, that in the standard situation one always observes linear behavior $\tau_q = q\alpha_{\min,\max}$ starting from some value of $|q|$, see the formula (2.42) in [1] and discussions around it.

2. Extrema of random landscapes and ergodicity breaking

Another set of recent works on the random Gaussian model with correlations specified by Eq. (2) which deserves mentioning is a continuing attempt [20] to relate the phenomenon of ergodicity breaking occurring at the level of statistical mechanics to statistical properties of the minima (and other stationary points) of high-dimensional Gaussian random surfaces $\mathcal{H}(\boldsymbol{x})$, see [21] for introduction to the problematic. The authors managed to show that for a generic smooth, concave confining potentials $V_{\mathrm{con}}(\boldsymbol{x})$ the condition of the *zero-temperature* replica symmetry breaking coincides with one signaling that both mean total number of stationary points in the energy landscape, and the mean number of minima are exponential in N. For a generic system of this sort the (annealed) complexity of minima vanishes cubically when approaching the transition, whereas the cumulative annealed complexity vanishes quadratically. One also can investigate how the complexity depends on the index of stationary points [20, 22]. In particular, in the vicinity of the transition the saddle-points with a positive annealed complexity must be close to minima, as they were found to have a vanishing *fraction* of negative eigenvalues in the corresponding Hessian.

3. Statistical mechanics in a sum of squared
Gaussian-distributed potentials

Finally, let us mention recent work [23] on the statistical mechanics in the energy landscape given by $\mathcal{H}(\boldsymbol{x}) = \frac{\mu}{2}\boldsymbol{x}^2 + \sum_{i=1}^{K} W_i^2(\boldsymbol{x})$. Here $W_i(\boldsymbol{x})$, with $i = 1, \ldots, K$ are assumed to be independent, identically distributed Gaussian functions with zero mean, the variance $\langle W_i^2 \rangle = \sigma\left(\frac{\boldsymbol{x}^2}{N}\right)$ and the structure function $\left\langle [W_i(\boldsymbol{x}_1) - W_i(\boldsymbol{x}_2)]^2 \right\rangle = 2\phi_W\left(\frac{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2}{N}\right)$. Note important differences from the Gaussian case: *(i)* the absence of the factor N in front of the (co)variance, in contrast to Eq. (2), and *(ii)* necessity of specifying both functions σ and ϕ_W, as the phase diagram will actually depend on both of them, in contrast to the the discussion in the footnote 1. The free energy of such a system turns out to have a well-defined large-N limit provided we scale $K = N\kappa$, and consider the parameter $0 < \kappa < \infty$. Naively one may think that the central limit theorem (CLT) would imply that the sum of $K = O(N)$ random terms effectively behaves as a Gaussian potential. A thorough consideration shows that such a reasoning is however deficient for the statistical mechanics problem in hand. Indeed, with lowering the temperature deep minima of the resulting potential start playing most prominent role, and the description of those minima goes beyond the applicability of CLT. This fact suggests that the statistical mechanics of such model may have features rather different from the former Gaussian case due to different statistics of deep minima [8]. The dynamics in this case may also be rather different, see interesting related results in [24].

The free energy can be evaluated in the limit $N \to \infty$ by extending the methods of [9], and the system shows both similarities and dissimilarities to the Gaussian case. In particular, the difference between the short-range and long-range potentials remains to be important, but manifests itself in a somewhat different way. One again finds that the replica-symmetric solution is unstable at low enough temperatures. Let us discuss here only the simplest case of a short-ranged potential with position-independent variance $\sigma\left(\frac{\boldsymbol{x}^2}{N}\right) = \sigma \equiv f_W(0)$, where $f_W(u)$ stands for the covariance function of the field W, related to the structure function as $\phi_W(u) = f_W(0) - f_W(u)$. The equation for the transition (de-Almeida–Thouless) line $T_{\mathrm{AT}}(\mu)$ is then given in terms of the structure function by:

$$\frac{[\phi_W'(\tau_{\mathrm{AT}})]^2 - \phi_W''(\tau_{\mathrm{AT}})\left[\sigma - \phi_W(\tau_{\mathrm{AT}})\right]}{\left[1 + \frac{\phi_W(\tau_{\mathrm{AT}})}{T_{\mathrm{AT}}}\right]^2} = \frac{\mu^2}{\kappa}, \qquad \tau_{\mathrm{AT}} = \frac{T_{\mathrm{AT}}}{\mu}. \qquad (19)$$

By investigating this expression one finds that the phase with broken replica symmetry may exist only as long as the parameter κ exceeds some critical

value

$$\kappa > \kappa_{\mathrm{cr}} = \frac{1}{1 + \frac{f_W(0) f_W''(0)}{f_W'^2(0)}} \, . \qquad (20)$$

Moreover, for every such κ the curvature of the confining potential must satisfy the inequality

$$\mu < \mu_{\mathrm{cr}} = \sqrt{\kappa \left[f_W(0) f_W''(0) + f_W'^2(0) \right]} + f_W'(0) \, . \qquad (21)$$

In the case of long-range potentials one has to take into account the fact of position-dependent variance, which makes the analysis more complicated, and the corresponding phase diagram quite intricate. These features are currently under investigation [23].

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