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AVERAGES OF SPECTRAL DETERMINANTS AND 'SINGLE RING THEOREM' OF FEINBERG AND ZEE*

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We compute $\langle \det(Iz-H)(Iz-H)^{\dagger}\rangle_H$ in the limit of infinite matrix dimension N for complex random matrices H with invariant matrix distribution in terms of the eigenvalue distribution of the Hermitian random matrices $HH^{\dagger}$. Under the assumption that $\frac{1}{N}\ln\langle \det(Iz-H)(Iz-H)^{\dagger}\rangle_H$ is asymptotically equal to $\frac{1}{N}\langle \ln\det(Iz-H)(Iz-H)^{\dagger}\rangle_H$ we reproduce the eigenvalue distribution of H obtained previously by Feinberg and Zee, *Nucl. Phys.* **B501**, 643 (1997).

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Recently there has been shown considerable interest to various averages of products [1–5] and ratios [6–11] of the characteristic polynomials of random matrices. This interest is partially motivated by applications to number theory [1, 4], quantum chaos [12, 13] and quantum chromodynamics [14–19] and is well deserved in the light of emerging links to combinatorics [20, 21], representation theory [8–10] and integrable systems [16, 22, 23].

In this short note we want to address another aspect of averages of characteristic polynomials, namely their relation to the eigenvalue distribution of random matrices. For Gaussian random matrices, such a relation comes in very naturally. The joint eigenvalue probability density function has the form of a multiplicative function of eigenvalues times a power of

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the Vandermonde determinant, see *e.g.* [24]. By rearranging the factors in the Vandermonde determinant one can express the eigenvalue correlation functions in terms of averages of products of characteristic polynomials of Gaussian random matrices of smaller dimensions. This relation, which is exact for finite matrix dimensions, is however rarely used as the eigenvalue correlation functions can be computed by the method of orthogonal polynomials [24]. One notable exception are matrices with complex eigenvalues, especially the Ginibre ensemble of real matrices, where such representation of the mean eigenvalue density in terms of averages of the characteristic polynomials proved to be useful [25–27]. Another case when this relation was successfully employed can be found in [28].

There is another sort of relation, asymptotic in nature, between the eigenvalue distribution and the averaged spectral determinant. It was discovered by Berezin [29] in the 1970s. He proved that in the limit of infinite matrix dimension N

$$\frac{1}{N}\langle \ln\det(Iz-H)\rangle_H = \frac{1}{N}\ln\langle \det(Iz-H)\rangle_H \,. \tag{1}$$

The mean density $w(\lambda)=\frac{1}{N}\langle\sum_{j=1}^N \delta(\lambda-\lambda_j)\rangle_H$ of eigenvalues of H is encoded in the left-hand side of (1),

$$\frac{1}{N}\langle \ln\det(Iz-H)\rangle_H = \int \ln(z-\lambda)w(\lambda)d\lambda\,,$$

whilst the right hand side has no evident direct relation to the eigenvalue distribution of H. Berezin considered a generalised Wigner ensemble of Hermitian random matrices H, but, as we will argue below, the asymptotic relation (1) holds for a variety of Hermitian random matrix ensembles.

The asymptotic relation (1) asserts that the limiting distribution of eigenvalues of H can be read out from the rate of growth of $\langle\det(Iz-H)\rangle$ with N. This latter quantity is much more convenient for computations, and that was Berezin's strategy for obtaining distribution of eigenvalues in the generalised Wigner ensemble.

Berezin also observed that the asymptotic relation (1) implies a faster rate of decay of fluctuations of $\sum_{j=1}^N \ln(z-\lambda_j)$ around its mean value than that typical of the sum of independent random variables in the limit $N\to\infty$. His observation preceded studies of fluctuations of linear eigenvalue statistics in the 1990's. These studies established that in a variety of random matrix ensembles the variance of the linear eigenvalue statistic $\sum_{j=1}^N f(\lambda_j)$ is of the order of unity and the fluctuations are asymptotically Gaussian, see *e.g.* recent publication [30] and references therein. Ignoring the tails of eigenvalue distribution and applying this limit theorem to the test function $\ln(z-\lambda)$, one concludes that

$$\det(Iz-H) = \exp\left[\sum_{j=1}^{N}\ln(z-\lambda_j)\right] = \exp\left[N\langle \ln\det(Iz-H)\rangle_H + \xi\right], \quad (2)$$

where the random variable ξ is asymptotically Gaussian with zero mean and finite variance. The asymptotic relation (1) follows immediately from (2). This argument can be made mathematically rigorous by taking care of the tails of the eigenvalue distribution.

If the random matrices H have no symmetry conditions imposed on them then their eigenvalues are complex and the function $\ln\det(Iz-H)$ is ill-defined. This, however, can be easily remedied. Recall that the density of eigenvalues z_j in the complex plane $z = x + iy$,

$$\rho(x,y) = \frac{1}{N}\sum_{j=1}^{N}\delta(x-x_j)\delta(y-y_j),$$

can be obtained from the logarithmic potential of the eigenvalue distribution by the Poisson equation:

$$\rho(x,y) = \frac{1}{4\pi}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\iint \ln[(x-x')^2+(y-y')^2]\rho(x',y')\,dx'dy'. \quad (3)$$

By writing

$$\begin{aligned}\iint \ln[(x-x')^2+(y-y')^2]\rho(x',y')\,dx'dy' &= \frac{1}{N}\sum_{j=1}^{N}\ln|z-z_j|^2 \\ &= \frac{1}{N}\ln\det(Iz-H)(Iz-H)^\dagger,\end{aligned}$$

one gets back to Hermitian matrices. This procedure is sometimes referred to as Hermitisation. It is natural to expect that the limit theorem for the linear statistics of the eigenvalues of Hermitian matrices extends to the matrices $(Iz-H)(Iz-H)^\dagger$. Under this assumption,

$$\frac{1}{N}\langle \ln\det(Iz-H)(Iz-H)^\dagger\rangle_H = \frac{1}{N}\ln\langle\det(Iz-H)(Iz-H)^\dagger\rangle_H \quad (4)$$

in the limit $N\to\infty$. There is one technicality here, though. In (1) one can ignore the singularity of the logarithmic function at zero by the analyticity argument. Namely, the function $\int\ln(z-\lambda)w(\lambda)d\lambda$ is an analytic function of z and as such is fully determined by its values high in the complex plane, thus avoiding singularities on the real line. However, in (4) one does not have such a luxury and in order to justify the derivation of (4) on the *mathematical* level of rigour, one has to prove that the singularity of the logarithmic function at zero can be ignored. This is a difficult and challenging problem which at the moment has only been solved for matrices with stochastically independent matrix entries [31, 32]. In the context of invariant matrix distributions (6) independence of matrix entries implies $V(t) = t$, or in other words the distribution is Gaussian. It is not clear how to tackle this problem for invariant non-Gaussian distributions.

The aim of this paper is much more modest. We test the validity of the asymptotic relation (4) by calculating

$$\Phi(|z|^2) = \frac{1}{N} \ln\langle \det(Iz - H) \det(Iz - H)^\dagger \rangle_H \tag{5}$$

in the limit $N \to \infty$ for a class of complex random matrices defined by the probability density

$$P(H) = \text{const} \times e^{-N \operatorname{Tr} V(HH^\dagger)} \tag{6}$$

and then verifying that it reproduces the same formulae for the eigenvalue distribution as those obtained by Feinberg and Zee [33], see also [34, 35] for a discussion of features of this eigenvalue distribution.

Our starting point is formula (10) which tells us how to perform the integration over the 'angular' part of H in the spectral determinant $\det(Iz - H)(Iz - H)^\dagger$. The singular value decomposition asserts that H can be written as VDU where U and V are unitary and D is the diagonal matrix of the singular values of H. The Jacobian of the transformation from H to (U, V, D) does not include U or V and is proportional to the square of the Vandermonde determinant composed of the entries of D, see *e.g.* [24, 36]. Hence

$$\langle \det(Iz - H)(Iz - H)^\dagger \rangle_H = \langle \langle \det(Iz - DU)(Iz - DU)^\dagger \rangle_U \rangle_D \,,$$

where the D-average includes the weight function $e^{-N \operatorname{Tr} V(DD^\dagger})$ and the Jacobian, and the U-average is over the unitary group $U(N)$ with respect to the Haar measure. The matrices V disappear because of the invariance of the Haar measure.

The integration in U can be performed explicitly. Indeed, the coefficient c_k of the characteristic polynomial $\det(Iz - DU) = \sum (-1)^k c_k z^{N-k}$ is equal to the sum of all principal minors of order k of the matrix DU. But D is diagonal, and therefore,

$$\det(zI - DU) = \sum_{k=0}^{N} (-1)^k z^{N-k} \sum_{1 \le i_1 < \ldots < i_k \le N} d_{i_1} \cdots d_{i_k} U(i_1, \ldots i_k) \,, \tag{7}$$

where d_i are the diagonal entries of D and $U(i_1, \dots i_k)$ are the principal minors of order k of U,

$$U(i_1, \dots i_k) = \begin{vmatrix} U_{i_1 i_1} & \dots & U_{i_1 i_k} \\ \dots & \dots & \dots \\ U_{i_k i_1} & \dots & U_{i_k i_k} \end{vmatrix}.$$

Multiplying the right-hand side of (7) by its complex conjugate and integrating over U, one gets

$$\begin{aligned} &\langle \det(zI - DU) \det(zI - DU)^\dagger \rangle_U \\ &= \sum_{k=0}^{N} |z|^{2(N-k)} \sum_{1 \le i_1 < \dots < i_k \le N} d_{i_1}^2 \cdots d_{i_k}^2 \langle |U(i_1, \dots, i_k)|^2 \rangle_U . \end{aligned}$$

The cross-product terms $\langle U(i_1, \dots, i_k) \overline{U(j_1, \dots, j_m)} \rangle_U$ vanish because of the invariance of the Haar measure. Another consequence of this invariance is that the mean square of the principal minors of U do not depend on the choice of indices,

$$\langle |U(i_1, \dots i_k)|^2 \rangle_U = \langle |U(1, \dots, k)|^2 \rangle_U .$$

The numbers $\langle |U(1, \dots, k)|^2 \rangle_U$, $k = 1, \dots, N$, can be obtained from the generating function

$$\langle \det(zI - U)(zI - U)^\dagger \rangle_U = \sum_{k=0}^{N} |z|^{2(N-k)} C_N^k \langle |U(1, \dots, k)|^2 \rangle_U . \qquad (8)$$

The evaluation of the left-hand side is a standard random matrix computation. This integral is effectively in the eigenvalues of U,

$$\begin{aligned} &\langle \det(zI - U)(zI - U)^\dagger \rangle_U \\ &= \frac{1}{N!} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \int_0^{2\pi} \frac{d\theta_N}{2\pi} \prod_{j=1}^{N} |z - e^{\theta_j}|^2 \prod_{1 \le j < k \le N} |e^{\theta_j} - e^{\theta_k}|^2 \\ &= \frac{1}{N!} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \int_0^{2\pi} \frac{d\theta_N}{2\pi} \det[f_j(\theta_k)]_{j,k=1}^N \det[\bar{f}_j(\theta_k)]_{j,k=1}^N , \end{aligned}$$

where $f_j(\theta) = (z - e^{i\theta}) e^{i(j-1)\theta}$, $j = 1, \dots, N$, see *e.g.* [24]. An application of the Gramm formula,

$$\int \dots \int \det[f_j(x_k)] \det[g_j(x_k)] \prod_{k=1}^{N} dx_k = N! \det \left[\int f_j(x) g_k(x) dx \right],$$

then yields

$$\langle \det(zI-U)(zI-U)^\dagger\rangle_U = \sum_{k=1}^{N} |z|^{2(N-k)}.$$

Hence $\langle |U(1,\ldots,k)|^2\rangle_U = \frac{1}{C_N^k}$, and

$$\begin{aligned}\langle \det(zI-DU)\det(zI-DU)^\dagger\rangle_U &= \sum_{k=0}^{N} |z|^{2(N-k)}\frac{1}{C_N^k}\sum_{1\le i_1<\ldots<i_k\le N} d_{i_1}^2\cdots d_{i_k}^2\\ &= (N+1)\int_0^\infty \frac{\det(I|z|^2+tD^2)}{(1+t)^{N+2}}dt\,, \qquad (9)\end{aligned}$$

where we have used the integral representation for the Beta function,

$$\frac{1}{C_N^k} = (N+1)\int_0^\infty \frac{t^k dt}{(1+t)^{N+2}}.$$

Making now the substitution $t\to t/|z|^2$ in the obtained integral (9) and going back to the matrices H we arrive at the desired formula

$$\langle \det(Iz-H)\det(Iz-H)^\dagger\rangle_H = (N+1)|z|^{2(N+1)}\int_0^\infty \frac{\langle\det(It+HH^\dagger)\rangle_H}{(t+|z|^2)^{N+2}}dt\,. \qquad (10)$$

It reduces the integration over the eigenvalues and eigenvectors of H in the spectral determinant $\langle\det(Iz-H)(Iz-H)^\dagger$ which is very difficult to perform to an integration over the eigenvalues of $W=HH^\dagger$. Conveniently, the latter problem is Hermitian. The matrices W belong to the well studied class of unitary invariant ensembles and various formulas are known for the eigenvalue distribution. This makes the computation of the right-hand side of (10) in the limit $N\to\infty$ easy.

Before we proceed to this computation we would like to make one remark. The derivation of (10) given here can be generalised to the higher moments of $\det(Iz-H)$. This generalisation uses Schur function expansions and a Selberg-type integral and leads to the following formula [37, 38]

$$\langle \det(I-AU)^m\det(I-U^\dagger B^\dagger)^n\rangle_U \propto \int_{\mathbb{C}^{n\times m}} \frac{\det(I+QQ^\dagger\otimes AB^\dagger)}{\det(I+QQ^\dagger)^{N+n+m}}dQ\,,$$

where the integration on the left is over the unitary group $U(N)$ and the integration on the right is over the complex matrices Q of size $n\times m$, $n,m\ge 1$.

This formula can also be obtained by using Zirnbauer's colour–flavour transformation [39, 40]. In fact, the two approaches are closely related, for a discussion see [38].

Now, we can to compute the function $\Phi(|z|^2)$ of equation (5) in the limit $N \to \infty$. In this limit

$$\langle \det(It + HH^\dagger) \rangle_H = \exp\left[N \int \ln(t+\lambda) w(\lambda) d\lambda \right],$$

where $w(\lambda)d\lambda$ is the limiting distribution of eigenvalues of $W = HH^\dagger$. Therefore, by (10),

$$\Phi(|z|^2) = \ln|z|^2 + \frac{1}{N} \ln \int_0^\infty \frac{\exp\left\{ N \left[\int \ln(t+\lambda) w(\lambda) d\lambda - \ln(t+|z|^2) \right] \right\}}{(t+|z|^2)^2} dt \,. \tag{11}$$

For large N, the main contribution to the integral on the right comes from the small intervals around the points of maximum of

$$\phi(t) = \int \ln(t+\lambda) w(\lambda) d\lambda - \ln(t+|z|^2) \,.$$

The equation for the stationary points of $\phi(t)$ reads

$$\int \frac{w(\lambda)d\lambda}{t+\lambda} - \frac{1}{t+|z|^2} = 0 \,, \tag{12}$$

or, by rearranging,

$$\frac{1 - t \int \frac{w(\lambda)d\lambda}{t+\lambda}}{\int \frac{w(\lambda)d\lambda}{t+\lambda}} = |z|^2 \,.$$

If $w(\lambda)$ is not a delta-function, then the left-hand side in the above equation increases monotonically with t, taking the value $1/m_{-1}$ at $t = 0$, and the value m_1 at $t = +\infty$,

$$m_{\pm 1} = \int \lambda^{\pm 1} w(\lambda) d\lambda \,.$$

Thus, if $|z|^2 < 1/m_{-1}$ then $\phi(t)$ increases monotonically with t and the weight of the integral on the right-hand side in (11) is concentrated in the vicinity of $t = 0$. Hence, in the limit $N \to \infty$,

$$\Phi(|z|^2) = \int (\ln \lambda) w(\lambda) d\lambda \,, \qquad |z|^2 < \frac{1}{m_{-1}} \,. \tag{13}$$

If $|z|^2 > m_1$ then $\phi(t)$ decreases monotonically with t and the weight of the integral on the right-hand side in (11) is concentrated in the vicinity of $t = +\infty$. A quick calculation shows that, in the limit $N \to \infty$,

$$\Phi(|z|^2) = \ln |z|^2 , \quad |z|^2 > m_1 . \tag{14}$$

For $1/m_{-1} < |z|^2 < m_1$, the equation for the stationary points of $\phi(t)$ has exactly one solution. At this point the function $\phi(t)$ is maximal. Therefore, in the limit $N \to \infty$,

$$\Phi(|z|^2) = \ln |z|^2 + \int \ln(t_0 + \lambda) w(\lambda) d\lambda - \ln(t_0 + |z|^2) , \quad \frac{1}{m_{-1}} < |z|^2 < m_1 , \tag{15}$$

where t_0 is determined by equation (12).

Let us now check whether the obtained formulas reproduce the eigenvalue distribution found by Feinberg and Zee [33]. Because of the invariance of the matrix distribution (6), the mean eigenvalue density is invariant with respect to the rotations $z \to z e^{i\theta}$,

$$\langle \rho(x, y) \rangle_H = \rho(r), \quad r^2 = x^2 + y^2,$$

and the eigenvalue distribution can be described by the integrated density of the radial parts of eigenvalues,

$$\gamma(r) = \int \int_{x^2+y^2 \le r^2} \langle \rho(x, y) \rangle_H dx dy = 2\pi \int_0^r \rho(r') r' dr' . \tag{16}$$

In other words, $N\gamma(r)$ is the expected number of the eigenvalues of H in the disk $|z| \le r$. The function $\gamma(r)$ can be easily found from the electrostatic potential

$$\tilde{\Phi}(|z|^2) = \frac{1}{N} \left\langle \ln \det(Iz - H)(Iz - H)^\dagger \right\rangle_H .$$

By the Poisson equation (3),

$$\rho(r) = \frac{1}{\pi} \left[\frac{\partial}{\partial r^2} + r^2 \frac{\partial^2}{(\partial r^2)^2} \right] \tilde{\Phi}(r^2) .$$

On substituting this back into equation (16), we obtain the expression for the integrated radial eigenvalue density in terms of the logarithmic potential, $\gamma(r) = r^2 \partial \tilde{\Phi}(r^2)/\partial r^2$. Referring now to the asymptotic relation (4), we replace $\tilde{\Phi}(|z|^2)$ in the limit $N \to \infty$ by the function $\Phi(|z|^2)$ of equation (5),

$$\gamma(r) = r^2 \, \frac{\partial \Phi(r^2)}{\partial r^2} .$$

From (13) and (14) we immediately conclude that $\gamma(r) = 0$ for all $r^2 < 1/m_{-1}$ and $\gamma(r) = 1$ for all $r^2 > m_1$, thus recovering the mean feature of the eigenvalue distribution found by Feinberg and Zee. Namely, depending on the tail at the $\lambda = 0$ boundary of the eigenvalue distribution of the *radial* part of H, the eigenvalues of H are distributed either in the disk $r^2 \leq m_1$, or, in the annulus $1/m_{-1} \leq r^2 \leq m_1$. One cannot have the eigenvalue distribution of H supported by several disjoint rings. This is in a stark contrast to Hermitian invariant ensembles where the eigenvalue distribution can be supported by disjoint intervals on the real axis, see the discussion in [34].

On the support of the eigenvalue distribution,

$$\gamma(r) = \frac{t_0}{t_0 + r^2}\,, \qquad \frac{1}{m_{-1}} < r^2 < m_1\,, \tag{17}$$

where t_0 is the unique solution of equation (12). One arrives at this formula by differentiating the function Φ given by (15) and noting that $d\phi(t)/dt = 0$ at $t = t_0$.

Equation (17) gives the radial density $\gamma(r)$ of the eigenvalue distribution of H in terms of the solution of the equation

$$\int \frac{w(\lambda) d\lambda}{t_0 + \lambda} = \frac{1}{t_0 + r^2}\,. \tag{18}$$

There is a one-to-one correspondence between γ and t_0. This can be used to rewrite the equation for t_0 as an equation for γ:

$$\frac{\gamma}{1-\gamma}\left[1 - \gamma - r^2 \int \frac{w(\lambda) d\lambda}{\lambda + \frac{r^2\gamma}{1-\gamma}}\right] = 0\,. \tag{19}$$

This is the equation obtained by Feinberg and Zee.

Of course equations (18) and (19) can only be solved explicitly in exceptional cases. One such case is the Ginibre ensemble of complex matrices [41] which corresponds to $V(t) = t$ in (6). In this case $W = HH^\dagger$ is the Wishart ensemble of random matrices. The eigenvalue distribution in this ensemble obeys the Marchenko–Pastur law,

$$w(\lambda) = \frac{1}{2\pi}\sqrt{\frac{4-\lambda}{\lambda}}\,, \qquad 0 < \lambda < 4\,.$$

A quick calculation shows that $m_1 = 0$, $m_{-1} = \infty$ and

$$\int \frac{w(\lambda) d\lambda}{t + \lambda} = \frac{2}{\sqrt{4t + t^2} + t}\,.$$

Hence the solution of the equation for t_0 is $t_0 = r^4/(1-r^2)$. On substituting this into (17) one gets $\gamma(r) = r^2$, $0 < r < 1$. This is nothing else the Ginibre circular law [41].

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